

On Subharmonic Functions which are Bounded Above by Certain Functions

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1. Introduction

Let $X=(x_1, x_2, \dots, x_k)$ denote a point in the k -dimensional Euclidean space R^k ($k \geq 1$) and $\|X\|$ denote the norm of X .

$$\|X\| = \sqrt{x_1^2 + x_2^2 + \dots + x_k^2}.$$

The k -dimensional Lebesgue measure of a set S in R^k is denoted by $|S|$. With a non-negative measurable function $f(X)$ defined on R^m ($m \geq 1$), we associate a non-increasing function $\eta = F_f(\xi)$ on the interval $(0, +\infty)$ such that for every $t \geq 0$ the m -dimensional measure $|S_f(t)|$ of the set

$$S_f(t) = \{X \in R^m \mid f(X) \geq t\}$$

is equal to the one-dimensional Lebesgue measure of the set

$$\{\xi \mid 0 < \xi < +\infty, F_f(\xi) \geq t\}.$$

Such a function $F_f(\xi)$ is obtained by considering the inverse function of $\xi = |S_f(\eta)|$ and is uniquely determined except on a countable set. A non-negative measurable function $f(X)$ on R^m is said to grow slimly, if

$$(1) \quad \int_0^{\infty} \xi^{-(m-1)/m} \log^+ F_f(\xi) d\xi < +\infty.$$

We note that for a function $f(x)$ defined on R (R^1 is simply denoted by R), (1) is equivalent to the condition

$$\int_{-\infty}^{+\infty} \log^+ f(x) dx < +\infty$$

from the definition of the Lebesgue integral.

Domar [4, Theorem 3] proved the following fact: Let a function $f(X)$ be a slimly growing function on a domain D in R^m and $u(P)$ be subharmonic on the cylinder

$$E = \{P=(X,y) \mid X \in D, 0 < y < c\},$$

where c is a positive constant, such that

$$u(P) \leq f(X)$$

for any $P=(X,y)$, $X \in D$, $0 < y < c$. Then,

$$u(P) \leq K$$

on every compact subset of E , where K is a constant independent of $u(P)$.

In this paper, given a slimly growing function $f(X)$ on R^m and some function $h(y)$ on $(0, +\infty)$, we consider an analogous problem to Domar's with respect to a subharmonic function $u(P)$ defined on the $(m+n)$ -dimensional Euclidean space R^{m+n} such that

$$u(P) \leq f(X)h(\|Y\|)$$

for any $P=(X,Y)$, $X \in R^m$, $Y \in R^n$. Using an obtained result, we give a sharpened Phragmén-Lindelöf theorem which extends a result of Deny and Lelong [1], [2] and a result of Brawn [3, Theorem 1].

2. Statements of fundamental results

The proofs of all theorems in this section will be given in the last section. Let $y_0 \geq 0$ be a constant. A positive non-decreasing function $h(y)$ defined for $(y_0, +\infty)$ is said to

grow regularly, if there is a constant $\mu \geq 1$ such that

$$h(y+1) \leq \mu h(y)$$

for any $y > y_0$.

The following result is essentially based on Domar's idea in [4].

Theorem 1. Let $f(X)$ be a slimly growing function on R^m and $h(y)$ be a regularly growing function on $(y_0, +\infty)$, $y_0 \geq 0$, i.e.

$$h(y+1) \leq \mu h(y)$$

for any $y > y_0$. Suppose that $u(P)$ is a subharmonic function on R^{m+n} such that

$$u(P) \leq f(X)h(\|Y\|)$$

for any $P=(X,Y)$, $X \in R^m$, $Y \in R^n$, $\|Y\| > y_0$.

Then, there exists a constant K dependent only on $f(X)$ and μ such that

$$u(P) \leq Kh(\|Y\|)$$

at every $P=(X,Y)$, $X \in R^m$, $Y \in R^n$, $\|Y\| > y_0 + 2$.

Remark 1. If a function $h(y)$ grows regularly, we can find two positive constants A and B such that

$$h(y) \leq Ae^{By}$$

to every $y > y_0$. In fact, let $y, y > y_0$, be any number and take a non-negative integer n satisfying

$$n \leq y - y_0 < n+1.$$

Then,

$$h(y) \leq h(y_0 + (n+1)) \leq \mu^n h(y_0 + 1) \leq \mu^{(y-y_0)} h(y_0 + 1) = Ae^{By},$$

where

$$A = \mu^{-y_0} h(y_0+1), \quad B = \log \mu.$$

But, the converse is not always true. Consider the non-decreasing function $h(y)$ on $(0, +\infty)$ defined by

$$h(y) = \int_0^y \phi(t) dt$$

where

$$\phi(t) = \begin{cases} t & t \in (0, 1) \\ (n-1)! & t \in [(n-1)!, n!) \quad (n=2, 3, \dots). \end{cases}$$

Then, since $\phi(t) \leq t$, we have

$$h(y) \leq 2^{-1} e^{2y}.$$

On the other hand, for a sequence $\{y_n\}$, $y_n = \log n!$ ($n \geq 2$),

$$h(1+y_n) = \int_0^{en!} \phi(t) dt > \int_{n!}^{2n!} \phi(t) dt \geq (n!)^2$$

and

$$nh(y_n) = n \int_0^{n!} \phi(t) dt \leq n(n-1)!n! = (n!)^2.$$

This shows that $h(y)$ does not grow regularly.

It follows from Remark 1 that $h(y)$ in Theorem 1 must satisfy the growth condition

$$(2) \quad h(y) = O(e^{By}) \quad (y \rightarrow \infty)$$

for some constant $B > 0$. The following Theorem 2 analogous to Otsuka's [6] shows that (2) is almost sharp.

Theorem 2. For any $\varepsilon > 0$, there exists a subharmonic function $u_\varepsilon(P)$ on R^{m+n} satisfying the following conditions (i) and (ii);

(i) for a slimly growing function $f_\varepsilon(X)$ on R^m

$$u_\varepsilon(P) \leq f_\varepsilon(X) e^{\|Y\|^{1+\varepsilon}}$$

at any $P=(X,Y)$, $X \in R^m$, $Y \in R^n$,

$$(ii) \quad \sup_{P=(X,Y), X \in R^m, Y \in R^n} u_\varepsilon(P) e^{-\|Y\|^{1+\varepsilon}} = +\infty.$$

Question. The function $h(y)=e^{y^{1+\varepsilon}}$ does not grows regularly because it grows quickly. Is it possible to find any result similar to Theorem 2 for a slowly growing function $h(y)$ which does not grow regularly?

The following Theorem 3 shows that the exponent $-(m-1)/m$ of the condition (1) for slim growth of $f(X)$ is best value in Theorem 1.

Theorem 3. There exists a subharmonic function $u(P)$ on R^{m+n} satisfying the following two conditions (i) and (ii);

(i) for a non-negative measurable function $f(X)$ satisfying

$$\int_0^\infty \xi^{-\ell} \log^+ F_f(\xi) d\xi < +\infty \quad \text{for any } \ell < (m-1)/m$$

and a regularly growing function $h(y)$ on $(0, +\infty)$,

$$u(P) \leq f(X)h(\|Y\|)$$

at every $P=(X,Y)$, $X \in R^m$, $Y \in R^n$, $\|Y\| \neq 0$.

$$(ii) \quad \sup_{P=(X,Y), X \in R^m, Y \in R^n, \|Y\| \neq 0} u(P)h(\|Y\|)^{-1} = +\infty.$$

3. Extended Phragmén-Lindelöf theorems

By R^+ , we denote the set of positive real numbers. Let G

be a domain in R^k ($k \geq 2$) and denote the boundary of G by ∂G . When a function $u(P)$ on G is given, we say that $u(P)$ satisfies the Phragmén-Lindelöf boundary condition on ∂G , if

$$\overline{\lim}_{P \in G, P \rightarrow Q} u(P) \leq 0$$

for every $Q \in \partial G$. When a domain D in R^m and a function $u(P) = u(X, Y)$ on

$$D \times R^n = \{P = (X, Y) \in R^{m+n} \mid X \in D, Y \in R^n\}$$

are given, the maximum modulus $M(u, y)$ of $u(P)$ is defined on R^+ by

$$M(u, y) = \sup_{X \in D, Y \in R^n, \|Y\| = y} u(X, Y),$$

Hardy and Rogosinski [5] proved:

Theorem HR. Let D be an open interval (α, β) and $u(z)$ be a subharmonic function in the half-strip

$$\Lambda = \{z = x + iy \mid x \in D, y \in R^+\}$$

such that $u(z)$ satisfies the Phragmén-Lindelöf boundary condition on $\partial \Lambda$ and

$$\overline{\lim}_{y \rightarrow \infty} M(u, y) e^{-(\beta - \alpha)^{-1} \pi y} \leq 0.$$

Then

$$u(z) \leq 0$$

on Λ .

Deny and Lelong [1], [2] generalized Theorem HR to a function defined on a half-cylinder in the Euclidean space of higher dimension. In the following, a bounded domain in R^m having sufficiently smooth boundary (if $m=1$, an interval) is

called a bounded regular domain. For a given bounded regular domain D , let $\lambda_D > 0$ be the first eigenvalue of the boundary value problem with respect to D :

$$\Delta f + \lambda_D f = 0 \quad \text{in } D, \quad f = 0 \quad \text{on } \partial D$$

where Δ denotes the Laplace operator (if $m=1$, $\Delta = \frac{d^2}{dx^2}$). If D is an interval (α, β) in R , we easily see

$$\sqrt{\lambda_D} = (\beta - \alpha)^{-1} \pi.$$

Theorem DL. Let D be a bounded regular domain in R^m ($m \geq 1$) and $u(P)$ be a subharmonic function in $\Gamma = D \times R^+$ such that $u(P)$ satisfies the Phragmén-Lindelöf boundary condition on $\partial\Gamma$ and

$$\overline{\lim}_{y \rightarrow \infty} M(u, y) e^{-\sqrt{\lambda_D} y} \leq 0.$$

Then,

$$u(P) \leq 0$$

on Γ .

On the other hand, Brawn [3, Theorem 1] generalized Theorem HR to a subharmonic function in the strip $(0, 1) \times R^n$ in R^{n+1} ($n \geq 1$).

Theorem B. Let $u(P)$ be a subharmonic function in

$$\Omega = (0, 1) \times R^n \quad (n \geq 1)$$

such that $u(P)$ satisfies the Phragmén-Lindelöf boundary condition on $\partial\Omega$ and

$$\overline{\lim}_{y \rightarrow \infty} M(u, y) e^{-\pi y^{(n-1)/2}} \leq 0.$$

Then

$$u(P) \leq 0$$

on Ω .

Now, we shall give a generalized form of Theorem DL and Theorem B.

Theorem 4. Let D be a bounded regular domain in R^m ($m \geq 1$) and $u(P)$ be a subharmonic function on the domain $\Pi = D \times R^n$ in R^{m+n} such that $u(P)$ satisfies the Phragmén-Lindelöf boundary condition on $\partial\Pi$ and

$$\lim_{y \rightarrow \infty} M(u, y) e^{-\sqrt{\lambda_D} y} y^{(n-1)/2} \leq 0.$$

Then,

$$u(P) \leq 0$$

on Π .

Now, we shall give an extension of Theorem 4.

Theorem 5. Let D be a bounded regular domain in R^m ($m \geq 1$) and $u(P)$ be a subharmonic function on the domain $\Pi = D \times R^n$ such that $u(P)$ satisfies the Phragmén-Lindelöf boundary condition on $\partial\Pi$. Suppose that for a slimly growing function $f(X)$ on R^m

$$u(P) \leq \varepsilon(\|Y\|) f(X) e^{\sqrt{\lambda_D} \|Y\|} \|Y\|^{(1-n)/2}$$

at every $P=(X, Y)$, $X \in D$, $Y \in R^n$, $\|Y\| \neq 0$, where $\varepsilon(t)$ is a function on R^+ satisfying

$$\varepsilon(t) \rightarrow 0 \quad (t \rightarrow \infty).$$

Then,

$$u(P) \leq 0$$

on Π .

Remark 2. If $n=1$, Theorem 5 extends Theorem DL. If D is $(0,1)$ in R , Theorem 5 extends Theorem B.

The following Theorem 6 shows that the exponent $-(m-1)/m$ in the condition (1) for slim growth of $f(X)$ is best value in Theorem 5.

Theorem 6. There exists an unbounded subharmonic function $u(P)$ on the domain $\Pi_0 = D_0 \times R^n$ ($n \geq 1$),

$$D_0 = \{X \in R^m \mid \|X\| < 2^{-1}\pi\} \quad (m \geq 1)$$

which satisfies the following conditions (i) and (ii):

(i) $u(P)$ satisfies the Phragmén-Lindelöf boundary condition on $\partial\Pi_0$,

(ii) for a function $\varepsilon(t)$ on R^+ satisfying

$$\varepsilon(t) \rightarrow 0 \quad (t \rightarrow \infty)$$

and a non-negative measurable function $f(X)$ on R^m satisfying

$$\int_0^\infty \xi^{-\ell} \log^+ F_f(\xi) d\xi < +\infty \quad \text{for any } \ell < (m-1)/m,$$

$$u(P) \leq \varepsilon(\|Y\|) f(X) e^{\sqrt{\lambda_D} \|Y\|} \|Y\|^{(1-n)/2}$$

at every $P=(X,Y)$, $X \in D$, $Y \in R^n$, $\|Y\| \neq 0$.

4. Proofs of theorems

By $C_{m+n}(P,r)$, we denote the $(m+n)$ -dimensional ball having a center $P \in R^{m+n}$ and a radius r . To prove Theorem 1, we need the following Lemma which is analogous to Domar's [4, Lemma

2].

Lemma. Let $f(X)$ be a slimly growing function on R^m and $h(y)$ be a regularly growing function on $(y_0, +\infty)$, $y_0 \geq 0$, i.e.

$$h(y+1) \leq \mu h(y)$$

for any $y > y_0$. Suppose that $u(P)$ is a subharmonic function on R^{m+n} such that

$$(3) \quad u(P) \leq f(X)h(\|Y\|)$$

for any $P=(X,Y)$, $X \in R^m$, $Y \in R^n$, $\|Y\| > y_0$. Let Q and λ be positive integers satisfying

$$e A_n^{-1} A_{m+n}^{-1} Q^{-m} + e^{-\lambda} < \mu^{-1}$$

where

$$A_k = \pi^{k/2} / \Gamma(2^{-1}k+1).$$

If there are an integer v satisfying

$$0 < Q |S_f(e^{v-\lambda})|^{1/m} < 1$$

and a point $P = (X_v, Y_v)$, $X_v \in R^m$, $Y_v \in R^n$, $\|Y_v\| > y_0 + 1$

such that

$$u(P_v) \geq e^v h(\|Y_v\|),$$

then there also exists a point $P_{v+1} = (X_{v+1}, Y_{v+1}) \in C_{m+n}(P_v, r_v)$,

$X_{v+1} \in R^m$, $Y_{v+1} \in R^n$,

$$r_v = Q |S_f(e^{v-\lambda})|^{1/m},$$

such that

$$u(P_{v+1}) \geq e^{v+1} h(\|Y_{v+1}\|).$$

Proof. First of all, we note that

$$(4) \quad e^v h(\|Y_v\|) \leq u(P_v) \leq A_{m+n}^{-1} r_v^{-(m+n)} \int_{C_{m+n}(P_v, r_v)} u(P) dP$$

where dP denotes the $(m+n)$ -dimensional volume element (see e.g. Rado [7]).

Now, assume that

$$u(P) < e^{v+1} h(\|Y\|)$$

for every $P=(X,Y) \in C_{m+n}(P_v, r_v)$, $X \in R^m$, $Y \in R^n$. Then,

$$(5) \quad u(P) \leq e^{v+1} h(\|Y_v\| + r_v) \leq \mu e^{v+1} h(\|Y_v\|)$$

for every $P \in C_{m+n}(P_v, r_v)$. If we put

$$S = C_{m+n}(P_v, r_v) \cap \{S_f(e^{v-\lambda}) \times R^n\},$$

we have

$$(6) \quad |S| \leq A_n r_v^n |S_f(e^{v-\lambda})| = A_n Q^{-m} r_v^{m+n}$$

and

$$(7) \quad u(P) \leq e^{v-\lambda} h(\|Y\|) \leq e^{v-\lambda} h(\|Y_v\| + r_v) \leq \mu e^{v-\lambda} h(\|Y_v\|)$$

for every $P=(X,Y) \in C_{m+n}(P_v, r_v) - S$, from (3). Thus, we obtain

$$\begin{aligned} & A_{m+n}^{-1} r_v^{-(m+n)} \int_{C_{m+n}(P_v, r_v)} u(P) dP = \\ & A_{m+n}^{-1} r_v^{-(m+n)} \left(\int_S u(P) dP + A_{m+n}^{-1} r_v^{-(m+n)} \int_{C_{m+n}(P_v, r_v) - S} u(P) dP \right) \\ & \leq A_{m+n}^{-1} r_v^{-(m+n)} \mu e^{v+1} h(\|Y_v\|) |S| \\ & \quad + A_{m+n}^{-1} r_v^{-(m+n)} \mu e^{v-\lambda} h(\|Y_v\|) |C_{m+n}(P_v, r_v) - S| \\ & \leq (e A_{m+n}^{-1} A_n Q^{-m} + e^{-\lambda}) \mu e^v h(\|Y_v\|) < e^v h(\|Y_v\|), \end{aligned}$$

from (5), (6) and (7). But, this contradicts (4).

Proof of Theorem 1. If we put

$$a_k = |S_f(e^k)|,$$

then

$$\begin{aligned} \sum_{k=1}^{\infty} |S_f(e^k)|^{1/m} &= m \sum_{k=1}^{\infty} \int_0^{a_k} \xi^{-(m-1)/m} d\xi = m \sum_{k=1}^{\infty} \int_{a_{k+1}}^{a_k} k \xi^{-(m-1)/m} d\xi \\ &\leq m \sum_{k=1}^{\infty} \int_{a_{k+1}}^{a_k} \xi^{-(m-1)/m} \log^+ F_f(\xi) d\xi \leq m \int_0^{\infty} \xi^{-(m-1)/m} \log^+ F_f(\xi) d\xi. \end{aligned}$$

Hence, we see that the series

$$\sum_{k=1}^{\infty} |S_f(e^k)|^{1/m}$$

converges.

Now, we shall prove by dividing into two cases.

(Case 1) We consider the case where

$$|S_f(e^k)| > 0$$

for any positive integer k . For the integer Q and λ (which are dependent on μ) chosen in Lemma, take a sufficiently large integer v_0 such that

$$(8) \quad \sum_{v=v_0}^{\infty} |S_f(e^{v-\lambda})|^{1/m} < Q^{-1}.$$

Here, we remark that v_0 depends on $f(X)$ and μ .

Now, assume that there is a point $P_{v_0} = (X_{v_0}, Y_{v_0})$, $X_{v_0} \in R^m$, $Y_{v_0} \in R^n$, $\|Y_{v_0}\| > y_0 + 2$, such that

$$u(P_{v_0}) \geq e^{v_0 h(\|Y_{v_0}\|)}.$$

If we put

$$r_{v_0} = Q |S_f(e^{v_0 - \lambda})|^{1/m}$$

and apply Lemma, we can find a point

$$P_{v_0+1} = (X_{v_0+1}, Y_{v_0+1}) \in C_{m+n}(P_{v_0}, r_{v_0}), \quad X_{v_0+1} \in R^m, \quad Y_{v_0+1} \in R^n,$$

such that

$$u(P_{v_0+1}) \geq e^{v_0+1 h(\|Y_{v_0+1}\|)}.$$

Here, if we see

$$\|Y_{v_0+1}\| \geq \|Y_{v_0}\| - r_{v_0} > y_0 + 1$$

and put

$$r_{v_0+1} = Q |S_f(e^{v_0+1-\lambda})|^{1/m},$$

we can also apply Lemma and find a point

$$P_{v_0+2} = (X_{v_0+2}, Y_{v_0+2}) \in C_{m+n}(P_{v_0+1}, r_{v_0+1}), \quad X_{v_0+2} \in R^m, \quad Y_{v_0+2} \in R^n,$$

such that

$$u(P_{v_0+2}) \geq e^{v_0+2} h(\|Y_{v_0+2}\|).$$

Here, we see

$$\begin{aligned} \|P_{v_0+2} - P_{v_0}\| &\leq r_{v_0} + r_{v_0+1} \\ &= Q(|S_f(e^{v_0-\lambda})|^{1/m} + |S_f(e^{v_0+1-\lambda})|^{1/m}) < 1 \end{aligned}$$

from (8), which gives

$$\|Y_{v_0+2}\| \geq \|Y_{v_0}\| - 1 > y_0 + 1.$$

Thus, if we continue this process, we can obtain a sequence of points

$$\{P_{v_0+i}\}_{i=0}^{\infty}, P_{v_0+i} = (X_{v_0+i}, Y_{v_0+i}), X_{v_0+i} \in R^m, Y_{v_0+i} \in R^n,$$

such that

$$\|P_{v_0+i} - P_{v_0}\| < 1$$

and

$$u(P_{v_0+i}) \geq e^{v_0+i} h(\|Y_{v_0+i}\|) \geq e^{v_0+i} h(y_0+1) \rightarrow \infty \quad (i \rightarrow \infty).$$

These show that $u(P)$ is unbounded above on $C_{m+n}(P_{v_0}, 1)$. This contradicts the boundedness of $u(P)$ on $C_{m+n}(P_{v_0}, 1)$.

Thus, if we put $K = e^{v_0}$, we have

$$u(P) \leq Kh(\|Y\|)$$

for any $P = (X, Y)$, $X \in R^m$, $Y \in R^n$, $\|Y\| > y_0 + 2$.

(Case 2) Suppose that Case 1 does not happen i.e., there is a k_0 such that

$$|S_f(e^{k_0})| = 0.$$

Take any $P' = (X', Y')$, $X' \in R^m$, $Y' \in R^n$, $\|Y'\| > y_0$, and a positive number δ' , $\delta' < \min(1, \|Y'\| - y_0)$.

If we put

$$S' = C_{m+n}(P', \delta') \cap \{(X, Y) \mid X \in R^m, Y \in R^n, X \in S_f(e^{k_0})\},$$

we have

$$(9) \quad |S'| \leq |S_f(e^{k_0})| A_n \delta'^n = 0$$

and

$$(10) \quad u(P) \leq h(\|Y\|)f(X) < h(\|Y'\| + \delta')e^{k_0} \leq \mu h(\|Y'\|)e^{k_0}$$

for any $P=(X,Y) \in C_{m+n}(P', \delta') - S'$, $X \in R^m$, $Y \in R^n$. Hence, if we

denote by M' the maximum of $u(P)$ on $C_{m+n}(P', \delta')$,

$$\begin{aligned} u(P') &\leq A_{m+n}^{-1} \delta'^{-(m+n)} \int_{C_{m+n}(P', \delta')} u(P) dP \\ &= A_{m+n}^{-1} \delta'^{-(m+n)} \int_{S'} u(P) dP + A_{m+n}^{-1} \delta'^{-(m+n)} \int_{C_{m+n}(P', \delta') - S'} u(P) dP \\ &\leq M' A_{m+n}^{-1} \delta'^{-(m+n)} |S'| + \mu h(\|Y'\|)e^{k_0} = \mu h(\|Y'\|)e^{k_0} \end{aligned}$$

from (9) and (10).

Thus, putting $\mu e^{k_0} = K$, we have

$$u(P) \leq Kh(\|Y\|)$$

for any $P=(X,Y)$, $X \in R^m$, $Y \in R^n$, $\|Y\| > Y_0$.

Proof of Theorem 2. Given any $\varepsilon > 0$, consider the function $u_\varepsilon^*(P)$ on R^{m+n} defined by

$$u_\varepsilon^*(P) = \begin{cases} \|Y\|^\varepsilon (\cos \|X\|) \exp(\|Y\|^{1+\varepsilon} - \|X\|^2 \|Y\|^\varepsilon) & \text{on } \{P=(X,Y) \mid X \in R^m, \|X\| < 2^{-1}\pi, Y \in R^n\} \\ 0 & \text{elsewhere.} \end{cases}$$

If we write $\|X\|=x$, $\|Y\|=y$ and

$$g(x,y) = \exp(y^{1+\varepsilon} - x^2 y^\varepsilon)$$

for simplicity, we have

$$\begin{aligned} \Delta u_\varepsilon^* &= \frac{\partial^2 u_\varepsilon^*}{\partial x^2} + \frac{m-1}{x} \frac{\partial u_\varepsilon^*}{\partial x} + \frac{n-1}{y} \frac{\partial u_\varepsilon^*}{\partial y} + \frac{\partial^2 u_\varepsilon^*}{\partial y^2} \\ &\geq g(x,y) [y^{3\varepsilon} \{(1+\varepsilon)^2 - o(1)\} \cos x + y^{2\varepsilon} \{4 - x^{-2}(m-1)y^{-\varepsilon}\}] x \sin x \end{aligned}$$

$$\geq \begin{cases} g(x,y)[y^{3\varepsilon}\{(1+\varepsilon)^2 - o(1)\}\cos x + y^{2\varepsilon}\{2^{-1}\sqrt{2} - o(1)\}] \\ \quad (4^{-1}\pi \leq x \leq 2^{-1}\pi, y \rightarrow \infty) \\ g(x,y)y^{3\varepsilon}\{2^{-1}\sqrt{2}(1+\varepsilon)^2 - o(1) - (m-1)y^{-2\varepsilon}x^{-1}\sin x\} \\ \geq g(x,y)y^{3\varepsilon}\{2^{-1}\sqrt{2}(1+\varepsilon)^2 - o(1)\} \quad (0 < x \leq 4^{-1}\pi, y \rightarrow \infty) \end{cases}$$

by an elementary computation. This shows that $u_\varepsilon^*(P)$ is subharmonic on $\{P=(X,Y) \mid X \in \mathbb{R}^m, Y \in \mathbb{R}^n, \|Y\| > a\}$ for a sufficiently large a . Here, choose a constant M_ε so that

$$u_\varepsilon^*(P) \leq M_\varepsilon \quad \text{on } \{P=(X,Y) \mid X \in \mathbb{R}^m, Y \in \mathbb{R}^n, \|Y\| < 2a\}$$

and define $u_\varepsilon(P)$ by

$$u_\varepsilon(P) = \max\{u_\varepsilon^*(P), M_\varepsilon\}.$$

Then, $u_\varepsilon(P)$ is a subharmonic function on $\mathbb{R}^{m+n}$ which is requested in Theorem 2.

First, for the function

$$(11) \quad f_\varepsilon(X) = \max\{\|X\|^{-2}, M_\varepsilon\}$$

on $\mathbb{R}^m$, we shall show the inequality of (i) in Theorem 2.

Set

$$\psi(x,y) = x^{-2}y^\varepsilon \exp(-x^2y^\varepsilon)$$

for (x,y) , $x \in \mathbb{R}^+$, $y \in \mathbb{R}^+$. Then, we have

$$\frac{\partial \psi}{\partial y} = (-\varepsilon y^{\varepsilon-1} + \varepsilon x^2 y^{2\varepsilon-1}) \exp(-x^2 y^\varepsilon)$$

which vanishes at $y_0 = x^{-2/\varepsilon}$. Further,

$$\psi(x, y_0) = x^{-2} e^{-1} x^{-2} > 0$$

and

$$\psi(x,y) \rightarrow x^{-2} \quad \text{as } y \rightarrow 0 \text{ and } y \rightarrow \infty.$$

Hence,

$$\psi(x,y) > 0, \text{ i.e. } x^{-2} > y^\varepsilon \exp(-x^2 y^\varepsilon)$$

on $\mathbb{R}^+ \times \mathbb{R}^+$. From this fact, the required inequality immediately follows.

Here, it is easy to see that $f_\varepsilon(X)$ in (11) is a slimly

growing function on R^m , because

$$F_{f_\varepsilon}(\xi) = (A_m \xi^{-1})^{2/m}$$

at every $\xi < A_m M_\varepsilon^{-m/2}$.

To obtain (ii) in Theorem 2, observe

$$u_\varepsilon(0, Y) e^{-\|Y\|^{1+\varepsilon}} \rightarrow +\infty$$

uniformly as $\|Y\| \rightarrow +\infty$.

Proof of Theorem 3. Put

$$V(P) = \exp(e^{\|Y\|} \cos \|X\|) \cos(e^{\|Y\|} \sin \|X\|)$$

for any $P=(X, Y)$, $X \in R^m$, $Y \in R^n$ and consider the function

$$U^*(P) = \{V(P)\}^{2m-1}$$

defined on $R^m \times R^n$. If we write $\|X\|=x$ and $\|Y\|=y$, we have

$$\begin{aligned} \Delta U^* &= (2m-1)V^{2m-2}[(2m-2)\{(\frac{\partial V}{\partial x})^2 + (\frac{\partial V}{\partial y})^2\} + V\Delta V] \\ &= (2m-1)V^{2m-2} \exp(y+2e^y \cos x) g(x, y) \end{aligned}$$

where

$$\begin{aligned} g(x, y) &= (2m-2)e^y \\ &+ \cos(e^y \sin x) \left\{ \frac{n-1}{y} \cos(x+e^y \sin x) - \frac{m-1}{x} \sin(x+e^y \sin x) \right\}. \end{aligned}$$

Here, if

$$0 < x < \pi/2 \quad \text{and} \quad x + e^y \sin x < \pi/2,$$

we see that

$$\sin(x+e^y \sin x) \leq x+e^y \sin x \leq x(1+e^y)$$

and hence

$$g(x, y) \geq (m-1)(e^y-1) + \frac{n-1}{y} \cos(e^y \sin x) \cos(x+e^y \sin x) \geq 0.$$

Hence, we have

$$\Delta U^* \geq 0$$

for any $P=(X, Y)$, $X \in R^m$, $Y \in R^n$, $\|X\| < \pi/2$, $\|X\| + e^{\|Y\|} \sin \|X\| < \pi/2$.

Let

$$D_0 = \{X \in R^m \mid \|X\| < \pi/2\}$$

and

$S = \{(X, Y) \in \mathbb{R}^{m+n} \mid X \in D_0, Y \in \mathbb{R}^n, \sin \|X\| < 2^{-1} \pi e^{-\|Y\|}, \|Y\| > Y_0\}$,
where $Y_0 = \log 2^{-1} \pi$. Choose a positive constant M such that

$$U^*(P) \leq M$$

on $D_0 \times \{Y \in \mathbb{R}^n \mid \|Y\| < 2Y_0\}$ and define the function $u(P)$ on $\mathbb{R}^{m+n}$ by

$$u(P) = \begin{cases} M^{-1} \max\{U^*(P), M\} & \text{on } S \\ 1 & \text{elsewhere,} \end{cases}$$

which is a subharmonic function requested in Theorem 3.

Now, if we define $f(X)$ on $\mathbb{R}^m$ by

$$(12) \quad f(X) = \sup_{Y \in \mathbb{R}^n} u(X, Y)$$

and $h(y)$ on $\mathbb{R}^+$ by

$$h(y) \equiv 1,$$

we have the inequality of (i) in Theorem 3. Here, it is evident that $h(y)$ is a regularly growing function on $\mathbb{R}^+$.

Hence, we shall show that

$$(13) \quad \int_0^\infty \xi^{-\ell} \log^+ F_f(\xi) d\xi < +\infty \quad \text{for any } \ell, \ell < (m-1)/m.$$

Put

$$v(x, y) = \exp(e^y \cos x) \cos(e^y \sin x)$$

for $x \in \mathbb{R}$, $y \in \mathbb{R}$, $y > Y_0 = \log 2^{-1} \pi$. Then, for any fixed y , $v(x, y)$ increases from 0 to $\exp(e^y)$ as x decreases from $\sin^{-1}(2^{-1} \pi e^{-y})$ to 0. This fact gives that

$$u(P) > t$$

on the domain which is surrounded by the set

$$\{P \in D_0 \times \mathbb{R}^n \mid P \in S, V(P) = t\}$$

for a sufficiently large t . For a given t , consider the curve

$$L = \{(x, y) \in \mathbb{R}^2 \mid v(x, y) = t, 0 \leq x < \pi/2\}$$

in the plane and put

$$x_0 = \max_{(x,y) \in L} x.$$

Since

$$\frac{dy}{dx} = -\tan(x + e^y \sin x)$$

along L , we have

$$x_0 + e^{y \sin x_0} = \pi/2.$$

Hence, x_0 satisfies

$$\exp\{(2^{-1}\pi - x_0)\cot x_0\} \sin x_0 = t.$$

Since

$$|S_f(t)| = A_m x_0^m$$

for a sufficiently large t from the definition (12) of $f(X)$,

we have

$$F_f(\xi) = \exp\{[2^{-1}\pi - (A_m^{-1}\xi)^{1/m}]\cot\{(A_m^{-1}\xi)^{1/m}\}\}\sin\{(A_m^{-1}\xi)^{1/m}\}.$$

Thus, for a sufficiently small $\xi > 0$,

$$K_1 \xi^{-1/m} \leq \log F_f(\xi) \leq K_2 \xi^{-1/m}$$

where K_1 and K_2 are two positive constants. This gives (13).

The conclusion (ii) in Theorem 3 immediately follows for these $u(P)$ and $h(y)$ from the fact

$$u(0, Y) = M^{-1} \exp\{(2m-1)e^{\|Y\|}\}$$

at any $Y \in \mathbb{R}^n$ having sufficiently large $\|Y\|$.

Proof of Theorem 4. This theorem is proved by following both methods used to prove Theorem DL and Theorem B. For a given bounded regular domain D , we denote the positive eigenfunction corresponding to the eigenvalue λ_D by $f_D(X)$ and define $h_D(P)$ on

$$D \times \mathbb{R}^n = \{P=(X, Y) \mid X \in D, Y \in \mathbb{R}^n\}$$

by

$$h_D(P) = f_D(X) \|Y\|^{1-n/2} I_{n/2-1}(\sqrt{\lambda_D} \|Y\|),$$

where $I_{n/2-1}(y)$ is the Bessel function of the third kind, of order $n/2-1$ (see e.g. Watson [8, p.77]). It is easy to see that $h_D(P)$ is harmonic on $D \times \mathbb{R}^n$. We also remark that

$$I_{n/2-1}(y) = (2\pi y)^{-1/2} e^y (1+o(1)) \quad (y \rightarrow +\infty)$$

(see Watson [8, p.203]).

Now, consider the subharmonic function $u_1(P)$ on Π defined by

$$u_1(P) = u(P) - \eta_1 h_D(P) \quad (\eta_1 > 0).$$

Take a closed ball $B \subset D$ and choose a positive constant ε_1 such that

$$f_D(X) \geq \varepsilon_1 \quad \text{on } B.$$

If we choose a positive constant y_1 such that

$$M(u, y) < 2^{-1} \varepsilon_1 \eta_1 C_D e^{\sqrt{\lambda_D} y} y^{(1-n)/2}$$

for any $y \geq y_1$, where

$$C_D = (2\pi \sqrt{\lambda_D})^{-1/2},$$

we see that

$$u_1(P) \leq \varepsilon_1 \eta_1 C_D \{-2^{-1} - o(1)\} e^{\sqrt{\lambda_D} \|Y\|} \|Y\|^{(1-n)/2}$$

for any $P=(X, Y)$, $X \in B$, $\|Y\| \geq y_1$. Hence, there are a value M and a point $P_0 \in B \times \mathbb{R}^n$ such that

$$(14) \quad u_1(P_0) = M \quad \text{and} \quad u_1(P) \leq M \quad \text{on } B.$$

Next, take a bounded regular domain D^* , $D^* \subset \mathbb{R}^m$ such that

$$\partial(D-B) \cup (D-B) \subset D^* \quad \text{and} \quad \lambda_D < \lambda_{D^*} < \lambda_{D-B}.$$

Consider the subharmonic function $u_2(P)$ on $(D^*-B) \times \mathbb{R}^n$ defined by

$$u_2(P) = u_1(P) - \eta_2 h_{D^*}(P) \quad (\eta_2 > 0).$$

If we take a positive number ε_2 such that

$$f_{D^*}(X) \geq \varepsilon_2 \quad \text{on } \partial(D-B) \cup (D-B)$$

and a number y_2 such that

$$M(u, y) < \varepsilon_2 \eta_2 C_{D^*} e^{\sqrt{\lambda_D} y^{(1-n)/2}}$$

for any $y \geq y_2$, we have that

$$\begin{aligned} u_2(P) &\leq u(P) - \eta_2 h_{D^*}(P) \\ &\leq \varepsilon_2 \eta_2 C_{D^*} \{ e^{(\sqrt{\lambda_D} - \sqrt{\lambda_{D^*}}) \|Y\|} - (1+o(1)) \} e^{\sqrt{\lambda_{D^*}} \|Y\|} \|Y\|^{(1-n)/2} \end{aligned}$$

for any $P=(X, Y) \in D \times R^n - B$, $\|Y\| \geq y_2$. Hence, with (14) the maximal principle gives that

$$u_2(P) \leq \max(0, M) \quad \text{on } D-B.$$

Thus, we have that

$$u_1(P) \leq \max(0, M) \quad \text{on } D-B,$$

because η_2 is chosen arbitrarily small. Further, we have from (14) that

$$u_1(P) \leq \max(0, M) \quad \text{on } D.$$

By (14) and the maximal principle, this gives that

$$M \leq 0 \quad \text{and hence } u_1(P) \leq 0 \quad \text{on } D.$$

As $\eta_1 \rightarrow 0$, we can conclude that

$$u(P) \leq 0 \quad \text{on } D.$$

Proof of Theorem 5. For each positive integer m , take a number t_m such that

$$\varepsilon(t) \leq 1/m$$

for every $t \geq t_m$. Then

$$u(P) \leq f(X) \{ m^{-1} e^{\sqrt{\lambda_D} \|Y\|} \|Y\|^{(1-n)/2} \}$$

at every $P=(X, Y)$, $X \in D$, $Y \in R^n$, $\|Y\| \geq t_m$. If we put

$$h_m(y) = m^{-1} e^{\sqrt{\lambda_D} y^{(1-n)/2}}$$

it is easy to see that $h_m(y)$ is a regularly growing function on $(t_m, +\infty)$, i.e.

$$h_m(y+1) \leq e^{\sqrt{\lambda_D} h_m(y)}$$

for every $y > t_m$. Hence, if we also put $u(P)=0$ on $R^{m+n}-\Pi$ and apply Theorem 1, there exists a constant K independent of m such that

$$u(P) \leq K h_m(\|Y\|) = m^{-1} K e^{\sqrt{\lambda_D} \|Y\|} \|Y\|^{(1-n)/2}$$

for every $P=(X,Y)$, $X \in D$, $Y \in R^n$, $\|Y\| > t_m + 2$. This gives that

$$\lim_{y \rightarrow \infty} M(u, y) e^{-\sqrt{\lambda_D} y} y^{(n-1)/2} \leq 0.$$

Hence, from Theorem 4, the conclusion follows.

Proof of Theorem 6. For the function $u(P)$ and the constant taken in the proof of Theorem 3, consider the function $u(P)-1$ on $\Pi_0 = D_0 \times R^n$. When we represent this function by $u(P)$ again, we shall show that $u(P)$ is the subharmonic function requested in Theorem 6. The statement (i) in Theorem 6 is evident. To prove the statement (ii) in Theorem 6, define $f(X)$ on R^m by

$$f(X) = \begin{cases} \sup_{Y \in R^n} u(X, Y) & \text{on } D \\ 0 & \text{elsewhere} \end{cases}$$

and $\varepsilon(t)$ on R^+ by

$$\varepsilon(t) = e^{-\sqrt{\lambda_D} t} t^{(n-1)/2}.$$

Then,

$$u(P) \leq \varepsilon(\|Y\|) f(X) e^{\sqrt{\lambda_D} \|Y\|} \|Y\|^{(1-n)/2}$$

for any $P=(X,Y)$, $X \in D_0$, $Y \in R^n$, $\|Y\| \neq 0$. The finiteness of the integral

$$\int_0^\infty \xi^{-\lambda} \log^+ F_f(\xi) d\xi$$

for any $\ell < (m-1)/m$ follows immediately from the proof of Theorem 3.

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