

ON BLOCH FUNCTIONS AND THE CONTRACTION OF TEICHMÜLLER METRICS

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ABSTRACT. In this note, we consider the properties of Bloch functions determined by Beltrami coefficient. A sufficient condition for extremal quasiconformal mapping with nonexistence of degenerating sequence is obtained. As a result, we consider the contraction or preserved of Teichmüller metrics for the related Beltrami lines under the projection mapping π .

1. INTRODUCTION

Let Q_I be the class of quasiconformal mappings f of the unit disk $D = \{z \mid |z| \leq 1\}$ onto itself with $f(0) = f(1) - 1 = 0$, μ_f be the complex dilatation of f , $k_f = \|\mu_f\|_\infty = \text{esssup}_{z \in D} |\mu_f|$, $k_0(f) = \inf_g k_g$, where $g \in Q_I$ with $g|_{\partial D} = f|_{\partial D}$. We say that $f(z)$ is extremal if $k_f = k_0(f)$, and the corresponding μ_f is called extremal.

We know that the universal Teichmüller space $T(1)$ can be represented as a quotient space of QS by the Möbius group $PSL(2, R)$, where QS is the group of all quasi-symmetric homeomorphisms of a circle, and the Teichmüller distance $d([f], [g])$, from a point $[g]$ to another point $[f]$ in $T(1)$, is equal to

$$(1.1) \quad d([f], [g]) = \frac{1}{2} \log \frac{1 + k_0(g \circ f^{-1})}{1 - k_0(g \circ f^{-1})}.$$

QS contains another topological subgroup, which is much larger than $PSL(2, R)$, the subgroup S of symmetric homeomorphisms. Gardiner-Sullivan [1] showed that $QSm\text{od}S$ also has a natural complex Banach manifold structure and a natural quotient metric $\bar{d}$, called the Teichmüller metric in $QSm\text{od}S$. Let $\bar{k}_f = \inf_U \text{esssup}_{z \in U} |\mu_f(z)|$, where U moves all neighborhoods of ∂D in D , $\bar{k}_f$ is called the boundary dilatation of f . Set $\bar{k}_0(f) = \inf_g \bar{k}_g$, where g moves all quasiconformal mappings of D with the same boundary values as f . If $\bar{k}_0(f) = \bar{k}_f$, then $f(z)$ is called extremal in $QSm\text{od}S$. The distance between two points $\pi[f]$ and $\pi[g]$ in $QSm\text{od}S$ is equal to

$$(1.2) \quad \bar{d}(\pi[f], \pi[g]) = \frac{1}{2} \log \frac{1 + \bar{k}_0(g \circ f^{-1})}{1 - \bar{k}_0(g \circ f^{-1})}.$$

Suppose $\mu(z)$ is a given Beltrami coefficient, we consider the Beltrami line $C_\mu = \{[f^t] \mid -1 \leq t \leq 1\}$ or $\pi C_\mu = \{\pi[f^t] \mid -1 \leq t \leq 1\}$, where $\mu_{f^t} = t \frac{\mu}{\|\mu\|_\infty}$. If μ is

extremal in $T(1)$ or in $QSm\text{od}S$, then the natural mapping $t \mapsto t \frac{\mu}{\|\mu\|_\infty}$ from the open interval $(-1, 1)$ with the Poincaré metric onto C_μ or πC_μ with the Teichmüller metric is an isometry. Whether μ is extremal or not, such mapping is weakly contracting. The following problem is very interesting and considered by many authors (cf. [2], [3]):

For which points $[f] \in T(1)$, does the Teichmüller distance from 0 to $[f]$ in QS strictly greater than the distance from 0 to $\pi[f]$ in $QSm\text{od}S$?

In this note, we will investigate some properties for Bloch functions determined by μ and partially solve the above problem.

2. MAIN RESULTS AND THEIR PROOFS

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in D , $f(z)$ is called a Bloch function if

$$(2.1) \quad \|f\|_B = \sup_{z \in D} (1 - |z|^2) |f'(z)| < \infty.$$

The Bloch functions will be denoted by B . B_0 will be the subset of B with

$$(2.2) \quad \|f\|_{B_0} = \lim_{|z| \rightarrow 1} \sup (1 - |z|^2) |f'(z)| = 0.$$

$A(D) = \{f(z) | f(z) \text{ is analytic in } D, \|f(z)\|_1 = \frac{1}{\pi} \iint_D |f(z)| dx dy < \infty\}$. The quasi-conformal mapping f from D onto itself is called a Teichmüller mapping of finite type, if $\mu_f = \|\mu(z)\|_\infty \frac{\bar{\varphi}_0}{|\varphi_0|}$, $\varphi_0 \in A(D)$. From Reich's example (cf. [4]), we know that even the point $[f]$ corresponds to a Teichmüller mapping of finite type, the distance from 0 to $[f]$ under the projection π may not contract. However, if $[f] \in T(1)$, and $\bar{d}(0, \pi[f]) < d(0, [f])$, then $[f]$ contains a Teichmüller mapping of finite type. This makes the above problem more complicated.

Suppose $\kappa(z) \in L^\infty(D)$, the space of complex-valued bounded measurable functions in D with $\|\kappa\|_\infty = \text{esssup}_{z \in D} |\kappa(z)|$, we consider a linear functional L_κ on $A(D)$

$$(2.3) \quad L_\kappa(f) = \frac{1}{\pi} \iint_D \kappa(z) f(z) dx dy, \quad f(z) \in A(D),$$

then

$$(2.4) \quad \|L_\kappa\| \leq \|\kappa\|_\infty.$$

Hamilton, Reich and Streble [5, 6] showed that

Theorem A. A Beltrami coefficient μ is extremal if and only if one of the following statements holds:

1) There exist $\varphi \in A(D)$ and $k \in [0, 1)$ such that $\mu = k\bar{\varphi}/|\varphi|$ for almost everywhere on D .

2) There is a degeneration sequence $\{\varphi_n\} \in A(D)$, $\|\varphi_n\|_1 = 1$, converging to 0 locally uniformly in D , such that

$$(2.5) \quad \lim_{n \rightarrow \infty} \left| \iint_D \varphi_n \mu \, dx dy \right| = \|\mu\|_\infty.$$

For a given Beltrami coefficient $\mu(z)$, let

$$(2.6) \quad b_n = \frac{n+2}{\pi} \iint_D \mu(z) z^n \, dx dy, \quad g(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^n,$$

it is clearly that $|b_n| \leq 2\|\mu(z)\|_\infty$ and $g(\zeta)$ is analytic in D . We call that the analytic function $g(\zeta)$ is determined by $\mu(z)$.

Let $G(\zeta) = \zeta g(\zeta)$, Anderson proved in [7] the following

Theorem B. For a given $\mu(z) \in L^\infty(D)$, then

$$(2.7) \quad \|L_\mu\| \leq \|G(\zeta)\|_B \leq 4\|L_\mu\|,$$

where $G'(\zeta) = \frac{2}{\pi} \iint_D \frac{\mu(z)}{(1-\zeta z)^3} \, dx dy$.

Theorem C. If $\mu(z)$ possesses a degenerating sequence, then

$$(2.8) \quad \|L_\mu\| \leq \lim_{|z| \rightarrow 1} \sup (1 - |z|^2) |G'(z)|,$$

where $G'(\zeta) = \frac{2}{\pi} \iint_D \frac{\mu(z)}{(1-\zeta z)^3} \, dx dy$. In particular, if

$$(2.9) \quad \iint_D \frac{\mu(z)}{(1-\zeta z)^3} \, dx dy = o(1 - |\zeta|^2)^{-1} \quad (|\zeta| \rightarrow 1^-),$$

then $\mu(z) = \|\mu\|_\infty \frac{\bar{\varphi}_0(z)}{|\varphi_0(z)|}$, $\varphi_0 \in A(D)$, for almost all $z \in D$.

Theorem C means that if $\mu(z)$ is extremal and $\lim_{|z| \rightarrow 1} \sup (1 - |z|^2) |G'(z)| = 0$, then

$$\mu(z) = \|\mu\|_\infty \bar{\varphi}_0 / |\varphi_0|, \quad \varphi_0(z) \in A(D),$$

for almost everywhere $z \in D$. For an extremal quasiconformal mapping $f^{\mu(z)} \in Q_I$, in what case, is it a finite type Teichmüller mapping or even has it no degenerating sequence? This problem is very interesting itself (cf. [8, 9] and the references cited there). First, we will prove the following

Theorem 1. Suppose $\mu(z)$ is extremal, let $g(z)$ be defined in (2.6), if there exists a ρ_0 , $0 < \rho_0 < 1$, such that

$$(2.10) \quad \sup_{\rho_0 < |z| < 1} (1 - |z|^2) |g'(z)| < 1,$$

then there exists a $\varphi_0 \in A(D)$ with $\mu(z) = \|\mu(z)\|_\infty \frac{\bar{\varphi}_0}{|\varphi_0|}$ for almost all $z \in D$. In particular, $\mu(z)$ possesses no degenerating sequence.

The proof of Theorem 1. If $\mu(z)$ is an extremal Beltrami coefficient, let $g(\zeta)$ be defined in (2.6), if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A(D)$, $0 < \rho < 1$, we have

$$L_\mu(f(\rho z)) = \sum_{n=0}^{\infty} a_n \rho^n L_\mu(z^n) = \sum_{n=0}^{\infty} \frac{a_n b_n}{n+2} \rho^n.$$

Since $\|f(\rho z) - f(z)\|_1 \rightarrow 0$, when $\rho \rightarrow 1^-$, then we have

$$L_\mu(f) = \lim_{\rho \rightarrow 1^-} \sum_{n=0}^{\infty} \frac{a_n b_n}{n+2} \rho^n.$$

On the other hand, if $G(\zeta) = \zeta g(\zeta)$, then

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) G'(\zeta re^{-i\theta}) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{n=0}^{\infty} a_n r^n e^{in\theta} \right) \left(\sum_{n=0}^{\infty} (n+1) b_n \zeta^n r^n e^{-in\theta} \right) d\theta \\ &= \sum_{n=0}^{\infty} (n+1) a_n b_n \zeta^n r^{2n}. \end{aligned}$$

Thus, we have

$$(2.11) \quad \lim_{\rho \rightarrow 1^-} \sum_{n=0}^{\infty} \frac{a_n b_n}{n+2} \rho^n = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} f(re^{i\theta}) G'(\zeta re^{-i\theta}) (1-r^2) r, dr d\theta,$$

for any $f(z) \in A(D)$. Since

$$\begin{aligned} g(\zeta) &= \sum_{n=0}^{\infty} b_n \zeta^n = \sum_{n=0}^{\infty} \left(\frac{n+2}{\pi} \iint_D z^n \mu(z) dx dy \right) \zeta^n \\ &= \frac{1}{\pi} \iint_D \left(\sum_{n=0}^{\infty} (n+2) z^n \zeta^n \mu(z) \right) dx dy \\ &= \frac{1}{\pi} \iint_D \left[\frac{2-z\zeta}{(1-z\zeta)^2} \right] \mu(z) dx dy, \end{aligned}$$

then,

$$(2.11) \quad |g(\zeta)| \leq \frac{3\|\mu\|_\infty}{\pi|\zeta|} \log \frac{1+|\zeta|}{1-|\zeta|} = o((1-|\zeta|^2)^{-1}), \quad |\zeta| \rightarrow 1^-.$$

If $\{f_n(z)\}$ is a degenerating sequence for $\mu(z)$ with $\|f_n\|_1 = 1$, by Theorem B and (2.11), we can choose a ρ' with $\rho_0 < \rho' < 1$ such that

$$\begin{aligned} |L_\mu(f_n)| &\leq \frac{4\|\mu\|_\infty}{\pi} \iint_{|z| \leq \rho'} |f_n(re^{i\theta})| r dr d\theta + \sup_{\rho' < |z| < 1} (1-|z|^2) |g(z)| \\ &\quad + \sup_{\rho' < |z| < 1} (1-|z|^2) |g'(z)| < 1, \quad \text{for } n \rightarrow \infty, \end{aligned}$$

which contradicts that $\{f_n(z)\}$ is a degenerating sequence. By Theorem A, Theorem 1 is proved.

The following example 1 shows that there is non-extremal Beltrami coefficient $\mu(z)$ with the bound $\sup_{\rho_0 < |z| < 1} (1-|z|^2) |g'(z)| = \frac{2}{\pi}$.

Example 1. Set Beltrami coefficient

$$\mu(z) = \begin{cases} 1, & \text{for } \Im z \geq 0, |z| < 1 \\ 0, & \text{for } \Im z < 0, |z| < 1. \end{cases}$$

Then by [8, Theorem 1], we see that $\mu(z)$ is not extremal. In this case, by calculation, we have

$$g'(z) = 2 + \frac{2i}{\pi} \left[z + \frac{1}{3}z^3 + \cdots + \frac{1}{2n-1}z^{2n-1} + \cdots \right]$$

and $\lim_{|z| \rightarrow 1} (1 - |z|^2) |g'(z)| = \frac{2}{\pi}$.

Next we will investigate the relationship between extremal Beltrami coefficient μ and the coefficients of $g(z)$ defined in (2.6).

From [11] and Theorem 1, we know that if $\mu(z)$ is extremal and the determined analytic function $g(z) \in B_0$, then $\lim_{n \rightarrow \infty} |b_n| = 0$. However, we also know that even if $f(z) \in B$ and $\lim_{n \rightarrow \infty} |b_n| = 0$, one can not derive that $f(z) \in B_0$. From this we will prove the following

Corollary 1. Suppose $\mu(z)$ is extremal, and let $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be defined in (2.6), if there exist a positive number N_0 and l , $0 < l < \frac{1}{2}$, such that

$$|b_n| < \frac{l}{n}, \quad \text{holds for } n > N_0,$$

then there exists a $\varphi_0(z) \in A(D)$ with

$$\mu(z) = \|\mu\|_{\infty} \bar{\varphi}_0 / |\varphi_0|, \quad \text{for almost all } z \in D.$$

The proof of Corollary 1. If $\mu(z)$ is extremal, and let $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be defined in (2.6), we have

$$\begin{aligned} |g'(z)| &\leq \left| \sum_{n=0}^{N_0} n b_n z^n \right| + \sum_{n=N_0+1}^{\infty} l |z|^n \\ &= \left| \sum_{n=0}^{N_0} n b_n z^n \right| + l \frac{|z|^{N_0+1}}{1 - |z|}, \end{aligned}$$

thus there exists a $\rho_0 > 0$, such that $\sup_{\rho_0 < |z| < 1} (1 - |z|^2) |g'(z)| < 1$, by Theorem 1, we obtain the assertion.

Let Π denote the subset of $T(1)$ consisting of elements of $[f]$ which correspond to Teichmüller mappings of finite type whose complex dilatations $\mu = \mu_f$ satisfy the following condition: There exists a ρ_0 , $0 < \rho_0 < 1$, such that $\sup_{\rho_0 < |\zeta| < 1} (1 - |\zeta|^2) |g'(\zeta)| < 1$, where $g(\zeta)$ is defined in (2.6). We will prove the following

Theorem 2. For $[f] \in \Pi$, then $\bar{d}(0, \pi([f])) < d(0, [f])$.

In order to prove Theorem 2, we need the following Theorem D due to Gardiner [2].

Theorem D. For every $[f] \in T(1)$, then $\bar{k}_f = \bar{k}_0(f)$ if and only if

$$\sup_{\{\varphi_n\}} \limsup_{n \rightarrow \infty} |\operatorname{Re} \iint_D \varphi_n \mu_f dx dy| = \bar{k}_f,$$

where the supremum is taken over all degenerating sequences $\{\varphi_n\}$ for μ_f with $\|\varphi_n\|_1 = 1$ in $A(D)$.

The proof of Theorem 2. We use the same way as in [3] to prove Theorem 2. If $[f] \in \Pi$, then we conclude that $\bar{k}_0(f) = k_0(f)$. On the contrary, by Theorem D, we can find a degenerating sequence $\{\varphi_n\}$ with $\|\varphi_n\|_1 = 1$ such that

$$\lim_{n \rightarrow \infty} \operatorname{Re} \iint_D \varphi_n \mu_f dx dy = \|\mu_f\|_\infty = k_0(f) = \bar{k}_0(f),$$

which is impossible by Theorem 1.

Thus we have $\bar{k}_0(f) < k_0(f)$, which is equivalent to $\bar{d}(0, \pi([f])) < d(0, [f])$.

On the other hand, comparing with Theorem 2, we will prove the following

Theorem 3. Suppose $[f] \in T(1)$, and $b_n = \frac{n+2}{\pi} \iint_D \mu_f z^n dx dy$, if $\overline{\lim}_{n \rightarrow \infty} b_n = 2\|\mu_f\|_\infty$, then $\bar{d}(0, \pi([f])) = d(0, [f])$. The constant 2 is the best.

The proof of Theorem 3. First, from Fehlmann and Sakan's paper in [10], we know that the subset of $T(1)$ satisfying the conditions in Theorem 3 is not empty, and by the example of Fehlmann and Sakan made in [10], there exists an extremal Beltrami coefficient μ such that the coefficients of $g(z)$ satisfy $\overline{\lim}_{n \rightarrow \infty} b_n = 2\|\mu\|_\infty$, thus the constant 2 is the best. Now, if $\overline{\lim}_{n \rightarrow \infty} b_n = 2\|\mu_f\|_\infty$, then we have $\lim_{j \rightarrow \infty} b_{n_j} = 2\|\mu_f\|_\infty$, and the sequence $\{\varphi_{n_j}(z) = \frac{n_j+2}{2} z^{n_j}\}$ is a degenerating sequence for the Beltrami coefficient μ_f , with $\|\varphi_{n_j}\|_1 = 1$, by Theorem D, we conclude that $\bar{k}_0(f) = k_0(f)$, thus $d(0, \pi([f])) = d(0, [f])$.

To consider the contraction of Teichmüller metrics, we need the following Principle of Teichmüller contraction due to Gardiner [2].

Principle of Teichmüller contraction. Assume $\|\mu\| = 1$, $0 < k_1 < k_2 < 1$, and $d(0, [f^{k_1}]) \leq \lambda_1 d_p(0, k_1)$ or $\bar{d}(0, \pi([f^{k_1}])) \leq \lambda_1 d_p(0, k_1)$ with some $\lambda_1 < 1$, where and in the sequel, f^k is the quasiconformal mapping of D on to itself such that $\mu_f = k\mu$ for every positive $k < 1$. Then there exists a $\lambda_2 < 1$ depending only on k_1, k_2 , and λ_1 such that

$$d(0, [f^k]) \leq \lambda_2 d_p(0, k) \quad \text{or} \quad \bar{d}(0, \pi([f^k])) \leq \lambda_2 d_p(0, k)$$

respectively, for all k with $0 \leq k \leq k_2$.

Using Theorem 2 and the Principle of Teichmüller contraction, we can obtain the following

Corollary 2. Under the same circumstance as in Theorem 2, let $k = \|\mu_f\|_\infty$ and $\lambda = \bar{d}(0, \pi([f]))/d(0, [f])$. Fix $k' < 1$ and let f' be the quasiconformal mapping of D onto itself such that $\mu_{f'} = (t/k)\mu_f$ for every $t \in [0, k']$. Then there exists $\lambda' < 1$ depending only on k, k' , and λ such that

$$\bar{d}(0, \pi([f'])) \leq \lambda' d_p(0, t),$$

for every t with $0 \leq t \leq k'$, where d_p denotes the Poincaré metric on D .

The proof of Corollary 2. By Theorem 2, we have $d(0, [f]) = d_p(0, k)$ and $\lambda = \bar{d}(0, \pi([f]))/d(0, [f]) < 1$, using the principle of Teichmüller contraction, the Corollary 2 is obtained.

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