

One Criterion on a Class of Certain Analytic Functions

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Let Λ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $U = \{z; |z| < 1\}$.

A function belonging to Λ is said to be a member of the class $S(\alpha)$ if it satisfies

$$(1) \quad \frac{zf'(z)}{f(z)} \prec 1 + (1 - \alpha)z$$

for some α ($0 \leq \alpha < 1$) and for all $z \in U$. The symbol $\prec$ denotes the subordination. It is easily confirmed that the condition (1) is equivalent to the following

$$(2) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha$$

for all $z \in U$.

In [1], Fukui obtained the following result

Theorem A. If $f(z) \in \Lambda$ satisfies

$$(3) \quad \left| \beta \frac{zf'(z)}{f(z)} - 1 + (1 - \beta) \frac{zf''(z)}{f'(z)} \right| < 1 - \alpha$$

for some α ($0 \leq \alpha < 1$), β ($0 \leq \beta < 1$), and for all $z \in U$, then $f(z) \in S(\alpha)$.

Making a lemma, we will improve Theorem A.

In order to derive our result, we need the following lemma due to Jack[2] (or Miller and Mocanu[3]).

Lemma 1. Let $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0 , then we have

$$z_0 w'(z_0) = k w(z_0)$$

where k is real and $k \geq 1$.

Applying Lemma 1, we have

Main Theorem. Let $p(z)$ be analytic in U , $p(0) = 1$ and suppose that

$$(4) \quad |\beta(p(z) - 1) + (1 - \beta)(p^2(z) - p(z) + zp'(z))| < (1 - \alpha)(1 + \alpha - \alpha\beta)$$

for some α ($0 \leq \alpha < 1$), β ($0 \leq \beta < 1$) and for all $z \in U$. Then we have

$$|p(z) - 1| < 1 - \alpha$$

for all $z \in U$.

Proof. Let us put

$$(1 - \alpha)w(z) = (p(z) - 1).$$

Then we have $w(0) = 0$.

By an easy calculation, we have

$$\begin{aligned} & |\beta(p(z) - 1) + (1 - \beta)(p^2(z) - p(z) + zp'(z))| \\ &= |\beta(1 - \alpha)w(z) + (1 - \beta)(1 - \alpha)\{(1 - \alpha)w^2(z) + w(z) + zw'(z)\}| \\ &= \left| (1 - \alpha)w(z) \left\{ 1 + (1 - \alpha)(1 - \beta)w(z) + (1 - \beta) \frac{zw'(z)}{w(z)} \right\} \right| \end{aligned}$$

If there exists a point z_0 such that

$$\max_{z < z_0} |w(z)| = |w(z_0)| = 1,$$

then from Lemma 1, we have

$$\begin{aligned} & \left| (1 - \alpha)w(z_0) \left\{ 1 + (1 - \beta)((1 - \alpha)w(z_0) + \frac{z_0 w'(z_0)}{w(z_0)}) \right\} \right| \\ &= (1 - \alpha) |1 + k(1 - \beta) + (1 - \alpha)(1 - \beta)w(z_0)| \\ &\geq (1 - \alpha)(1 + 1 - \beta - (1 - \alpha)(1 - \beta)) \\ &= (1 - \alpha)(1 + \alpha - \alpha\beta). \end{aligned}$$

This contradicts to (4). This shows that

$$|p(z) - 1| < 1 - \alpha$$

for all $z \in U$. This completes our proof.

Putting

$$p(z) = \frac{zf'(z)}{f(z)}$$

then we have

$$p^2(z) - p(z) + zp'(z) = \frac{zf''(z)}{f(z)}.$$

Therefore, from the Main theorem, we have

C o r o l l a r y 1. If $f(z) \in \Lambda$ satisfies

$$\left| \beta \frac{zf'(z)}{f(z)} - 1 + (1 - \beta) \frac{zf''(z)}{f(z)} \right| < (1 - \alpha)(1 + \alpha - \alpha\beta)$$

for some α ($0 \leq \alpha < 1$), β ($0 \leq \beta < 1$) and for all $z \in U$, then we have $f(z) \in S(\alpha)$.

This is an improvement of Theorem A.

Taking $\beta = 0$ in Corollary 1, we have

C o r o l l a r y 2. If $f(z) \in \Lambda$ satisfies

$$\left| \frac{z^2 f''(z)}{f(z)} \right| < 1 - \alpha^2$$

for some α ($0 \leq \alpha < 1$) and for all $z \in U$, then we have $f(z) \in S(\alpha)$.

This is an improvement of [1, Corollary 1].

Taking $\beta = 1/2$ in Corollary 1, we have

C o r o l l a r y 3. If $f(z) \in \Lambda$ satisfies

$$\left| \frac{z f'(z)}{f(z)} - 1 + \frac{z^2 f''(z)}{f(z)} \right| < (2 - \alpha + \alpha^2)$$

for some α ($0 \leq \alpha < 1$) and for all $z \in U$, then we have $f(z) \in S(\alpha)$.

This is an improvement of [1, Corollary 2].

Taking $\beta = 0$ in Main theorem, we have

C o r o l l a r y 4. Let $p(z)$ be analytic in U , $p(0) = 1$ and suppose that

$$| p^2(z) - p(z) + z p'(z) | < 1 - \alpha^2$$

for all $z \in U$. Then we have

$$| p(z) - 1 | < 1 - \alpha$$

for all $z \in U$.

References

- [1] S. Fukui: A Remark on a Class of Certain Analytic Functions. Proc. Japan Acad., 66, Ser. A, 191-192 (1990)
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- [3] S. S. Miller and P. T. Mocanu: Second order differential inequalities in complex plane. J. Math. Anal. Appl., 65, 289-305 (1978)