

What can be said about w -vectors of
finite partially ordered sets ?

Takayuki Hibi

Department of Mathematics
Faculty of Science
Nagoya University
Chikusa-ku, Nagoya 464, Japan

Any partially ordered set (poset for short) to be considered is finite. The cardinality of a finite set X is denoted by $\#(X)$. Let N be the set of non-negative integers and Z the set of integers.

§1. w -vectors

Let P be a poset with elements $x_1, x_2, \dots, x_p$ labeled so that if $x_i < x_j$ in P then $i < j$ in Z . Given an integer i , $0 \leq i < p$, write $w_i = w_i(P)$ for the number of permutations $\pi = \begin{pmatrix} 1 & 2 & \dots & p \\ a_1 & a_2 & \dots & a_p \end{pmatrix}$ such that (a) if $x_{a_r} < x_{a_s}$ in P , then $r < s$ (i.e., π is a linear extension of P) and (b) $\#\{r; a_r > a_{r+1}\}$, the number of descents of π , is equal to i . We say that the vector $w(P) := (w_0, w_1, \dots, w_{p-1})$ is the w -vector

of P . Consult [Stanley [Sta₂, pp. 211-221] for combinatorial background of w -vectors.

§2. Notation and terminology

A chain is a poset in which any two elements are comparable. The length of a chain C is defined by $\ell(C) := \#(C) - 1$. The rank of a poset P , denoted by $\text{rank}(P)$, is the supremum of lengths of chains contained in P . If $\alpha \leq \beta$ in P , we write $\ell(\alpha, \beta)$ for the rank of the subposet $P_\alpha^\beta := \{x \in P; \alpha \leq x \leq \beta\}$ of P . A poset P is called pure if every maximal chain of P has the same length. We say that P satisfies the $\delta^{(n)}$ -chain condition, $n \in \mathbb{N}$, if (a) for any $\xi \in P$, the subposet $P_\xi := \{y \in P; y \geq \xi\}$ of P is pure and (b) $\text{rank}(P) - \min\{\ell(C); C \text{ is a maximal chain of } P\} = n$. Thus P satisfies the $\delta^{(0)}$ -chain condition if and only if P is pure.

Give a poset P , we write $P^\wedge$ for the poset obtained by adjoining a new pair of elements, $0^\wedge$ and $1^\wedge$, to P such that $0^\wedge < x < 1^\wedge$ for any $x \in P$. A sequence $A = (\alpha_0, \beta_0, \alpha_1, \beta_1, \dots, \alpha_t, \beta_t)$, which consists of elements of $P^\wedge$, is called rhythmical if (a) $\alpha_0 = 0^\wedge$, $\beta_t = 1^\wedge$, (b) $\alpha_i < \beta_i$ for any i , $0 \leq i \leq t$, (c) $\alpha_{i+1} < \beta_i$ for any i , $0 \leq i < t$ and (d) $\alpha_{i+2} \neq \beta_i$ for any i , $0 \leq i \leq t-2$. Let $\ell(A) := \sum_{0 \leq i \leq t} \ell(\alpha_i, \beta_i) - \sum_{0 \leq i \leq t-1} \ell(\alpha_{i+1}, \beta_i)$. We say that P satisfies the Δ -chain condition if $\ell(A) \leq \text{rank}(P^\wedge)$ for any rhythmical sequence A of $P^\wedge$. We easily see that, for any $n \in \mathbb{N}$, the $\delta^{(n)}$ -chain condition implies the Δ -chain condition.

§3. Results.

Now, what can be said about w -vectors of posets ? In the following, let $w(P) = (w_0, w_1, \dots, w_{p-1})$ be the w -vector of a poset P with $\#(P) = p$ and $s := \max\{i; w_i \neq 0\}$.

THEOREM (Stanley [Sta₂, (4.5.17)]). The sequence $w_0, w_1, \dots, w_s$ is symmetric, i.e., $w_i = w_{s-i}$ for any i , $0 \leq i \leq s$, if and only if P is pure.

THEOREM (Stanley). The inequality

$$w_0 + w_1 + \dots + w_i \leq w_s + w_{s-1} + \dots + w_{s-i}$$

holds for any i , $0 \leq i \leq [s/2]$.

THEOREM ($[H_2]$). Assume that P satisfies the Δ -chain condition. If i and j are non-negative integers with $i + j \leq s$, then $w_i \leq w_j w_{i+j}$.

THEOREM ($[H_2]$). Assume that P satisfies the $\delta^{(n)}$ -chain condition. Then, the inequality

$$w_s + w_{s-1} + \dots + w_{s-i} \leq w_0 + w_1 + \dots + w_i + \dots + w_{i+n}$$

holds for any i , $0 \leq i \leq [(s-n)/2]$.

Our technique $[H_2]$, which originated in $[H_1]$, is heavily based on commutative algebra, especially the theory of canonical modules $[Sta_1]$ of invariant subrings of tori $[Hoc]$.

It would, of course, be of great interest to find a characterization of w -vectors of posets.

The author would like to thank Prof. Richard P. Stanley for helpful discussions relevant to this work.

References

- $[H_1]$ T.Hibi, Canonical ideals of Cohen-Macaulay partially ordered sets, to appear.
- $[H_2]$ _____, Linear diophantine equations and Stanley's (P, ω) -partitions, in preparation.
- $[Hoc]$ M.Hochster, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes, Ann. of Math. 96 (1972), 318-337.
- $[Sta_1]$ R.Stanley, Hilbert functions of graded algebras, Advances in Math. 28 (1978), 57-83.
- $[Sta_2]$ _____, "Enumerative Combinatorics, Volume I", Wadsworth, Monterey, Calif., 1986.