

Metrizability of spaces having certain k -networks

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As is well-known, each of the following properties implies that X is metrizable.

- (A) X is a paracompact developable space (R. H. Bing [2]).
- (B) X is a paracompact space having a σ -locally countable base (V. V. Fedorcuk [9]).
- (C) X has a σ -hereditarily closure-preserving base (D. Burke, R. Engelking and D. Lutzer [5]).
- (D) X is a paracompact M -space, and a σ -space (A. Okuyama [15]; the paracompactness can be omitted; see F. Siwiec and J. Nagata [16]).
- (E) X is an M -space having a point-countable base (V. V. Filippov [10]).

In terms of these properties, we give some metrization theorems by means of certain k -networks, or generalizations of M -spaces, etc.

We assume that spaces are *regular and T_1* .

Definitions. (1) A cover $\mathcal{C}$ of a space is a k -network if, whenever $K \subset U$ with K compact and U open in X , then $K \subset \bigcup \mathcal{C}' \subset U$ for some finite $\mathcal{C}' \subset \mathcal{C}$. If the K is a single point, then such a cover is called network (or net). Recall that a space is an κ -space (resp. σ -space) if it has a σ -locally finite k -network (resp. network).

(2) A space is *countably bi-quasi-k* [14] if, whenever $\{F_n: n \in \mathbb{N}\}$ is a decreasing sequence with $x \in \overline{F_n}$, there exists a decreasing sequence $\{A_n: n \in \mathbb{N}\}$ such that $x \in \overline{A_n \cap F_n}$ for each $n \in \mathbb{N}$, and if $x_n \in A_n$ for each $n \in \mathbb{N}$, then the sequence $\{x_n: n \in \mathbb{N}\}$ has a cluster point in $\bigcap \{A_n: n \in \mathbb{N}\}$. Recall that a space X is a q -space if each point has a sequence $\{V_n: n \in \mathbb{N}\}$ of nbds such that if $x_n \in V_n$, the sequence $\{x_n: n \in \mathbb{N}\}$ has a cluster point in X . Every q -space is a countably bi-quasi-k-space. Every countably bi-quasi-k-space is precisely the countably bi-quotient image of an M -space; see [14].

(3) A space X is a *monotonic $w\Delta$ -space* (simply, $mw\Delta$ -space) [18] (resp. *monotonic developable* [7] (equivalently, space having a base of countable order in the sense of Arhangel'skii [1])), if there exists a sequence $(\mathcal{B}_n)$ of bases for X such that any *decreasing* sequence $\{B_n: n \in \mathbb{N}\}$ with $B_n \in \mathcal{B}_n$ satisfies (*):

(*) If $x \in \bigcap B_n$, and $x_n \in B_n$, then the sequence $\{x_n: n \in \mathbb{N}\}$ has a cluster point in X (resp. the cluster point x).

If the sequence $\{B_n: n \in \mathbb{N}\}$ with $B_n \in \mathcal{B}_n$ is not necessarily decreasing, then such a space is called a $w\Delta$ -space; developable space respectively.

Every $w\Delta$ -space or every (monotonic) p -space [7] is an $mw\Delta$ -space. Every $mw\Delta$ -space is a q -space, hence countably bi-quasi-k. Among θ -refinable (= submetacompact) spaces, $mw\Delta$ -spaces (resp. monotonic developable spaces) are $w\Delta$ -spaces (resp. developable spaces); see [18] (resp. [7]).

The lemma below holds by means of [13; Proposition 3.2], [17; Section 4], [4; Theorem 4.1], and [18], etc. Here, we note that, in a space having a σ -locally countable k -network, each point is a G_δ -set. So, every countably bi-quasi-k-space with a σ -locally countable k -network is countably bi-k by [14; Theorem 7.3].

Lemma. (1) Suppose that X is a k -space; a normal space in which every closed countably compact set is compact; or each point of X is a G_δ -set. Then (i) and (ii) below hold.

(i) X has a point-countable base if and only if X is a countably bi-quasi- k -space with a point-countable k -network.

(ii) X is a monotonically developable space with a point-countable base if and only if X is an $mw\Delta$ -space with a point-countable k -network (cf. [18]).

(2) (i) A space has a σ -locally countable base if and only if it is a countably bi-quasi- k -space with a σ -locally countable k -network.

(ii) A space is a monotonically developable space with a σ -locally countable base if and only if it is an $mw\Delta$ -space with a σ -locally countable k -network.

Metrizability Theorem The following are equivalent.

- (a) X is metrizable,
- (b) X is a paracompact M -space having a point-countable k -network,
- (c) X is an M -space, and a k -space having a point-countable k -network,
- (d) X is an M -space having a point-countable k -network, and having a σ -locally countable *network*.
- (e) X is an M -space having a σ -locally countable k -network,
- (f) X is a paracompact, countably bi-quasi- k -space having a σ -locally countable k -network,
- (g) X is a countably bi-quasi- k -space having a point-countable k -network, and having a σ -closure preserving k -network; cf. [12].
- (h) X is an $mw\Delta$ -space having a σ -closure preserving k -network; see [18].
- (i) X is a countably bi-quasi- k -space having a σ -hereditarily closure preserving k -network.

Remark. In the previous theorem, it is possible to replace “ countably bi-quasi-k-space ” by “ countably bi-quasi-k-space, but the sequence $\{x_n: n \in \mathbb{N}\}$ has a cluster point in X (instead of $\bigcap \{A_n: n \in \mathbb{N}\}$) in the definition of countably bi-quasi-k-spaces ”.

We see that each condition in (b) $\sim$ (h) of Metrization Theorem is essential by means of the following examples.

Examples. (1) Not every countably compact (resp. countably compact, first countable) space having a point-countable k-network (resp. locally countable *network*) is a σ -space.

(2) Not every Čech-complete (resp. metacompact developable) space having a σ -locally countable base is a σ -space [8] (resp. metrizable; cf. [11]).

(3) Not every first countable, Lindelöf space having a σ -closure preserving base is developable [6].

But, the following holds by means of [3] and [18], etc.

Proposition. (1) Every θ -refinable space X is developable if (a) or (b) below holds.

(a) X is an $\text{mw}\Delta$ -space having a point-countable k-network, or having a σ -locally countable *network*.

(b) X is a countably bi-quasi-k-space having a σ -locally countable k-network.

(2) Every $\text{mw}\Delta$ -space which is the quotient compact image (resp. quotient s-image) of a metric space is developable (resp. monotonically developable); see [18].

We note that every $mw\Delta$ -space having a σ -locally finite *network* (resp. σ -locally countable k -network) is developable (resp. monotonically developable; cf. [18]).

In view of the above results, we have the following questions.

Questions. (1) Every $w\Delta$ -space having a σ -locally countable k -network (resp. σ -locally countable *network*) is developable (resp. monotonically developable) ?

(2) Every $w\Delta$ -space which is the quotient countable-to-one image of a metric space is developable ?

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