数理解析研究所講究録
第1729巻 2011年 117-122

The notions between Martin-Löf randomness and 2-randomness *

NingNing Peng

Mathematical Institute, Tohoku University
Sendai-shi, Miyagi-ken, 980-8578, Japan
sa8m42@math.tohoku.ac.jp

Abstract

In this paper, we study randomness notions between Martin-Löf randomness and 2-randomness. A best-known example of such notions is weak 2-randomness: a set is weakly 2-random if it is not in any Π^0_2 null class. We propose a new notion of randomness, called $\mathbb{L}$-randomness, that is Martin-Löf randomness relative to any low set.

1 Introduction

To the definition of an algorithmically random sequences, *measure-theoretic* approach is one of the most popular approach. According to this approach, a random sequence should have certain stochastic properties. This approach can be traced at least back to Von Mises' work [18]. When computability theory emerged two decades later, Church [2] made the connection with the theory of computability by suggesting that one should take all *computable* stochastic properties. Later, this approach developed by Martin-Löf, which makes the notion of randomness clear.

The present paper is concerned with the notion of randomness as originally by P. Martin-Löf [11] in 1966, that is nowadays regarded as central. The relativized randomness was first studied by Gaifman and Snir [7]. We say that a set is n-random if it is ML-random relative to $\emptyset^{(n-1)}$. So it is 1-random if it is ML-random. 2-random if it is ML-random relative to $\emptyset'$. 2-randomness was first studied by Kurtz [8], and more recently in [13], where a characterization was given using the plain Kolmogorov complexity of the initial segments. He also considered weak 2-randomness, an interesting notion lying strictly between Martin-Löf randomness and 2-randomness. Other randomness notion between Martin-Löf randomness and 2-randomness

*This research was partially supported by a Global COE Program funded by MEXT, and National Natural Science Foundation of China (Grant: 10801110).

is Demuth randomness. This notion introduced and studied by Demuth [4] [5], and about 30 years later, it getting interesting among logicians. Since Demuth random set can be Δ^0_2, and all Demuth random sets are in GL_1, this implies that Demuth randomness and weak 2-randomness are incomparable.

As we know now, there are several notions of randomness lying between Martin-Löf randomness and 2-randomness have been discovered. The two of them were weak 2-randomness and Demuth randomness discussed above. The others are weak Demuth randomness, Balanced randomness, Difference randomness and $\emptyset'$-Schnorr randomness, which introduced recently.

In [9], Kučera and Nies defined weak Demuth randomness, they show that weak Demuth randomness is stronger than ML-randomness. Demuth tests generalize Martin-Löf tests $(G_m)_{m\in\mathbb{N}}$ so that one can exchange the m-th component (a Σ^0_1 set in Cantor space of measure at most 2^{-m}) for a computably bounded number of times. A set Z fails a Demuth test if Z is in infinitely many of the G_m. The weak Demuth random tests only allow Demuth tests such that $G_m \supseteq G_{m+1}$ for each m. It is not hard to see that every Demuth random set is weak Demuth random, and that every weak Demuth random set is ML-random. They also show that a weakly Demuth random set can be high and Δ^0_2, yet not superhigh. Another randomness notion weaker than weak Demuth is Balanced randomness, which introduced in [6], interpolates between weak Demuth and ML-randomness.

Difference randomness [15], lies in middle of Martin-Löf randomness and 2-randomness, and stronger than Demuth randomness and weak 2-randomness. This notion of randomness based on the difference hierarchy. See [15] for more on difference randomness.

In [1], Barmpalias, Miller and Nies have been studied Martin-Löf randomness, Schnorr randomness relative to $\emptyset'$, weak randomness relative to $\emptyset'$. They show that within the Martin-Löf randomness sets, weak randomness relative to any oracle can be separated from weak 2-randomness. Also, they prove the following implications hold:

$$ML[\emptyset'] \Rightarrow SR[\emptyset'] \Rightarrow W2R \Rightarrow Kurtz[\emptyset'] \cap ML \Rightarrow ML.$$

None of the implications can be reversed.

For more background on algorithmic randomness and unexplained notions we refer to [3] and [12].

The notions between ML-randomness and 2-randomness has been extensively investigated in the literature by many researchers. But the notion of randomness is far from being fully understood. There are many open problems in this area. We believe there must still exist other randomness notions lying between Martin-Löf randomness and 2-randomness.

2 Preliminaries

The collection of binary strings is denoted by $2^{<\mathbb{N}}$, i.e. the set of all functions from $\{0,\dots,n\}$ to $\{0,1\}$ for some $n \in \mathbb{N}$. We use $\sigma, \tau, \cdots$ to denote the elements of $2^{<\mathbb{N}}$. Let $2^{\mathbb{N}}$ denote the set of infinite binary sequences. Subsets of $\mathbb{N}$ can be identified with element of $2^{\mathbb{N}}$. These

are also called *reals*. For sets A, B, Let $A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\}$, namely the set which is A on the even bit positions and B on the odd positions.

For $\sigma \in 2^{<\mathbb{N}}$, we write $|\sigma|$ for the length of σ. Equivalently, $|\sigma| = \#\mathrm{dom}(\sigma)$. Here the cardinality of a set A is denoted by $\#A$. The empty string is denoted by λ. For strings σ and τ, let $\sigma \preceq \tau$ denotes that σ is a prefix of τ, i.e., $\mathrm{dom}(\sigma) \subseteq \mathrm{dom}(\tau)$ and $\sigma(m) = \tau(m)$ holds for each $m \in \mathrm{dom}(\sigma)$. The concatenation of two strings σ and τ is denoted by $\sigma\tau$. For a set A, $A \upharpoonright n$ is the prefix of A of length n. A topology of $2^{\mathbb{N}}$ is induced by basic open sets $[\sigma] = \{X \in 2^{\mathbb{N}} : X \succeq \sigma\}$ for all strings $\sigma \in 2^{<\mathbb{N}}$. So each open set of $2^{\mathbb{N}}$ is generated by a subset of $2^{<\mathbb{N}}$, that is $[S]^{\prec} = \{X \in 2^{\mathbb{N}} : \exists \sigma \in S\ \sigma \preceq X\}$. With this topology, $2^{\mathbb{N}}$ is called *the Cantor space.*

The *Lebesgue measure* on $2^{\mathbb{N}}$ is induced by giving each basic open set $[\sigma]$ measure $\mu([\sigma]) := 2^{-|\sigma|}$. for each string σ. If a class $G \subseteq 2^{\mathbb{N}}$ is open then $\mu(G) = \sum_{\sigma \in B} 2^{-|\sigma|}$ where B is a prefix-free set of strings such that $G = \bigcup_{\sigma \in B}[\sigma]$. A class $\mathcal{C} \subseteq 2^{\mathbb{N}}$ is called *null* if $\mu(\mathcal{C}) = 0$. If $2^{\mathbb{N}} - \mathcal{C}$ is null we say that $\mathcal{C}$ is *conull.*

ML-randomness is a central notion of algorithmic randomness for subsets of $\mathbb{N}$, which defined in the following way.

Definition 1 (Martin-Löf [11]). (i) A *Martin-Löf* test, or ML-test for short, is a uniformly c.e. sequence $(G_m)_{m \in \mathbb{N}}$ of open sets such that $\forall m \in \mathbb{N}\ \mu(G_m) \leq 2^{-m}$.

(ii) A set $Z \subseteq \mathbb{N}$ *fails* the test if $Z \in \bigcap_m G_m$, otherwise Z *passes* the test.

(iii) Z is *ML-random* if Z passes each ML-test. Let *MLR* denote the class of ML-random sets. Let *non-MLR* denote its complement in $2^{\mathbb{N}}$.

In this way, we are presenting the ML-random sets as the sets that pass all reasonable statistical tests in the form of effectively presented null sets. The randomness notions which stronger than ML-randomness have been studied. To get such a notion, we place weaker effective conditions on the presentation of the test. Weak 2-randomness, like ML-randomness, is defined in terms of tests.

Definition 2 (Kurtz [8]). (i) A *generalized ML-test* is a uniformly c.e. sequence $(G_m)_{m \in \mathbb{N}}$ of open sets such that $\mu(\bigcap_m G_m) = 0$.

(ii) Z is *weakly 2-random* if it passes every generalized ML-test.

Fact 1. (i) 2-randomness $\Rightarrow$ weak 2-randomness $\Rightarrow$ ML-randomness.

(ii) The reverse implications fail (Kurtz, Kautz).

3 Definition of $\mathbb{L}$-randomness

In this section, we propose a new notion of randomness: $\mathbb{L}$-randomness, that is, Martin-Löf randomness in any low set. We will study some properties of this notion along this work.

Recall that A set A is low if $A' \leq_T \emptyset'$.

Definition 3. A set Z is Γ-*random* if Z is ML-random relative to A for all $A \in \Gamma$. In particular, Γ-randomness is called $\mathbb{L}$-*randomness* if Γ is the set of low sets.

Since a *Martin-Löf test* is a uniformly c.e. sequence $(G_m)_{m\in\mathbb{N}}$ of open sets such that $\forall m \in \mathbb{N}\ \mu G_m \leq 2^{-m}$. Thus, we can define an $\mathbb{L}$-test to be a sequence $(G_m)_{m\in\mathbb{N}}$ of open sets, which is uniformly c.e in some low set, such that $\forall m \in \mathbb{N}\ \ \mu G_m \leq 2^{-m}$.

In fact, $\mathbb{L}$-randomness lying strictly between Martin-Löf randomness and 2-randomness.

By Schnorr' theorem, it is easy to see there is a characterization of $\mathbb{L}$-randomness via a growth condition on the initial segment complexity.

Theorem 1. $Z \in 2^{\mathbb{N}}$ *is* $\mathbb{L}$-random *if*

$$(\forall n)(K^A(Z \upharpoonright n) > n - O(1))$$

for any set A *such that* $A' \equiv_T \emptyset'$.

Here, K denote the prefix-free Kolmogorov complexity which defined by $K(\sigma) = \min\{|\tau| : \mathbb{U}(\tau) = \sigma\}$, where $\mathbb{U}$ is a universal prefix-free machine. We refer to [3], [10] and [12] for more details.

3.1 Some facts on $\mathbb{L}$-randomness

Obviously, any $\mathbb{L}$-random set is ML-random. For any set Z, Z is not 1-random in Z. Thus, each low set is not $\mathbb{L}$-random. Hence, $\mathbb{L}$-randomness is strictly stronger than ML-randomness because there is a low ML-random.

If the randomness notions stronger than ML-randomness, then the notions are usually closed downwards under Turing reducibility within the random sets. We show that $\mathbb{L}$-randomness also have this property.

Proposition 1. *Let* X, Y *be ML-random sets. If* $X \leq_T Y$ *and* Y *is* $\mathbb{L}$-*random, then* X *is* $\mathbb{L}$-*random.*

It is not hard to prove the following result, then we show there is no universal $\mathbb{L}$-test.

Theorem 2. *For any low set* A*, there exists a low set* B *and a set* X *such that* X *is* A*-random and* X *is not* B*-random.*

Corollary 1. *There is no universal* $\mathbb{L}$*-test.*

Van Lambalgen's Theorem is one of the fundamental and important results in algorithmic randomness.

Theorem 3 (van Lambalgen [17])**.** *Let* $A, B \subseteq \mathbb{N}$*. Then* $A \oplus B$ *is ML-random* $\Leftrightarrow$ B *is ML-random and* A *is ML-random relative to* B*.*

Proof. See Theorem 3.4.6 of [12] □

The above theorem have many corollaries. For instance, A and B are Turing incomparable if $A \oplus B$ is ML-random. Since we know the importance of van Lambalgen's Theorem, it is natural to ask whether it holds for other notions of algorithmic randomness besides Martin-Löf randomness and its higher level versions. By relativization, we get van Lambalgen's Theorem for $\mathbb{L}$-randomness.

Theorem 4. *Let $A, B \subseteq \mathbb{N}$. Then $A \oplus B$ is $\mathbb{L}$-random $\Leftrightarrow$ B is $\mathbb{L}$-random and A is $\mathbb{L}$-random relative to B.*

The following results hold. For the proves, see my master thesis [16].

Proposition 2. *There is no $\mathbb{L}$-random which is Turing comparable with $\emptyset'$.*

Theorem 5. *Weak 2-randomness does not imply $\mathbb{L}$-randomness.*

Acknowledgments

The author would like to thank Prof. Kazuyuki Tanaka and Prof. Takeshi Yamazaki for many useful discussions and comments.

References

[1] George Barmpalias, Joseph S. Miller and André Nies: Randomness notions and partial relativization. *Submitted.*

[2] Alonzo Church: On the concept of a random sequence. *Bulletin Amer. Math. Soc., vol. 46, pp. 130-135, 1940.*

[3] Rod G. Downey, Denis R. Hirshfeldt: Algorithmic Randomness and Complexity. *Springer-Verlag, Berlin. November, 2010.*

[4] Osvald Demuth: Some classes of arithmetical real numbers (Russian). *Commentations Mathematicae Universitatis Carolinae, vol. 23, no. 3, pp. 453-465, 1982.*

[5] Osvald Demuth: Remarks on the structure of tt-degrees based on constructive measure theory. *Commentations Mathematicae Universitatis Carolinae, vol. 29, no. 2, pp. 233-247, 1988.*

[6] Santiago Figueira, Denis R. Hirschfeldt, Joseph S. Miller, Keng Meng Ng, and André Nies: Counting the changes of random Δ_2^0 sets. *In CIE 2010, pp. 1-10, 2010.*

[7] Haim Gaifmann and Marc Snir: Probabilities over rich languages. *The Journal of symbol logic, vol. 47, pp. 495-548, 1982.*

[8] Stuart A. Kurtz: Randomness and genericity in the degrees of unsolvability. *In: Ph.D. Dissertation, University of Illinois, Urbana*, 1981.

[9] Antonín Kučera and André Nies: Demuth randomness and computational complexity. *Submitted. 2010.*

[10] Li Ming and Paul Vitányi: An Introduction to Kolmogorov Complexity and its Applications. *2nd ed., Springer-Verlag, New York, 1997.*

[11] Per. Martin-Löf: The definition of random sequences. *Information and Control, vol. 9, no. 6, pp. 602-619, 1966.*

[12] André Nies: Computability and Randomness. *Oxford University Press, 2009.*

[13] André Nies, Frank Stephan, Sebastiaan A. Terwijn: Randomness, Relativization and Turing Degrees. *The Journal of Symbolic Logic, vol. 70, No.2, pp. 515-535, 2005.*

[14] Andre. Nies: Lowness properties and randomness. *Advances in Mathematics, vol. 197, pp. 274-305, 2005.*

[15] Johanna N.Y. Franklin and Keng Meng Ng: Difference randomness. *Proceedings of the American Mathematical Society. To appear.*

[16] NingNing Peng: Algorithmic randomness and lowness notions. *Master Thesis, Tohoku University, Sendai, Japan, August, 2010.*

[17] Michiel van Lambalgen: Random sequences. *Ph.D. Dissertation, University of Amsterdam, The Netherlands, 1987.*

[18] R. von Mises: Grundlagen der Wahrscheinlichkeitsrechnung. *Mathematische Zeitschrift, vol. 5, pp. 52-99, 1919.*