

On $k\beta$ -Spaces and Some Other Generalized Metric Spaces.

小 竹 義 朗
(Yoshiro KOTAKE)

Department of Mathematics, Faculty of Education
Gunma University

1. Introduction

In [11] Wu Lishing introduced the notion of $k\beta$ -spaces, which generalizes k -semi-stratifiable spaces due to Lutzer [7]. Recently Xia Shengxiang studied the conditions under which $k\beta$ -spaces to be k -semi-stratifiable. We investigate further properties of $k\beta$ -spaces, most of which are concerned with the metrization of $k\beta$ -spaces. Since the class of k -semi-stratifiable spaces is closely related to that of Nagata spaces, we also investigate the relationship between $k\beta$ -spaces and Nagata spaces.

Let (X, τ) be a space and let g be a function from $N \times X$ into τ such that $x \in g(n+1, x) \subset g(n, x)$ for each x in X and n in N . Such a function g is called a COC-function (= countable open covering function). In [3] Hodel introduced some important generalized metric spaces by means of a function COC-function $g: N \times X \rightarrow \tau$.

For the definitions of some generalized metric spaces which are not defined in this note, see [1], [2], and [3].

Unless otherwise stated, all topological spaces are assumed to be T_1 . The set of positive integers will be denoted by N .

2. Nagata spaces and $k\beta$ -spaces.

Instead of giving the original definitions of k -semi-stratifiable spaces [7] and Nagata spaces, we present an equivalent formulations which are used in this paper. For the actual definitions of these concepts, the reader is referred to [3] and [7].

Definition 2.1 ([11], [12]). (a): A space X is a k -semi-stratifiable space if there is a COC-function g such that $g(n, x_n) \cap K \neq \emptyset$ for $n=1, 2, \dots$, (where K is compact) then the sequence $\langle x_n \rangle$ has a cluster point in K .

(b): A space X is a $k\beta$ -space if there is a COC-function g such that $g(n, x_n) \cap K \neq \emptyset$ for $n=1, 2, \dots$, (where K is compact) then the sequence $\langle x_n \rangle$ has a cluster point.

Definition 2.2 ([3]). (a): A space X is a Nagata space if there is a COC-function g such that $g(n, p) \cap g(n, x_n) \neq \emptyset$ for $n=1, 2, \dots$, then p is a cluster point of the sequence $\langle x_n \rangle$.

(b): A space X is a wN -space if there is a COC-function g such that $g(n, p) \cap g(n, x_n) \neq \emptyset$ for $n=1, 2, \dots$, then the sequence $\langle x_n \rangle$ has a cluster point.

Theorem 2.3. Every wN -space is a $k\beta$ -space.

A space (X, τ) is called weakly subsequential if each sequence in X which has a cluster point has a subsequence with compact closure.

A space X is a $w\sigma$ -space if there is a COC-function g such that

$p \in g(n, y_n)$, $y_n \in g(n, x_n)$ for $n=1, 2, \dots$, then the sequence $\langle x_n \rangle$ has a cluster point, see [2] and [4].

Theorem 2.4. Every weakly subsequential $k\beta$ -space is a $w\sigma$ -space.

A space X is c -stratifiable if there is a COC -function g such that for each compact set K in X and $p \in X-K$, then there exists n which satisfies $p \notin Cl(g(n, K))$. A space X is called c -Nagata space if it is c -stratifiable and first countable.

Theorem 2.5 A space X is a Nagata space if and only if X is a c -Nagata, $k\beta$ -space.

Proof. Let f be a c -Nagata function and g be a $k\beta$ -function. Let $h: N \times X \rightarrow \tau$ be defined by $h(n, x) = f(n, x) \cap g(n, x)$. Since every first countable k -semi-stratifiable is a Nagata space, it suffices to show that X is k -semi-stratifiable. Let K be a compact subset in X and $h(n, x_n) \cap K \neq \emptyset \ni y_n$ for $n=1, 2, \dots$. As X is a $k\beta$ -space, $\langle x_n \rangle$ has a cluster point p . Since every c -Nagata space is first countable, there is a subsequence $\langle x_{n_k} \rangle$ which converges to p . Let $\{p\} \cup \{x_{n_k} \mid k=1, 2, \dots\} = C$. Suppose that $p \notin K$. Without loss of generality, we can assume that $K \cap C = \emptyset$. Since $y_{n_k} \in K$, $k=1, 2, \dots$, $\langle y_{n_k} \rangle$ has a cluster point q . Since f is a c -Nagata function, there is an m such that $Clf(m, C) \not\ni q$. Then $Clh(m, C) \not\ni q$. Let $V = X - Clh(m, C)$, then there is an i such that $V \ni y_{n_i}$, $n_i \geq m$. So we have $y_{n_i} \notin h(m, x_{n_i}) \supset h(n_i, x_{n_i})$ so that $y_{n_i} \notin h(n_i, x_{n_i})$. This is a contradiction. It follows that X is a k -semi-stratifiable space.

Corollary 2.6 (Lee [5]). A space X is a Nagata space if and only if X is a c -Nagata, wN -space.

A space X is said to have a G_δ^* -diagonal if there exists a

sequence $\langle \mathcal{G}_n \rangle$ of open covers of X such that, for each $x \in X$,
 $\bigcap_{n=1}^{\infty} \text{Cl}(\text{st}(x, \mathcal{G}_n)) = \{x\}$, see [3].

Theorem 2.7. A regular space X is a Nagata space if and only if X is a $q, k\beta$ -space with a G_δ^* -diagonal.

Proof. Since every regular q -space in which points are G_δ -sets is first countable, let $f: X \times \mathbb{N} \rightarrow \tau$ be a first countable function. Let $g: \mathbb{N} \times X \rightarrow \tau$ be a $k\beta$ -function. Let $h: \mathbb{N} \times X \rightarrow \tau$ be defined by $h(n, x) = f(n, x) \cap g(n, x)$. To show that h is a wN -function, let $h(n, p) \cap h(n, x_n) \neq \emptyset$, for $n=1, 2, \dots$. Then there is a sequence $\langle y_n \rangle$ such that $h(n, p) \cap h(n, x_n) \ni y_n$ for all $n \in \mathbb{N}$. Since f is a first countable function, the sequence $\langle y_n \rangle$ converges to p . Let $K = \{p\} \cup \{y_n \mid n=1, 2, \dots\}$, then K is compact and $g(n, x_n) \cap K \neq \emptyset$ for $n=1, 2, \dots$. Then $\langle x_n \rangle$ has a cluster point. So h is a wN -function. Note that a regular wN -space with a G_δ^* -diagonal is a Nagata space (see, Kotake [5]), whence X is a Nagata space.

A space X is said to have a regular G_δ -diagonal if the diagonal Δ is the intersection of countably many closures of open subsets of $X \times X$ (see [5]).

Theorem 2.8. Every regular $k\beta$ -space with a regular G_δ -diagonal is a k -semi-stratifiable space.

3. Metrizable of $k\beta$ -spaces.

Theorem 3.1. A space X is metrizable if and only if X is a Hausdorff γ , $k\beta$ -space.

Proof. Let f be a γ -function and g be a $k\beta$ -function. Let $h: N \times X \rightarrow \tau$ be defined by $h(n, x) = f(n, x) \cap g(n, x)$. To show that h is a k -semi-stratifiable function, let K be a compact subset of X and let $h(n, x_n) \cap K \neq \emptyset$, for $n=1, 2, \dots$. As g is a $k\beta$ -function, the sequence $\langle x_n \rangle$ has a cluster point p . Since X is a γ -space, X is first countable. Then there is a subsequence $\langle x_{n_k} \rangle$ that converges to p . Let $C = \{p\} \cup \{x_{n_k} \mid k=1, 2, \dots\}$. If $p \notin K$, we may assume without loss of generality that $C \cap K = \emptyset$. Since f is a γ -function, there is an n_0 such that $g(n_0, C) \cap K = \emptyset$. Now for $n_k \geq n_0$, $g(n_0, C) \supset g(n_0, x_{n_k}) \supset g(n_k, x_{n_k})$, so $h(n_k, x_{n_k}) \cap K = \emptyset$. A contradiction. It follows that X is k -semi-stratifiable. Since every k -semi-stratifiable, first countable space is a Nagata space (Lutzer [7]), X is paracompact. In [3], Hodel proved that every β , γ -space is developable. It is well known that every paracompact developable space metrizable, it follows that X is metrizable.

Theorem 3.2. A regular space X is metrizable if and only if X is a $w\theta$, $k\beta$ -space with a G_δ^* -diagonal.

Corollary 3.3 (Hodel [3]). A regular space X is metrizable if and only if X is a $w\theta$, wN -space with a G_δ^* -diagonal.

In [3] Hodel noted that every developable space is a $w\theta$ -space.

Therefore, we have the following corollary.

Corollary 3.4. A Hausdorff developable, $k\beta$ -space is metrizable

Corollary 3.5 (Hodel [3]). Every Hausdorff developable, wN -space is metrizable.

Since every k -semi-stratifiable space has a G_δ^* -diagonal, Theorem 3.2 generalizes the following result of Martin.

Corollary 3.6 (Martin [10]). A regular space X is metrizable if and only if X is a k -semi-stratifiable, quasi- γ -space.

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