

Boundary value problems and variational inequalities

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1. Introduction and notations

Let Ω be a bounded open set in $\mathbf{R}^n$ ($n \geq 2$). We consider the obstacle problem for a quasilinear elliptic operator of second order in Ω as follows:

$$(1.1) \quad L = -\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u).$$

Here A (*resp.* B) is a vector (*resp.* scalar) valued function defined on $\Omega \times \mathbf{R} \times \mathbf{R}^n$ and the functions A and B are assumed to satisfy the following inequalities:

$$\begin{aligned} |A(x, u, w)| &\leq a(|w|^{p-1} + |u|^{p-1} + 1), \\ |B(x, u, w)| &\leq b(|w|^{p-1} + |u|^{p-1} + 1), \\ w \cdot A(x, u, w) + uB(x, u, w) &\geq c_1|w|^p - c_2(|u|^p + 1), \\ (A(x, u, w_1) - A(x, u, w_2)) \cdot (w_1 - w_2) &> 0 \quad (w_1 \neq w_2) \end{aligned}$$

for all $x \in \Omega$, $u \in \mathbf{R}$ and $w_1, w_2 \in \mathbf{R}^n$, where a, b, c_1, p are positive real numbers satisfying $p > 1$ and c_2 is a nonnegative real number.

It has been known that for each continuous function f on $\partial\Omega$ and a function ϕ on Ω satisfying

$$\int_0^\infty t^{p-1} B_{1,p}(A(t)) dt < +\infty$$

there exists a solution $u \in W_{loc}^{1,p}(\Omega)$ to the obstacle problem with boundary data f , where

$$A(t) = \{y \in \Omega; \psi(y) > t\}.$$

The obstacle problem is to find a function $u \in W_{loc}^{1,p}(\Omega)$ such that

$$u \geq \psi \text{ on } \Omega \text{ except for a subset of } \Omega,$$

$$u = f \text{ weakly on } \partial\Omega$$

and

$$\int_{\Omega} A(y, u(y), \nabla u(y)) \cdot \nabla \phi(y) dy + \int_{\Omega} B(y, u(y), \nabla u(y)) \phi(y) dy \geq 0$$

for all $\phi \in C_0^\infty(\Omega)$ satisfying $\phi \geq \psi - u$ on Ω except for a subset of Ω .

In this paper we will prove the existence of a weak solution to the obstacle problem for (possibly) non-bounded boundary function f . To consider boundary functions which value $\pm\infty$, we must distinguish functions not up to a set of n -dimensional Lebesgue measure zero, but up to a more fine set, for example, a set of $B_{1,s}$ -capacity zero.

Recall that for $s > 1$ the Bessel capacity $B_{1,s}$ with order 1 is defined by

$$B_{1,s}(E) = \inf\{\|g\|_s^s; g \in L^s(\mathbf{R}^n), g \geq 0, G_1 * g \geq 1 \text{ on } E\}$$

for a subset E of $\mathbf{R}^n$. If a property holds on a subset X of $\mathbf{R}^n$ except for a set of $B_{1,s}$ -capacity zero, we say that it holds $B_{1,s}$ -q.e. on X . In the case $s = p$ we use simply "q.e." instead of " $B_{1,p}$ -q.e."

To distinguish functions up to a set of $B_{1,s}$ -capacity zero, we construct a family of functions defined on $\partial\Omega$, which contains all continuous functions and the restrictions of all Bessel potentials $G_1 * g$ ($g \in L^s(\mathbf{R}^n)$) to $\partial\Omega$, where G_1 is the Bessel function with order 1, i.e.,

$$G_1(x) = \frac{1}{(4\pi)^{1/2}} \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty \exp(-\frac{\pi|x|^2}{t}) \exp(-\frac{t}{4\pi}) t^{(1-n)/2} \cdot \frac{1}{t} dt.$$

Recall that the Fourier function of G_1 is equal to

$$\frac{1}{(1 + 4\pi|x|^2)^{1/2}}.$$

Let us define, for each extended real-valued function f on $\partial\Omega$,

$$\gamma_{1,s}(f) = \inf\{\|g\|_s^s; g \in L^s(\mathbf{R}^n), g \geq 0, G_1 * g \geq |f| \text{ on } \partial\Omega\}.$$

Furthermore, denote by $\mathcal{B}(\gamma_{1,s})$ the family of all Borel measurable functions on $\partial\Omega$ such that $\gamma_{1,s}(f) < +\infty$. We remark that $\mathcal{B}(\gamma_{1,s}) \supset C(\partial\Omega)$, where $C(\partial\Omega)$ is the family of all continuous real-valued functions on $\partial\Omega$.

We denote by $\mathcal{L}(\gamma_{1,s})$ the family of all $f \in \mathcal{B}(\gamma_{1,s})$ such that $\gamma_{1,s}(f - f_j) \rightarrow 0$ ($j \rightarrow \infty$) for some $\{f_j\} \subset C(\partial\Omega)$.

It is well-known that, if $1 < p < n$ and $v \in W^{1,p}(\Omega)$, then

$$(1.2) \quad \lim_{r \rightarrow 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} v(y) dy$$

exists (as a real number) for all $x \in \Omega$ except for a set E with $B_{1,p}(E) = 0$, where $B(x, r)$ is the ball with center x and radius r and $|B(x, r)|$ stands for the volume of the ball $B(x, r)$ (cf. [FZ]). For each $v \in W^{1,p}(\Omega)$ we denote by v^* the function defined by (1.2) q.e. on Ω .

Under these notations we will prove the following theorem.

Theorem. Let $1 < p < n$, $p < s \leq \frac{np}{n-p}$, $f \in \mathcal{L}(\gamma_{1,s})$ and ψ be a real-valued function on Ω such that $|\psi| \leq G_1 * g$ on Ω for some $g \in L^s(\mathbb{R}^n)^+$. If

$$\limsup_{y \rightarrow x} \psi(y) < +\infty \text{ and } \limsup_{y \rightarrow x} \psi(y) \leq f(y)$$

for all $x \in \partial\Omega$. Then there exists a function $u \in W_{loc}^{1,p}(\Omega)$ having the following properties:

- (i) $u \geq \psi$ q.e. on Ω ,
- (ii) If $\phi \in W^{1,p}(\Omega)$, $\text{supp } \phi \subset \Omega$ and $\phi^* \geq \psi - u$ q.e. on Ω , then

$$\int_{\Omega} \{A(y, u(y), \nabla u(y)) \cdot \nabla \phi(y) + B(y, u(y), \nabla u(y)) \phi(y)\} dy \geq 0.$$

- (iii) $u = f$ on $\partial\Omega$ in the following sense:

Let τ be a function on $\bar{\Omega}$ such that it is lower semi-continuous on $\bar{\Omega} \setminus K$ for some compact subset K of Ω and

$$\tau|_{\partial\Omega} \in W^{1,s}(\Omega), \quad \tau \geq f + \delta \text{ on } \partial\Omega \text{ for some } \delta > 0.$$

Then $(u - \tau)^+ \in W_0^{1,p}(\Omega)$. Further, let λ be a function on $\bar{\Omega}$ such that it is upper semicontinuous on $\bar{\Omega} \setminus K$ for some compact subset K of Ω and

$$\lambda|_{\partial\Omega} \in W^{1,s}(\Omega), \quad \lambda \leq f - \delta$$

for some $\delta > 0$ on $\partial\Omega$. Then $(u - \lambda)^- \in W_0^{1,p}(\Omega)$.

2. Properties of $\gamma_{1,s}$

In this section we study the properties of $\gamma_{1,s}$. It is easy to see that the functional $\gamma_{1,s}$ has the following properties similar to those of the upper integral.

Lemma 2.1. Let $s > 1$. Then the functional $\gamma_{1,s}$ has the following properties:

- (c₁) $\gamma_{1,s}(f) = \gamma_{1,s}(|f|)$,
- (c₂) $\gamma_{1,s}(bf) = b\gamma_{1,s}(f)$ for $b \in \mathbb{R}^+$,

- (c₃) $f_j \geq 0 \Rightarrow \gamma_{1,s}(\sum_{j=1}^{\infty} f_j) \leq \sum_{j=1}^{\infty} \gamma_{1,s}(f_j)$,
 (c₄) $\gamma_{1,s}(\chi_E) = B_{1,s}(E)^{1/s}$ for $E \subset \mathbb{R}^n$.

Using Lemma 2.1, we can show the following lemma.

Lemma 2.2. (i) If $\gamma_{1,s}(f) < +\infty$, then the set $\{x \in \partial\Omega; |f(x)| = +\infty\}$ is of $B_{1,s}$ -capacity zero.
 (ii) If $\gamma_{1,s}(f - g) = 0$, then $f = g$ $B_{1,s}$ -q.e. on $\partial\Omega$.

Lemma 2.3. Let g be a nonnegative function in $L^s(\mathbb{R}^n)$. Then the Bessel potential $G_1 * g$ belongs to $\mathcal{L}(\gamma_{1,s})$.

Proof. We can assume that g is nonnegative. Set

$$g_j = \min\{g, j\} \text{ and } h_j = g - g_j.$$

Noting $G_1 \in L^1(\mathbb{R}^n)$, we see that $G_1 * g_j$ is continuous on $\partial\Omega$. Since $|G_1 * g - G_1 * g_j| \leq G_1 * h_j$ and $\|h_j\|_s \rightarrow 0$ as $j \rightarrow \infty$, we have the conclusion. Q.E.D

Lemma 2.4. The set $\mathcal{L}$ of the restrictions of all Lipschitz functions on $\bar{\Omega}$ to $\partial\Omega$ is dense in $\mathcal{L}(\gamma_{1,s})$.

Proof. We can choose a nonnegative function $h = G_1 * g$ ($g \in L^s(\mathbb{R}^n)^+$) such that $h \geq 1$ on $\partial\Omega$. Since $\mathcal{L}$ is uniformly dense in $C(\partial\Omega)$, it is dense in $\mathcal{L}(\gamma_{1,s})$. Q.E.D.

Noting that

$$G_1(y) = O(e^{-c|y|}) \text{ for some } c > 0 \text{ as } |y| \rightarrow \infty,$$

we can easily show the following lemma.

Lemma 2.5. Let p, s be positive real numbers satisfying $1 < p \leq s$ and E be a relatively compact subset of $\mathbb{R}^n$. If $B_{1,s}(E) = 0$, then $B_{1,p}(E) = 0$.

For a function f defined on $\mathbb{R}^n$ we define

$$\gamma_{1,s}(f) = \inf\{\|f\|_s; g \in L^s(\mathbb{R}^n)^+; G_1 * g \geq |f| \text{ on } \mathbb{R}^n\}$$

It is easy to see that this functional $\gamma_{1,s}$ also has the properties in Lemmas 2.1 and 2.2 in which $\partial\Omega$ is replaced by $\mathbb{R}^n$.

Lemma 2.6. Let $\{f_j\}$ be a sequence of functions on $\mathbb{R}^n$ such that $\gamma_{1,s}(f_j) \rightarrow 0$ ($j \rightarrow \infty$). Then there exists a subsequence $\{g_k\}$ of $\{f_j\}$ such that $g_k \rightarrow 0$ pointwisely $B_{1,s}$ -q.e. on $\mathbb{R}^n$.

Proof. Choose a subsequence $\{g_k\}$ of $\{f_j\}$ satisfying

$$(2.1) \quad \sum_{k=1}^{\infty} 2^k \gamma_{1,s}(g_{k+1} - g_k) < +\infty.$$

To show that $\{g_k\}$ is the desired subsequence, set

$$E = \cup_{k=1}^{\infty} \{x \in \mathbb{R}^n; |g_k(x)| = +\infty\}.$$

Then we have $B_{1,s}(E) = 0$ by Lemmas 2.1 and 2.2. Further set

$$O'_k = \{x \in \mathbb{R}^n \setminus E; |g_{k+1}(x) - g_k(x)| \geq 2^{-k}\}$$

and

$$O_k = \cup_{i=k}^{\infty} O'_i \quad \text{and} \quad F_k = \mathbb{R}^n \setminus (O_k \cup E).$$

Setting $g_0 = 0$ and noting that

$$g_k = \sum_{i=0}^{k-1} (g_{i+1} - g_i)$$

on $\mathbb{R}^n \setminus E$, we see that $\{g_k\}$ converges to 0 on $\cup_{i=1}^{\infty} F_i$. We put

$$E_o = \mathbb{R}^n \setminus (\cup_{k=1}^{\infty} F_k \cup E) = \cap_{k=1}^{\infty} O_k.$$

From

$$\chi_{O_k} \leq \sum_{i=k}^{\infty} \chi_{O'_i} \leq \sum_{i=k}^{\infty} 2^i |g_{i+1} - g_i|$$

and Lemma 2.1 we deduce

$$\gamma_{1,s}(\chi_{O_k}) \leq \sum_{i=k}^{\infty} 2^i \gamma_{1,s}(g_{i+1} - g_i).$$

On account of (2.1) we have

$$\gamma_{1,s}(\chi_{O_k}) \rightarrow 0 \quad (k \rightarrow \infty)$$

and hence $\gamma_{1,s}(\chi_{E_o}) = 0$. Therefore we see by Lemma 2.1 that $B_{1,s}(E \cup E_o)^{1/s} = \gamma_{1,s}(\chi_{E \cup E_o}) = 0$ and $\{g_k\}$ converges to 0 on $\mathbb{R}^n \setminus (E \cup E_o)$.

Q.E.D.

3. Boundedness of solutions

For an open subset $\Omega_o \neq \emptyset$ of Ω we denote by $A(\Omega_o, \cdot)$ the mapping

$$W^{1,p}(\Omega_o) \rightarrow W^{1,p}(\Omega_o)'$$

defined by

$$\begin{aligned} \langle A(\Omega_o, v), w \rangle \\ = \int_{\Omega_o} \{A(y, v(y), \nabla v(y)) \cdot \nabla w(y) + B(y, v(y), \nabla v(y))w(y)\} \end{aligned}$$

The following theorem is fundamental.

Theorem A ([MZ, Theorem 3.1]). *Let $p < s$ and Ω_o be a nonempty open subset of Ω , $\varepsilon = +$ or $-$ and v, η be functions in $W^{1,p}(\Omega_o)$ such that $(v - \eta)^\varepsilon \in W^{1,p}(\Omega_o)$ and*

$$\langle A(\Omega_o, v), -\varepsilon(v - \eta)^\varepsilon \rangle \geq 0.$$

Then

$$\|(v - \eta)^\varepsilon\|_{W^{1,p}(\Omega_o)} \leq c + c \left(1 + \int_{\Omega_o} (|\eta|^s + \sum_{j=1}^n |\frac{\partial \eta}{\partial y_j}|^p) dy \right)^{1/p},$$

where c is a constant independent of v, η .

It is well-known that for each $s > 1$

$$W^{1,s}(\mathbf{R}^n) = \{G_1 * g; g \in L^s(\mathbf{R}^n)\}$$

and

$$(3.1) \quad \frac{1}{M} \|g\|_s \leq \|G_1 * g\|_{W^{1,s}(\mathbf{R}^n)} \leq M \|g\|_s,$$

where M is a constant independent of g (cf. [S, Theorem 3 on p.135]).

Lemma 3.1. *Let f be a Lipschitz function on $\bar{\Omega}$ such that $|f| \leq G_1 * g_1$ for some $g_1 \in L^p(\mathbf{R}^n)^+$. Furthermore, let ψ be a real-valued function on Ω such that*

$$(3.2) \quad \limsup_{y \rightarrow x, y \in \Omega} \psi(y) < f(x) - \delta$$

for all $x \in \partial\Omega$ and for some $\delta > 0$, and

$$|\psi| \leq G_1 * g_o \text{ for some } g_o \in L^p(\mathbf{R}^n)^+.$$

Then there exists a function $u \in W^{1,p}(\Omega)$ such that u has the properties:

- (i) $u \geq \psi$ q.e. on Ω ,
- (ii) If $\phi \in W^{1,p}_0(\Omega)$ and $\phi^* \geq \psi - u$ q.e. on Ω , then

$$\int_{\Omega} \{A(y, u(y), \nabla u(y)) \cdot \nabla \phi(y) + B(y, u(y), \nabla u(y)) \phi(y)\} dy \geq 0.$$

- (iii) $u - f \in W^{1,p}_0(\Omega)$,
- (iv) If η (resp. λ) is a function on $\bar{\Omega}$, such that it is lower (resp. upper) semicontinuous on $\bar{\Omega} \setminus K$ for some compact subset K of Ω and $\eta \in W^{1,p}(\Omega)$ (resp. $\lambda \in W^{1,p}(\Omega)$), $\eta(y) > f(y)$ (resp. $\lambda(y) < f(y)$) for all $y \in \partial\Omega$, then $(u - \eta)^+ \in W^{1,p}_0(\Omega)$ (resp. $(u - \lambda)^- \in W^{1,p}_0(\Omega)$).

Proof. Set

$$K = \{v \in W^{1,p}(\Omega); v - f \in W^{1,p}_0(\Omega), v^* \geq \psi \text{ q.e. on } \Omega\}.$$

We claim that K is not empty. Indeed, noting that $G_1 * g_1$ is lower semicontinuous, we can choose, by the aid of (3.2), an open set Ω_o such that $\bar{\Omega}_o \subset \Omega$ and

$$\psi(y) < f(y) \quad \text{for all } y \in \Omega \setminus \Omega_o.$$

Choose a Lipschitz function h such that

$$\text{supp } h \subset \Omega, \quad h = 1 \text{ on } \bar{\Omega}_o, \quad 0 \leq h \leq 1,$$

and define

$$\phi(y) = h(y)w(y) + (1 - h(y))f(y),$$

where $w = G_1 * g_o$. We note that $\text{supp } h$ stands for the closure of the set $\{y; h(y) \neq 0\}$. Then $\phi^* \geq \psi$ q.e. on Ω and $\phi - f = h(w - f) \in W^{1,p}_0(\Omega)$. Therefore we see that $\phi \in K$.

The family K is a convex closed subset of $W^{1,p}(\Omega)$ and hence weakly closed. The mapping $A(\Omega, \cdot)$ from $W^{1,p}(\Omega)$ to $W^{1,p}(\Omega)'$ is pseudomonotone by Theorem 3.9 in [MZ]. Furthermore we see that

$$\frac{\langle A(\Omega, v), v - v_o \rangle}{\|v\|_{W^{1,p}(\Omega)}} \rightarrow \infty$$

as $\|v\|_{W^{1,p}(\Omega)} \rightarrow \infty$ ($v \in K$). It follows from Theorem 8.2 on p.247 in [L] that there exists $u_o \in K$ such that

$$\langle A(\Omega, u_o), v - u_o \rangle \geq 0 \text{ for all } v \in K.$$

Setting $u = u_o^*$, we will show that u is the desired function. It is obvious that (i) and (iv) hold. To show (ii), let ϕ be a function in $W_o^{1,p}(\Omega)$ such that

$$\phi^* \geq \psi - u \text{ q.e. on } \Omega.$$

From $u - f \in W_o^{1,p}(\Omega)$ and $\phi + u - f \in W_o^{1,p}(\Omega)$, it follows that

$$\langle A(\Omega, u), \phi \rangle \geq 0.$$

Finally, to show (v), let η be a lower semicontinuous function on $\bar{\Omega}$ in $W^{1,p}(\Omega)$ such that $\eta > f$ on $\partial\Omega$. Since $f - \eta < 0$ outside a compact subset of Ω and $u - f \in W_o^{1,p}(\Omega)$, we have

$$(u - \eta)^+ \in W_o^{1,p}(\Omega).$$

Similarly we can show that $(u - \lambda)^- \in W_o^{1,p}(\Omega)$. Q.E.D.

4. Proof of Theorem

Let us prove Theorem. Suppose that $f \in \mathcal{L}(\gamma_{1,s})$. On account of Lemma 2.4 we can choose a sequence $\{f_j\}$ of Lipschitz functions on $\bar{\Omega}$ and a sequence $\{g_j\}$ of functions in $L^s(\mathbb{R}^n)^+$ such that

$$|f - f_j| \leq G_1 * g_j \text{ on } \partial\Omega \quad \|g_j\|_s < 2^{-j}.$$

Since $\gamma_{1,s}(G_1 * g_j) \rightarrow 0$, we can choose, by Lemma 2.6, a subsequence $\{G_1 * h_k\}$ converges pointwisely to 0 $B_{1,s}$ -q.e. on $\mathbb{R}^n$. Therefore, by Lemma 2.5, it converges to 0 q.e. on $\bar{\Omega}$. Noting that

$$\limsup_{y \rightarrow x} \psi(y) \leq f(x) \leq G_1 * h_k(x) + f_{j_k}(x)$$

for all $x \in \partial\Omega$, we define

$$\psi_k(y) = \psi(y) - G_1 * h_k(y) - 2^{-k} \quad \text{if } G_1 * h_k(y) < +\infty$$

and

$$\psi_k(y) = - \sup_{x \in \partial\Omega} |f_{j_k}(x)| - 1 \quad \text{otherwise.}$$

Then we have

$$\limsup_{y \rightarrow x} \psi_k(y) \leq f_{j_k}(x) - 2^{-k}.$$

Pick $h_o, h' \in L^s(\mathbb{R}^n)^+$ such that

$$G_1 * h_o \geq |f| \text{ on } \partial\Omega, \quad G_1 * h' \geq 1 \text{ on } \partial\Omega.$$

If we set

$$(4.1) \quad h = h' + \sum_{k=0}^{\infty} h_k,$$

then

$$(4.2) \quad |f_{j_k}| + 1 \leq |f| + G_1 * h_k + G_1 * h' \leq G_1 * h$$

on $\partial\Omega$.

We denote by u_k the solution u in Lemma 3.1 corresponding to $f = f_{j_k}$ and $\psi = \psi_k$. Let Ω_o be an arbitrary subset of Ω such that $\bar{\Omega}_o \subset \Omega$.

To show that

$$\{\|u_k\|_{W^{1,p}(\Omega_o)}\}$$

is uniformly bounded, let us take a Lipschitz function η such that

$$\text{supp } \eta \subset \Omega, \quad \eta = 1 \text{ on } \bar{\Omega}_o, \quad 0 \leq \eta \leq 1.$$

Further, take $g \in L^p(\mathbb{R}^n)^+$ satisfying $|\psi| \leq G_1 * g$ on Ω and define

$$\beta(y) = \eta(y)G_1 * g(y) + (1 - \eta(y))G_1 * h(y),$$

$$\phi(y) = \eta(y)\beta(y) - (1 - \eta(y))G_1 * h(y)$$

for $y \in \bar{\Omega}$. Then $\phi \in W^{1,p}(\Omega)$. Applying the Sobolev inequality and (3.1), and noting that $s \leq \frac{np}{n-p}$, we obtain

$$\int_{\Omega} |\beta(y)|^s + \sum_{j=1}^n \int_{\Omega} \left| \frac{\partial \beta}{\partial y_j}(y) \right|^p dy < +\infty$$

and

$$\int_{\Omega} |\phi(y)|^s + \sum_{j=1}^n \int_{\Omega} \left| \frac{\partial \phi}{\partial y_j}(y) \right|^p dy < +\infty.$$

Note that

$$(4.3) \quad (u_k - \phi)^- \geq 0 \geq \psi_k - u_k \quad \text{q.e. on } \Omega.$$

By the aid of (4.2) we have

$$f_{j_k} - 1 \geq -G_1 * h = \phi \quad \text{on } \partial\Omega.$$

From Lemma 3.1, (iv) we deduce $(u_k - \phi)^- \in W_o^{1,p}(\Omega)$, which and (4.3) lead to

$$\langle A(\Omega, u_k), (u_k - \phi)^- \rangle \geq 0.$$

Therefore we obtain, by Theorem A,

$$\begin{aligned} & \| (u_k - \phi)^- \|_{W^{1,p}(\Omega)} \\ & \leq c + c \left(\int_{\Omega} |\phi(y)|^s + \sum_{j=1}^n \int_{\Omega} \left| \frac{\partial \phi}{\partial y_j}(y) \right|^p dy \right)^{1/p} \end{aligned}$$

and hence

$$\begin{aligned} (4.4) \quad & \| (u_k - \beta)^- \|_{W^{1,p}(\Omega_o)} \\ & \leq c + c \left(\int_{\Omega} |\phi(y)|^s + \sum_{j=1}^n \int_{\Omega} \left| \frac{\partial \phi}{\partial y_j}(y) \right|^p dy \right)^{1/p}. \end{aligned}$$

On the other hand, since

$$f_{j_k} + 1 < G_1 * h \leq \beta \quad \text{on } \partial\Omega,$$

we have $(u_k - \beta)^+ \in W_o^{1,p}(\Omega)$. Moreover, if $u_k(y) \leq \beta(y)$, then

$$-(u_k - \beta)^+(y) = 0 \geq \psi_k(y) - u_k(y)$$

for q.e. y . If $u_k(y) > \beta(y)$, then

$$-(u_k - \beta)^+(y) = \beta(y) - u_k(y) \geq \psi_k(y) - u_k(y)$$

for q.e. y . Therefore we have

$$\langle A(\Omega, u_k), -(u_k - \beta)^+ \rangle \geq 0$$

by Lemma 3.1, (ii). This and Theorem A lead to

$$\begin{aligned} (4.5) \quad & \| (u_k - \beta)^+ \|_{W^{1,p}(\Omega)} \\ & \leq c + c \left(\int_{\Omega} |\beta(y)|^s + \sum_{j=1}^n \int_{\Omega} \left| \frac{\partial \beta}{\partial y_j}(y) \right|^p dy \right)^{1/p}. \end{aligned}$$

Thus we see by (4.4) and (4.5) that

$$\{\|u_k - \beta\|_{W^{1,p}(\Omega_o)}\}$$

and hence

$$\{\|u_k\|_{W^{1,p}(\Omega_o)}\}$$

is uniformly bounded for every open set Ω_o satisfying $\bar{\Omega}_o \subset \Omega$. Therefore we can choose a subsequence $\{u_{k_i}\}$ and $w \in W_{loc}^{1,p}(\Omega)$ such that for every open set Ω_o satisfying $\bar{\Omega}_o \subset \Omega$ the sequence $\{u_{k_i}\}$ converges weakly to w in $W^{1,p}(\Omega_o)$. Since identity mapping is a compact operator from $W^{1,p}(\Omega_o)$ to $L^p(\Omega_o)$, we may suppose that $\{u_{k_i}\}$ converges strongly to w in $L^p(\Omega_o)$. We note that

$$u_k = u_k^* \geq \psi_k \text{ q.e. on } \Omega, \quad \lim_{k \rightarrow \infty} \psi_k = \psi \text{ q.e. on } \Omega.$$

Since there exists a subsequence of convex combinations of the functions u_{k_i} which converges strongly to w in $W^{1,p}(\Omega_o)$, we conclude that

$$(4.6) \quad w^* \geq \psi \text{ q.e. on } \Omega.$$

Moreover, by the same method as in the proof of Lemma 4.5 in [MZ] we can show that

$$u_k \rightarrow w \text{ strongly in } W^{1,p}(\Omega_o).$$

Putting $u = w^*$, we will show that u is the desired function. The assertion (i) follows from (4.6). To show (ii), suppose that $\phi \in W^{1,p}(\Omega)$, $\text{supp } \phi \subset K$ and

$$\phi^* \geq \psi - u \text{ q.e. on } \Omega.$$

Take an open set Ω_o such that $\text{supp } \phi \subset \Omega_o \subset \bar{\Omega}_o \subset \Omega$ and choose a Lipschitz function τ on Ω such that

$$\text{supp } \tau \subset \Omega, \quad \tau = 1 \text{ on } \bar{\Omega}_o, \quad 0 \leq \tau \leq 1.$$

We define

$$\phi_i = \phi + \tau(u - u_{k_i})^+.$$

We note that

$$\phi_i^* = \phi^* + u - u_{k_i}^* \geq \psi_{k_i} - u_{k_i} \text{ q.e. on } \Omega_o \cap \{u \geq u_{k_i}\}$$

and

$$\phi_i^* = \phi^* \geq \psi - u \geq \psi_{k_i} - u_{k_i}^* \text{ q.e. on } \Omega_o \cap \{u < u_{k_i}\}.$$

On account of Lemma 3.1, (ii) we have

$$\langle A(\Omega_o, u_{k_i}), \phi_i \rangle \geq 0 \quad \text{for each } i.$$

Since

$$u_{k_i} \rightarrow u \text{ strongly in } W^{1,p}(\Omega_o)$$

and

$$\phi_i \rightarrow \phi \text{ strongly in } W^{1,p}(\Omega_o),$$

we see that

$$\langle A(\Omega_o, u), \phi \rangle \geq 0$$

and hence

$$\langle A(\Omega, u), \phi \rangle \geq 0.$$

Next, to show (iii), denote by E the set

$$\{x \in \partial\Omega; |f(x)| = +\infty\} \cup \{x \in \partial\Omega; \lim_{k \rightarrow \infty} G_1 * h_k(x) \neq 0\}.$$

Then $\gamma_{1,s}(\chi_E) = 0$ and

$$(4.7) \quad \lim_{k \rightarrow \infty} f_{j_k}(x) = f(x) \quad \text{for } x \in \partial\Omega \setminus E.$$

Suppose that τ is a function on $\bar{\Omega}$ such that it is lower semicontinuous on $\bar{\Omega} \setminus K$ for some compact subset K of Ω and $\tau \in W^{1,s}(\Omega)$, $\tau \geq f + \delta$ on $\partial\Omega$ for some $\delta > 0$. Since

$$\limsup_{y \rightarrow x} \psi(y) < +\infty \quad \text{and} \quad \limsup_{y \rightarrow x} \psi(y) \leq f(x)$$

for all $x \in \partial\Omega$, there is an open set Ω_1 satisfying $\bar{\Omega}_1 \subset \Omega$ and

$$\psi(y) < \tau(y) \quad \text{for all } y \in \Omega \setminus \Omega_1.$$

Take a Lipschitz function η such that

$$\text{supp } \eta \subset \Omega, \quad \eta = 1 \text{ on } \bar{\Omega}_1, \quad 0 \leq \eta \leq 1$$

and define

$$v_o = (G_1 * h)\eta,$$

where h is the function defined in (4.1).

Let us show that $(u - \tau)^+ \in W_o^{1,p}(\Omega)$. Since $\text{supp } v_o \subset \Omega$ and $v_o \in W^{1,p}(\Omega)$, it suffices to show that $(u - \tau - v_o)^+ \in W_o^{1,p}(\Omega)$. From the inequality

$$f_{j_k} < \tau + v_o + G_1 * h_k \quad \text{on } \partial\Omega$$

and lemma 3.1 we deduce

$$(u_k - (\tau + v_o + G_1 * h_k))^+ \in W_o^{1,p}(\Omega).$$

We will show that

$$\{\|(u_k - (\tau + v_o + G_1 * h_k))^+\|_{W^{1,p}(\Omega)}\}$$

is uniformly bounded. We claim that

$$-(u_k - (\tau + v_o + G_1 * h_k))^+ \geq \psi_k - u_k \quad \text{q.e. on } \Omega.$$

Indeed we have

$$-(u_k - (\tau + v_o + G_1 * h_k))^+ \geq -(u_k - (\tau + v_o + G_1 * h_k)) \geq \psi_k - u_k$$

q.e. on $\Omega \cap \{u_k \geq \tau + v_o + G_1 * h_k\}$ and

$$-(u_k - (\tau + v_o + G_1 * h_k))^+ = 0 \geq \psi_k - u_k$$

q.e. on $\Omega \cap \{u_k < \tau + v_o + G_1 * h_k\}$. Therefore, from Lemma 3.1, (ii) it follows that

$$\langle A(\Omega, u_k), -(u_k - (\tau + v_o + G_1 * h_k))^+ \rangle \geq 0.$$

Using Theorem A and (3.1), we have

$$\begin{aligned} & \|(u_k - (\tau + v_o + G_1 * h_k))^+\|_{W^{1,p}(\Omega)} \leq c_1 \\ & + c_1 \left(\int_{\Omega} |(\tau + v_o + G_1 * h_k)|^s + \sum_{j=1}^n \int_{\Omega} \left| \frac{\partial}{\partial y_j} (\tau + v_o + G_1 * h_k) \right|^p dy \right)^{1/p} \\ & \leq c_2 + c_2 M \left(\int_{\Omega} |(\tau + v_o)|^s + \sum_{j=1}^n \int_{\Omega} \left| \frac{\partial}{\partial y_j} (\tau + v_o) \right|^p dy \right)^{1/p}. \end{aligned}$$

Thus we see that

$$\{\|(u_k - (\tau + v_o + G_1 * h_k))^+\|_{W^{1,p}(\Omega)}\}$$

is uniformly bounded. We note that for every open set Ω_o satisfying $\bar{\Omega} \subset \Omega$, $\{u_{k_i}\}$ converges to u strongly in $W^{1,p}(\Omega_o)$ and

$$\|G_1 * h_{k_i}\|_{W^{1,p}(\Omega_o)} \leq M \|h_{k_i}\|_s \rightarrow 0$$

as $i \rightarrow \infty$. Using Lemma 4.6 in [MZ], we conclude that $(u - (\tau + v_o))^+ \in W_o^{1,p}(\Omega)$ and hence $(u - \tau)^+ \in W_o^{1,p}(\Omega)$.

Finally, suppose that λ is a function on $\bar{\Omega}$ such that it is upper semi-continuous on $\bar{\Omega} \setminus K$ for some compact subset K of Ω and $\lambda \in W^{1,p}(\Omega)$, $\lambda \leq f - \delta$ on $\partial\Omega$. In this case we can also show directly, without the aid of v_o , that $(u - \lambda)^- \in W_o^{1,p}(\Omega)$. Thus we see that (iii) also holds. This completes the proof. Q.E.D.

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