

ON THE ARITHMETICAL RANK OF SQUAREFREE MONOMIAL IDEALS CONCERNED WITH THE COMPLETE BIPARTITE GRAPH $K_{2,n}$

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Introduction

S : a polynomial ring over a field K .

$I, J \subset S$: squarefree monomial ideals.

Let $X = V(I)$ be an affine algebraic set. Then how many hypersurfaces need to express X as intersection of those?

$$X = X_0 \cap X_1 \cap \cdots \cap X_{s-1}, \quad X_i = V(f_i).$$

The arithmetical rank of I :

$$\text{ara } I := \min\{s : \exists f_0, f_1, \dots, f_{s-1} \in I \text{ s. t. } \sqrt{(f_0, f_1, \dots, f_{s-1})} = \sqrt{I}\}.$$

Fact (Lyubeznik).

$$(\star) \quad \text{height } I \leq \text{pd}_S S/I \leq \text{ara } I \leq \mu(I).$$

Problem 0.1. When does $\text{ara } I = \text{pd}_S S/I$ hold?

Known results:

- (1) $\mu(I) - \text{height } I = 0$ (clear).
- (2) (KTY) $\mu(I) - \text{height } I = 1$.
- (3) (KTY) $\mu(I) - \text{height } I = 2$.
- (4) $\mu(I) - \text{pd}_S S/I = 0$ (clear).
- (5) (KTY) $\mu(I) - \text{pd}_S S/I = 1$.
- (1)* (Schenzel-Vogel, Schmitt-Vogel)
 $\text{arithdeg } I - \text{indeg } I = 0$.
- (2)* (KTY) $\text{arithdeg } I - \text{indeg } I = 1$.
- (4)* (KTY) $\text{arithdeg } I - \text{reg } I = 0$.

Counterexamples (char $K \neq 2$):

- (6) (Yan) Stanley-Reisner ideal associated to Reisner's triangulation of the projective plane.
- (7) (KTY) An ideal with $\mu(I) - \text{height } I = 3$.
- (7)* (KTY) An ideal with $\text{arithdeg } I - \text{indeg } I = 3$.

$((\cdot)^*$ is Alexander dual of $(\cdot)$.)

In this poster, we focus on the case corresponding to (3)* and (5)*.

1 Alexander duality

Example 1.1 (Alexander dual ideal).

$$(\clubsuit) \quad I = (x_1, x_2, x_3) \cap (x_4, x_5, x_6) \cap (x_1, x_4) \cap (x_2, x_5) \cap (x_3, x_6) \\ \implies I^* = (x_1 x_2 x_3, x_4 x_5 x_6, x_1 x_4, x_2 x_5, x_3 x_6).$$

Properties:

$$\bullet I^{**} = I.$$

- (Frühbis-Krüger-Terai)

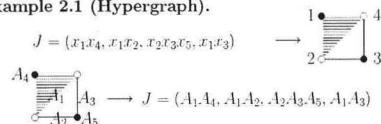
$$\text{indeg } I^* = \text{height } I, \quad \text{reg } I^* = \text{pd}_S S/I, \quad \text{arithdeg } I^* = \mu(I).$$

- (Hoa-Trung, Frühbis-Krüger-Terai)

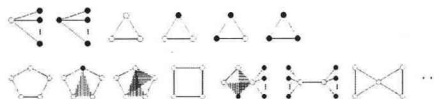
$$(\star)^* \quad \text{indeg } I \leq \text{reg } I \leq \text{arithdeg } I.$$

2 Hypergraphs

Example 2.1 (Hypergraph).



Using hypergraph, we can classify squarefree monomial ideals with $\mu(I) - \text{height } I \leq 2$ (KTY).



3 The case of (3)* arithdeg $I - \text{indeg } I = 2$

There are 3 cases by $(\star)^*$:

- (a) $\text{arithdeg } I = \text{reg } I = \text{indeg } I + 2$.
- (b) $\text{arithdeg } I = \text{reg } I + 1 = \text{indeg } I + 2$.
- (c) $\text{arithdeg } I = \text{reg } I + 2 = \text{indeg } I + 2$.

The case (a): contained in (4)*.

Theorem 3.1 (KTY). For the ideal I of the case (c),

$$\text{ara } I = \text{pd}_S S/I.$$

Proof. We determine the arithmetical rank according to the classification by hypergraphs. $\square$

The case (b) is an open problem. This case is a part of the case (5)*:

$$(5)^* \quad \text{arithdeg } I - \text{reg } I = 1.$$

$\iff \mathcal{H}(I^*)$ "contains" a complete bipartite graph (Terai).

Problem 3.2.

$$\mathcal{H}(J) - K_{2,n} -$$

Determine the arithmetical rank of $I = J^*$.

Case1: assign 1 variable to each edges.

Theorem 3.3 (KTY). For any n , the ideal I which is obtained by assigning 1 variable to each edge of hypergraphs in Problem 3.2 satisfies

$$\text{ara } I = \text{pd}_S S/I.$$

We shall see the case $n = 3$.

$$(\clubsuit) \quad I = (x_1, x_2, x_3) \cap (y_1, y_2, y_3) \cap (x_1, y_1) \cap (x_2, y_2) \cap (x_3, y_3).$$

In this case, $\text{pd}_S S/I = 3 = \text{ara } I$:

$$\begin{cases} f_0 = x_1 x_2 x_3 \cdot y_1 y_2 y_3, \\ f_1 = x_1 x_2 \cdot y_3 + x_1 x_3 \cdot y_2 + x_2 x_3 \cdot y_1, \\ f_2 = x_1 \cdot y_2 y_3 + x_2 \cdot y_1 y_3 + x_3 \cdot y_1 y_2. \end{cases}$$

Case2: general cases.

Theorem 3.4 (KTY). For $n = 2, 3, 4$, the ideal I in Problem 3.2 satisfies

$$\text{ara } I = \text{pd}_S S/I.$$

The idea of the proof:

Example 3.5 (Known result (4)*).

$$I = (y_1, x_1, x_2) \cap (y_2, x_1, x_3) \cap (y_3, x_2).$$

Then $\text{ara } I = \text{pd}_S S/I = 4$:

$$\begin{cases} h_0 = y_1 y_2 y_3, \\ h_1 = \mathbf{x}_1 \cdot y_3 + \mathbf{x}_2 \cdot y_2 + \mathbf{x}_3 \cdot y_1 y_3, \\ h_2 = \mathbf{x}_1 \mathbf{x}_2 + \mathbf{x}_1 \mathbf{x}_3 \cdot y_3 + \mathbf{x}_2 \mathbf{x}_3, \\ h_3 = \mathbf{x}_1 \mathbf{x}_2 \mathbf{x}_3. \end{cases}$$

Note that h_1, h_2, h_3 are modifications of elementary symmetric functions of x_1, x_2, x_3 .

The case $n = 3$ of Theorem 3.4:

$$(\clubsuit)^* \quad I = P_1 \cap P_2 \cap Q_1 \cap Q_2 \cap Q_3,$$

where

$$P_1 = (X_1, X_2, X_3), \quad P_2 = (Y_1, Y_2, Y_3), \quad Q_i = (X_i, Y_i),$$

$$X_i = x_i, \quad X'_i = x_i, x_{i2}, \dots, x_{id},$$

$$Y_i = y_i, \quad Y'_i = y_i, y_{i2}, \dots, y_{im}, \quad (i = 1, 2, 3).$$

Then $\text{pd}_S S/I = \sum_{i=1}^3 (\ell_i + m_i) - 3$.

k th elementary symmetric functions

$$g'_k = S_k(X'_1 \cup X'_2 \cup X'_3 \cup Y'_1 \cup Y'_2 \cup Y'_3) \\ (k = 1, 2, \dots, \text{pd}_S S/I - 3).$$

$f_0, f_1, f_2, \dots, \text{pd}_S S/I$ elements! and modify them.