

An application of Mellin-Barnes' type integrals to the mean square of L -functions

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1 Introduction

Let q be a positive integer, s a complex variable and $L(s, \chi)$ the Dirichlet L -function attached to a Dirichlet character $\chi \bmod q$. Note that $L(s, \chi)$ reduces to the Riemann zeta-function $\zeta(s)$ if $q = 1$.

Let $\varphi(q)$ be Euler's function. The mean square

$$\varphi(q)^{-1} \sum_{\chi \bmod q} |L(s, \chi)|^2, \quad (1.1)$$

summed over all characters $\chi \bmod q$, has been studied by various authors. Let $\mu(n)$ be Mobius' function. In the special case $s = \frac{1}{2}$, D. R. Heath-Brown [He] found the formula

$$\varphi(q)^{-1} \sum_{\chi \bmod q} |L(\tfrac{1}{2}, \chi)|^2 = q^{-1} \sum_{k|q} \mu(\tfrac{q}{k}) T(k), \quad (1.2)$$

where k runs through all positive divisors of q and $T(k)$ has the asymptotic expansion

$$T(k) = k \left(\log \frac{k}{8\pi} + \gamma \right) + 2\zeta^2(\tfrac{1}{2}) + \sum_{n=0}^{2N-1} c_n k^{-\frac{n}{2}} + O(k^{-N})$$

for any integer $N \geq 1$, with Euler's constant γ and unspecified numerical constants c_n . If $q = p$ is a prime, (1.2) gives an asymptotic series in terms of $p^{-\frac{1}{2}}$, since $T(1)$ can be evaluated in a closed form. On the other hand, Y. Motohashi [Mo1], in a series of his study on higher power moments for $\zeta(s)$ and $L(s, \chi)$, applied a classical idea of F. V. Atkinson [At] to (1.1) and proved for any prime $q = p$

$$\begin{aligned} & (p-1)^{-1} \sum_{\chi \bmod p} |L(\tfrac{1}{2} + it, \chi)|^2 \\ &= \log \frac{p}{2\pi} + 2\gamma + \operatorname{Re} \frac{\Gamma'}{\Gamma}(\tfrac{1}{2} + it) + 2p^{-\frac{1}{2}} |\zeta(\tfrac{1}{2} + it)|^2 \cos(\log p) \\ & \quad - p^{-1} |\zeta(\tfrac{1}{2} + it)|^2 + O(p^{-\frac{3}{2}}), \end{aligned}$$

where $\Gamma(s)$ is the gamma-function and the constant implied in the O -symbol depends on t . More general and precise formulae have been proved in [KM1], [Ka1] and [Ka2] by refining the argument of Atkinson and Motohashi.

and its meromorphic continuation.

Let $\sigma_a(n)$ denote the sum of the a -th powers of positive divisors of n . The error term $e_N(\sigma + it; k)$ in (1.6) is of the form

$$e_N(\sigma + it; k) = \operatorname{Re}\{k^{\sigma+it-N} R_N(\sigma + it, \sigma - it; k)\},$$

where $R_N(u, v; k)$ has the following expressions (cf. [Ka2, Lemma 2.2]):

For $\operatorname{Re} u < N$, $\operatorname{Re} v > -N + 1$ and $\operatorname{Re}(u + v) < 2$,

$$\begin{aligned} R_N(u, v; k) &= (-1)^N (2\pi)^{u+v-1} \frac{\Gamma(N+1-u)}{\Gamma(v)} \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} \sum_{l=1}^{\infty} \sigma_{u+v-1}(l) \\ &\quad \times \left\{ e^{\frac{\pi i}{2}(u+v-1)} J_-(\tau, l; k) + e^{-\frac{\pi i}{2}(u+v-1)} J_+(\tau, l; k) \right\} d\tau \end{aligned} \quad (1.8)$$

with

$$J_{\pm}(\tau, l; k) = \int_0^{\infty} y^{v+N-1} (1 + k^{-1}\tau y)^{u-N-1} e^{\pm 2\pi i l y} dy,$$

while for $\operatorname{Re} u < N$, $\operatorname{Re} v > -N + 1$ and $\operatorname{Re}(u + v) > 0$,

$$\begin{aligned} R_N(u, v; k) &= (-1)^N \frac{\Gamma(v+N)}{\Gamma(v)} \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} \sum_{l=1}^{\infty} \sigma_{1-u-v}(l) \\ &\quad \times \left\{ \tilde{J}_-(\tau, l; k) + \tilde{J}_+(\tau, l; k) \right\} d\tau \end{aligned} \quad (1.9)$$

with

$$\tilde{J}_{\pm}(\tau, l; k) = \int_0^{\infty} y^{-u+N} (1 + k^{-1}\tau y)^{-v-N} e^{\pm 2\pi i l y} dy.$$

It is in fact possible to obtain a more explicit estimate for $e_N(\sigma + it; k)$ by applying a saddle point lemma of Atkinson [At, Lemma 1] to $J_{\pm}(\tau, l; k)$ and $\tilde{J}_{\pm}(\tau, l; k)$.

Theorem 2 ([Ka2, Theorem 1 with $h = 0$]) *For any integer $N \geq 1$, the inequality*

$$e_N(\sigma + it; k) = O\{k^{\sigma-N}(|t| + 1)^{2N+\frac{1}{2}-\sigma}\} \quad (1.10)$$

holds in the region

$$\{\sigma + it; -N + 1 < \sigma < N, t: \text{real}\},$$

where the O -constant depends only on σ and N .

Remark. It is reasonable that such a bound as in (1.10) follows, since

$$\frac{(-1)^n}{n!} k^{\sigma+it-n} \frac{\Gamma(\sigma-it+n)}{\Gamma(\sigma-it)} \zeta(\sigma+it-n) \zeta(\sigma-it+n) \ll k^{\sigma-n}(|t| + 1)^{2n+\frac{1}{2}-\sigma} \quad (1.11)$$

for $-n + 1 < \sigma < n$ ($n \geq 1$), see (1.5). Note that (1.11) is the best-possible, since $\zeta(\sigma + it) = \Omega(1)$ for $\sigma > 1$ as $t \rightarrow \pm\infty$.

The main aim of this paper is to provide alternative simple proofs of Theorems 1 and 2. It should be remarked that the introduction of a Mellin transform (2.3) below is a key to the considerable simplification. In Sections 2 and 3, we shall prove Theorems 1 and 2, respectively. In the final section, the inner connections between different expressions for $R_N(u, v; k)$ (see (1.8), (1.9), (2.9) and (4.1)) will be examined.

Theorem 1 ([KM1, Theorem 1], [Ka1, Theorem 3], [Ka2, Theorem 3 with $h = 0$])
Let

$$E = \{1, 2, 3, \dots\} \cup \{\frac{n}{2} + it; n: \text{integer} \leq 2, t: \text{real}\}.$$

Then for any integer $N \geq 1$, in the region

$$\{\sigma + it; -N + 1 < \sigma < N + 1, t: \text{real}\} \quad (1.3)$$

except the points of E , the formula

$$\begin{aligned} & \varphi(q)^{-1} \sum_{\chi(\bmod q)} |L(\sigma + it, \chi)|^2 \\ &= \zeta(2\sigma) \prod_{p|q} (1 - p^{-2\sigma}) + 2q^{-2\sigma} \varphi(q) \Gamma(2\sigma - 1) \zeta(2\sigma - 1) \operatorname{Re} \left\{ \frac{\Gamma(1 - \sigma - it)}{\Gamma(\sigma - it)} \right\} \\ &+ 2q^{-2\sigma} \sum_{k|q} \mu\left(\frac{q}{k}\right) T(\sigma + it; k) \end{aligned} \quad (1.4)$$

holds, where p runs through all prime divisors of q and $T(\sigma + it; k)$ has the asymptotic expansion

$$\begin{aligned} T(\sigma + it; k) &= \sum_{n=0}^{N-1} \frac{(-1)^n k^{-n}}{n!} \operatorname{Re} \left\{ k^{\sigma+it} \frac{\Gamma(\sigma - it + n)}{\Gamma(\sigma - it)} \zeta(\sigma + it - n) \zeta(\sigma - it + n) \right\} \\ &+ e_N(\sigma + it; k). \end{aligned} \quad (1.5)$$

Here $e_N(\sigma + it; k)$ is the error term satisfying

$$e_N(\sigma + it; k) = O(k^{\sigma-N}) \quad (1.6)$$

in the region (1.3), with the O -constant depends only on σ , N and t . In particular, if $q = p$ is a prime, the asymptotic series

$$\begin{aligned} & (p-1)^{-1} \sum_{\chi(\bmod p)} |L(\sigma + it, \chi)|^2 \\ &= \zeta(2\sigma) + 2p^{1-2\sigma} \Gamma(2\sigma - 1) \zeta(2\sigma - 1) \operatorname{Re} \left\{ \frac{\Gamma(1 - \sigma - it)}{\Gamma(\sigma - it)} \right\} \\ &- p^{-2\sigma} |\zeta(\sigma + it)|^2 + 2p^{-2\sigma} T(\sigma + it; p) \end{aligned} \quad (1.7)$$

holds.

Remark 1. Asymptotic formulae as in (1.4) for the exceptional points $s \in E$ can be deduced as limiting cases of Theorem 1. Important cases $\operatorname{Re} s = \frac{1}{2}$ and $s = 1$ are treated in [KM1, Theorem 1] and [KM2, Theorems 1 and 4], respectively.

Remark 2. In this paper, the region (1.3) in which (1.4) remains valid will be slightly improved upon our earlier results [KM1, Theorem 1], [Ka1, Theorem 3] and [Ka2, Theorem 3].

Remark 3. Similar asymptotic results for (1.1) have been independently obtained by W. Zhang [Zh2]–[Zh7] and V. V. Rane [Ra]. Their proofs are based on the use of the Hurwitz zeta-function $\zeta(s, \alpha)$ defined by

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s} \quad (\operatorname{Re} s > 1, \alpha > 0),$$

2 Proof of Theorem 1

Let

$$Q(u, v; q) = \varphi(q)^{-1} \sum_{\chi(\bmod q)} L(u, \chi) L(v, \bar{\chi}).$$

We suppose first that $\operatorname{Re} u > 1$ and $\operatorname{Re} v > 1$. Then by the orthogonality and the periodicity of characters

$$Q(u, v; q) = \sum_{\substack{h, k=1 \\ h \equiv k(\bmod q) \\ (h, q) = (k, q) = 1}}^{\infty} h^{-u} k^{-v} = \sum_{\substack{a=1 \\ (a, q)=1}}^q \sum_{m, n=0}^{\infty} (qm + a)^{-u} (qn + a)^{-v}.$$

Classifying the last inner double sum according to the conditions $m = n$, $m < n$ and $m > n$, we get

$$Q(u, v; q) = L(u + v, \chi_0) + f(u, v; q) + f(v, u; q), \quad (2.1)$$

where χ_0 is the principal character mod q and

$$f(u, v; q) = \sum_{\substack{a=1 \\ (a, q)=1}}^q \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (qm + a)^{-u} (q(m + n) + a)^{-v}. \quad (2.2)$$

Atkinson succeeded in obtaining the analytic continuation of $f(u, v; 1)$ (namely in the case of $\zeta(s)$), which led him to the eventual application on taking $u = \frac{1}{2} + it$ and $v = \frac{1}{2} - it$. Several ways are known to prove the analytic continuation of $f(u, v; q)$. T. Meurman [Me] generalizes Atkinson's original proof to treat $f(u, v; q)$ by Poisson's summation formula, while Motohashi [Mo1] makes use of certain loop-integral expressions for $f(u, v; q)$. In this paper we apply

$$(qm + a)^{-u} (q(m + n) + a)^{-v} = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(-s)\Gamma(v + s)}{\Gamma(v)} (qm + a)^{-s-u-v} (qn)^s ds, \quad (2.3)$$

where c is a constant fixed with $-\operatorname{Re} v < c < -1$ and (c) denotes the vertical straight line from $c - i\infty$ to $c + i\infty$. This can be obtained by taking $-z = qn/(qm + a)$ in

$$\Gamma(\alpha)(1 - z)^{-\alpha} = \frac{1}{2\pi i} \int_{(b)} \Gamma(\alpha + s)\Gamma(-s)(-z)^s ds \quad (|\arg(-z)| < \pi, -\operatorname{Re} \alpha < b < 0),$$

which is a special case of Mellin-Barnes' integral expression for Gauss' hypergeometric function $F(\alpha, \beta; \gamma; z)$ (cf. [WW, p.289, 14.51 Corollary]). Integrals of the type (2.3) were firstly introduced by Motohashi [Mo2] to investigate the fourth power mean of $\zeta(s)$. Recently, A. Ivić [Iv2, Chapter 2] applied Motohashi's argument to treat the mean square of $\zeta(s)$.

We assume for brevity that all the singularities appearing in the following argument are at most simple poles, since other cases can be treated by taking limits (see Remark

1 of Theorem 1). Substituting (2.3) into each term in the right-hand side of (2.2), we obtain

$$f(u, v; q) = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(-s)\Gamma(v+s)}{\Gamma(v)} q^{-u-v} \sum_{\substack{a=1 \\ (a,q)=1}}^q \zeta(u+v+s, \frac{a}{q}) \zeta(-s) ds, \quad (2.4)$$

where the interchange of the order of summation and integration can be justified, since, by virtue of the choice of δ , the variables $u+v+s$ and $-s$ are both in the region of absolute convergence. As we shall see in the following, the formula (2.4) will provide the analytic continuation of $f(u, v; q)$ by deforming suitably the path of integration. Note that (c) separates the poles at $s = -1 + n$ ($n = 0, 1, 2, \dots$) from the poles at $s = 1 - u - v, -v - n$ ($n = 0, 1, 2, \dots$) of the integrand. If we replace (c) by the contour $\mathcal{C}$ which is suitably indented in such a manner as to separate the poles at $s = 1 - u - v, -1 + n$ ($n = 0, 1, 2, \dots$) from the poles at $s = -v - n$ ($n = 0, 1, 2, \dots$), then we get by the theorem of residues

$$f(u, v; q) = \frac{\Gamma(u+v-1)\Gamma(1-u)}{\Gamma(v)} \zeta(u+v-1) q^{1-u-v} \prod_{p|q} (1 - p^{-1}) + g(u, v; q), \quad (2.5)$$

where

$$\begin{aligned} g(u, v; q) &= q^{-u-v} \sum_{k|q} \mu\left(\frac{q}{k}\right) \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(-s)\Gamma(v+s)}{\Gamma(v)} \zeta(u+v+s) \zeta(-s) k^{u+v+s} ds \\ &= q^{-u-v} \sum_{k|q} \mu\left(\frac{q}{k}\right) S(u, v; k), \end{aligned} \quad (2.6)$$

say. Here we applied the identities

$$\sum_{\substack{a=1 \\ (a,q)=1}}^q \zeta(w, \frac{a}{q}) = \zeta(w) \sum_{k|q} \mu\left(\frac{q}{k}\right) k^w = \zeta(w) q^w \prod_{p|q} (1 - p^{-w}).$$

Hence from (2.1), (2.5) and (2.6),

$$\begin{aligned} Q(u, v; q) &= \zeta(u+v) \prod_{p|q} (1 - p^{-u-v}) + q^{-u-v} \varphi(q) \zeta(u+v-1) \Gamma(u+v-1) \times \\ &\quad \times \left\{ \frac{\Gamma(1-u)}{\Gamma(v)} + \frac{\Gamma(1-v)}{\Gamma(u)} \right\} + q^{-u-v} \sum_{k|q} \mu\left(\frac{q}{k}\right) \{S(u, v; k) + S(v, u; k)\} \end{aligned} \quad (2.7)$$

holds in the region $\operatorname{Re} u > 1$ and $\operatorname{Re} v > 1$, where $S(v, u; k)$ is expressed in the same manner as $S(u, v; k)$.

Next we shift the path of integration to the left. We suppose at this stage that $\operatorname{Re} u < 1$ and $\operatorname{Re} v > 1$, where $\mathcal{C}$ can be taken as a straight line (c_0) with $-\operatorname{Re} v < c_0 < \min(-1, 1 - \operatorname{Re}(u+v))$. Let N be a positive integer and c_N a constant fixed with $-\operatorname{Re} v - N < c_N < -\operatorname{Re} v - N + 1$. Since the order of the integrand in (2.6) is $O(|\operatorname{Im} s|^C e^{-\pi|\operatorname{Im} s|})$ as $\operatorname{Im} s \rightarrow \pm\infty$ (C is a positive constant depending only on $\operatorname{Re} s$,

$\operatorname{Re} u$ and $\operatorname{Re} v$), we can shift the path from (c_0) to (c_N) . Collecting the residues at the poles $s = -v - n$ ($n = 0, 1, \dots, N-1$), we obtain

$$S(u, v; k) = \sum_{n=0}^{N-1} \frac{(-1)^n}{n!} \frac{\Gamma(v+n)}{\Gamma(v)} \zeta(u-n) \zeta(v+n) k^{u-n} + r_N(u, v; k), \quad (2.8)$$

where

$$r_N(u, v; k) = \frac{1}{2\pi i} \int_{(c_N)} \frac{\Gamma(-s)\Gamma(v+s)}{\Gamma(v)} \zeta(u+v+s) \zeta(-s) k^{u+v+s} ds. \quad (2.9)$$

Here the condition on u and v can be relaxed as

$$\operatorname{Re} u < N+1 \quad \text{and} \quad \operatorname{Re} v > -N+1. \quad (2.10)$$

Under (2.10) we can choose c_N satisfying the condition

$$-\operatorname{Re} v - N < c_N < \min(-1, -\operatorname{Re} v - N + 1, 1 - \operatorname{Re}(u+v)),$$

by which (c_N) separates the poles at $s = -v - n$ ($n = N, N+1, N+2, \dots$) from the poles at $s = 1 - u - v, -1 + n$ ($n = 0, 1, 2, \dots$), $-v - n$ ($n = 0, 1, \dots, N-1$).

Now we proceed to prove Theorem 1. Taking $u = \sigma + it$ and $v = \sigma - it$ in (2.7), (2.8) and (2.9), we obtain (1.4) and (1.5), by noticing (2.10) and putting

$$T(\sigma + it; k) = \operatorname{Re}\{S(\sigma + it, \sigma - it; k)\} \quad \text{and} \quad e_N(\sigma + it; k) = \operatorname{Re}\{r_N(\sigma + it, \sigma - it; k)\}.$$

The error estimate (1.6) follows from

$$\begin{aligned} r_N(u, v; k) &= \frac{(-1)^N}{N!} \frac{\Gamma(v+N)}{\Gamma(v)} \zeta(u-N) \zeta(v+N) k^{u-N} \\ &\quad + \frac{1}{2\pi i} \int_{(c_{N+1})} \frac{\Gamma(-s)\Gamma(v+s)}{\Gamma(v)} \zeta(u+v+s) \zeta(-s) k^{u+v+s} ds \\ &\ll k^{\operatorname{Re} u - N} + k^{\operatorname{Re}(u+v) + c_{N+1}} \ll k^{\operatorname{Re} u - N}, \end{aligned}$$

by $-\operatorname{Re} v - N - 1 < c_{N+1} < -\operatorname{Re} v - N$. Furthermore (1.7) can be deduced from (2.7) by noting

$$\begin{aligned} S(u, v; 1) + S(v, u; 1) &= \zeta(u)\zeta(v) - \zeta(u+v) \\ &\quad - \zeta(u+v-1)\Gamma(u+v-1) \left\{ \frac{\Gamma(1-u)}{\Gamma(v)} + \frac{\Gamma(1-v)}{\Gamma(u)} \right\}, \end{aligned}$$

which is the special case $q = 1$ of (2.7). The proof of Theorem 1 is now complete.

3 Proof of Theorem 2

Throughout this section, let $-N+1 < \sigma < N$ and δ a constant fixed with $0 < \delta < \frac{1}{2} \min(N - \sigma, N - 1 + \sigma, 1)$. We write $s = -\sigma - N + \xi + i\tau$ in (2.9). For the proof of Theorem 2, we need

Lemma For any real τ, t and ξ with $|\xi| \leq \delta$, we have

$$\Gamma(\sigma + N - \xi - i\tau) \ll (|\tau| + 1)^{\sigma + N - \xi - \frac{1}{2}} e^{-\frac{\pi}{2}|\tau|}, \quad (3.1)$$

$$\Gamma(-N + \xi + i(\tau - t)) \ll \begin{cases} |\tau - t|^{-N + \xi - \frac{1}{2}} e^{-\frac{\pi}{2}|\tau - t|} & \text{for } |\tau - t| \geq 1, \\ |\xi + i(\tau - t)|^{-1} & \text{for } |\tau - t| \leq 1, \end{cases} \quad (3.2)$$

$$\Gamma(\sigma - it)^{-1} \ll (|t| + 1)^{\frac{1}{2} - \sigma} e^{\frac{\pi}{2}|t|}, \quad (3.3)$$

$$\zeta(\sigma - N + \xi + i\tau) \ll (|\tau| + 1)^{\frac{1}{2} - \sigma + N - \xi}, \quad (3.4)$$

$$\zeta(\sigma + N - \xi - i\tau) \ll 1. \quad (3.5)$$

Here and in what follows the implied constants depend at most on σ and N .

Proof. (3.1)–(3.3) follow from Stirling's formula (cf. [Iv1, p.492, (A.34)]) and the trivial bounds for $\Gamma(w)$ near the real axis. By virtue of the choice of δ , (3.5) is an immediate consequence of the inequality $\zeta(w) \ll 1$ for $\operatorname{Re} w > 1$, while (3.4) can be proved by applying the functional equation of $\zeta(w)$. $\square$

For the proof of Theorem 2 we may restrict ourselves to the case $t \geq 2$, since the case $t \leq -2$ follows from this case by the reflection principle, and the case $|t| \leq 2$ is a simple consequence of Theorem 1.

Let $\sigma_N = \sigma + N$ and L the infinite broken line joining the points $-\sigma_N - i\infty$, $-\sigma_N + i(t - \delta)$, $-\sigma_N + \delta + i(t - \delta)$, $-\sigma_N + \delta + i(t + \delta)$, $-\sigma_N + i(t + \delta)$ and $-\sigma_N + i\infty$. Taking $u = \sigma - it$ and $v = \sigma + it$ in (2.9), and then replacing the path (c_N) by L , we have

$$r_N(\sigma + it, \sigma - it; k) = \frac{1}{2\pi i} \int_L \frac{\Gamma(-s)\Gamma(\sigma - it + s)}{\Gamma(\sigma - it)} \zeta(2\sigma + s) \zeta(-s) k^{2\sigma + s} ds. \quad (3.6)$$

We shall estimate the right-hand integral in (3.6) by dividing

$$r_N(\sigma + it, \sigma - it; k) = \frac{1}{2\pi i} \left\{ \sum_{\substack{\mu=1 \\ \mu \neq 5}}^7 I_\mu + \sum_{\nu=1}^3 I_{5,\nu} \right\},$$

where

$$\begin{aligned} I_1 &= \int_{-\sigma_N - i\infty}^{-\sigma_N - i}, & I_2 &= \int_{-\sigma_N - i}^{-\sigma_N + i}, & I_3 &= \int_{-\sigma_N + i}^{-\sigma_N + i(t-1)}, & I_4 &= \int_{-\sigma_N + i(t-1)}^{-\sigma_N + i(t-\delta)}, \\ I_{5,1} &= \int_{-\sigma_N + i(t-\delta)}^{-\sigma_N + \delta + i(t-\delta)}, & I_{5,2} &= \int_{-\sigma_N + \delta + i(t-\delta)}^{-\sigma_N + \delta + i(t+\delta)}, & I_{5,3} &= \int_{-\sigma_N + \delta + i(t+\delta)}^{-\sigma_N + i(t+\delta)}, \\ I_6 &= \int_{-\sigma_N + i(t+\delta)}^{-\sigma_N + i(t+1)}, & I_7 &= \int_{-\sigma_N + i(t+1)}^{-\sigma_N + i\infty}. \end{aligned}$$

The treatment of $I_{5,\nu}$ ($\nu = 1, 2, 3$) is more delicate than that of other I_μ 's. By Lemma and the assumption $t \geq 2$, we get

$$I_1 \ll k^{\sigma - N} t^{\frac{1}{2} - \sigma} \int_{-\infty}^{-1} (-\tau)^{2N} (t - \tau)^{-N - \frac{1}{2}} e^{\pi\tau} d\tau \ll k^{\sigma - N} t^{-\sigma - N}, \quad (3.7)$$

$$I_2 \ll k^{\sigma - N} t^{\frac{1}{2} - \sigma} \int_{-1}^1 (t - \tau)^{-N - \frac{1}{2}} d\tau \ll k^{\sigma - N} t^{-\sigma - N}. \quad (3.8)$$

Moreover

$$I_3 \ll k^{\sigma-N} t^{\frac{1}{2}-\sigma} \int_1^{t-1} \tau^{2N} (t-\tau)^{-N-\frac{1}{2}} d\tau \ll k^{\sigma-N} t^{2N+\frac{1}{2}-\sigma}, \quad (3.9)$$

$$I_4 \ll k^{\sigma-N} t^{\frac{1}{2}-\sigma} \int_{t-1}^{t-\delta} \tau^{2N} (t-\tau)^{-1} d\tau \ll k^{\sigma-N} t^{2N+\frac{1}{2}-\sigma} \log \delta^{-1}, \quad (3.10)$$

where the last upper bounds in (3.9) and (3.10) are obtained by integrating by parts. Similarly to I_3 and I_4 ,

$$I_6 \ll k^{\sigma-N} t^{\frac{1}{2}-\sigma} \int_{t+\delta}^{t+1} \tau^{2N} (t-\tau)^{-1} d\tau \ll k^{\sigma-N} t^{2N+\frac{1}{2}-\sigma} \log \delta^{-1}, \quad (3.11)$$

$$I_7 \ll k^{\sigma-N} t^{\frac{1}{2}-\sigma} \int_{t+1}^{\infty} \tau^{2N} (\tau-t)^{-N-\frac{1}{2}} e^{-\pi(\tau-t)} d\tau \ll k^{\sigma-N} t^{2N+\frac{1}{2}-\sigma}. \quad (3.12)$$

For $I_{5,\nu}$ ($\nu = 1, 2, 3$), we proceed as follows. By $0 < \delta < \frac{1}{2}$,

$$\begin{aligned} I_{5,1} &\ll k^{\sigma-N} t^{\frac{1}{2}-\sigma} \int_0^\delta (t-\delta)^{2N-2\xi} e^{\frac{\pi}{2}\delta} |\xi - i\delta|^{-1} k^\xi d\xi \\ &\ll k^{\sigma-N} t^{2N+\frac{1}{2}-\sigma} \delta^{-1} \int_0^\delta (t^{-2}k)^\xi d\xi \leq k^{\sigma-N} t^{2N+\frac{1}{2}-\sigma} \max(1, (kt^{-2})^\delta), \end{aligned} \quad (3.13)$$

$$\begin{aligned} I_{5,2} &\ll k^{\sigma-N+\delta} t^{\frac{1}{2}-\sigma} \int_{t-\delta}^{t+\delta} \tau^{2N-2\delta} e^{\frac{\pi}{2}(t-\tau)} |\delta + i(t-\tau)|^{-1} d\tau \\ &\ll k^{\sigma-N+\delta} t^{\frac{1}{2}-\sigma} \delta^{-1} \int_{t-\delta}^{t+\delta} \tau^{2N-2\delta} d\tau \ll k^{\sigma-N} t^{2N+\frac{1}{2}-\sigma} (kt^{-2})^\delta, \end{aligned} \quad (3.14)$$

$$I_{5,3} \ll k^{\sigma-N} t^{2N+\frac{1}{2}-\sigma} \max(1, (kt^{-2})^\delta), \quad (3.15)$$

where the treatment of $I_{5,3}$ is similar to (3.13). Combining (3.7)–(3.15), we obtain

$$r_N(\sigma + it, \sigma - it; k) \ll k^{\sigma-N} t^{2N+\frac{1}{2}-\sigma} \{\log \delta^{-1} + \max(1, (kt^{-2})^\delta)\}. \quad (3.16)$$

Next let L' be the infinite broken line joining the points $-\sigma_N - i\infty$, $-\sigma_N + i(t - \delta)$, $-\sigma_N - \delta + i(t - \delta)$, $-\sigma_N - \delta + i(t + \delta)$, $-\sigma_N + i(t + \delta)$ and $-\sigma_N + i\infty$. Then

$$\begin{aligned} &r_N(\sigma + it, \sigma - it; k) \\ &= \frac{(-1)^N \Gamma(\sigma - it + N)}{N! \Gamma(\sigma - it)} \zeta(\sigma + it - N) \zeta(\sigma - it + N) k^{\sigma+it-N} \\ &\quad + \frac{1}{2\pi i} \int_{L'} \frac{\Gamma(-s) \Gamma(\sigma - it + s)}{\Gamma(\sigma - it)} \zeta(2\sigma + s) \zeta(-s) k^{2\sigma+s} ds. \end{aligned} \quad (3.17)$$

The first term in the right-hand side of (3.17) is bounded as $\ll k^{\sigma-N} t^{2N+\frac{1}{2}-\sigma}$ by (1.11). To estimate the second term, we divide

$$\int_{L'} = \sum_{\substack{\mu=1 \\ \mu \neq 5}}^7 I_\mu + \sum_{\nu=1}^3 I'_{5,\nu},$$

where

$$I'_{5,1} = \int_{-\sigma_N+i(t-\delta)}^{-\sigma_N-\delta+i(t-\delta)}, \quad I'_{5,2} = \int_{-\sigma_N-\delta+i(t-\delta)}^{-\sigma_N-\delta+i(t+\delta)}, \quad I'_{5,3} = \int_{-\sigma_N-\delta+i(t+\delta)}^{-\sigma_N+i(t+\delta)}.$$

Similarly to the previous case,

$$\begin{aligned} I'_{5,\nu} &\ll k^{\sigma-N} t^{2N+\frac{1}{2}-\sigma} \max(1, (k^{-1}t^2)^\delta) \quad (\nu = 1, 3), \\ I'_{5,2} &\ll k^{\sigma-N} t^{2N+\frac{1}{2}-\sigma} (k^{-1}t^2)^\delta. \end{aligned}$$

Therefore

$$r_N(\sigma + it, \sigma - it; k) \ll k^{\sigma-N} t^{2N+\frac{1}{2}-\sigma} \{\log \delta^{-1} + \max(1, (k^{-1}t^2)^\delta)\}. \quad (3.18)$$

Theorem 2 now follows from (3.16) if $t \geq k^{\frac{1}{2}}$, and from (3.18) if $t \leq k^{\frac{1}{2}}$, respectively.

4 Additional remarks

The first purpose of this section is to show how we can deduce (2.9) directly from (1.8) or (1.9). To do this we introduce a confluent hypergeometric function $\Psi(\alpha, \gamma; z)$ defined by

$$\Psi(\alpha, \gamma; z) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty e^{i\phi}} e^{-zw} w^{\alpha-1} (1+w)^{\gamma-\alpha-1} dw$$

for $\operatorname{Re} \alpha > 0$, $|\phi| < \pi$ and $|\phi + \arg z| < \frac{\pi}{2}$ (cf. [Er, p.256, 6.5(3)]). Rotating suitably the path of integrations for $J_\pm(\tau, l; k)$ and $\tilde{J}_\pm(\tau, l; k)$, we find

$$J_\pm(\tau, l; k) = (k\tau^{-1})^{v+N} \Gamma(v+N) \Psi(v+N, u+v; 2\pi kl\tau^{-1} e^{\mp \frac{\pi i}{2}})$$

and

$$\tilde{J}_\pm(\tau, l; k) = (k\tau^{-1})^{N+1-u} \Gamma(N+1-u) \Psi(N+1-u, 2-u-v; 2\pi kl\tau^{-1} e^{\mp \frac{\pi i}{2}}).$$

Furthermore, $J_\pm(\tau, l; k)$ and $\tilde{J}_\pm(\tau, l; k)$ can be expressed in terms of Mellin-Barnes' type integrals by using

$$\Psi(\alpha, \gamma; z) = \frac{1}{2\pi i} \int_{(b)} \frac{\Gamma(\alpha+s)\Gamma(-s)\Gamma(1-\gamma-s)}{\Gamma(\alpha)\Gamma(\alpha-\gamma+1)} z^s ds,$$

where $-\operatorname{Re} \alpha < b < \min(0, 1 - \operatorname{Re} \gamma)$ and $|\arg z| < \frac{3\pi}{2}$ (cf. [Er, p.256, 6.5(5)]). Substituting these integrals into each term in the right-hand infinite series in (1.8) and (1.9), respectively, and then applying the functional equation of $\zeta(w)$, we can see that either (1.8) or (1.9) directly yields (2.9), by noting

$$r_N(u, v; k) = k^{u-N} R_N(u, v; k). \quad (4.1)$$

On the other hand, (1.8) and (1.9) are connected by the transformation formula

$$\Psi(\alpha, \gamma; z) = z^{1-\gamma} \Psi(\alpha - \gamma + 1, 2 - \gamma; z)$$

(cf. [Er, p.257, 6.5(6)]), for details see [Ka1, Section 3]. In view of the consideration given above, this connection is embodied in (2.9) with the functional equation of $\zeta(w)$.

In this occasion we point out an error in the preceding article [KM3]. In Section 2, we should mention that the same result as R. Sitaramachandrarao [Si] with a slightly different log-factor was independently obtained by Zhang [Zh1, Corollary], whose main theorem is proved in a more general setting.

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