

STABLE PAIRS AND THEIR SPECTRAL CURVES

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Throughout this talk X will denote a compact Riemann surface of genus $g \geq 2$, and L a fixed holomorphic line bundle on X . Details of proofs may be found in [4,5].

(1) Definition: A stable pair on X with respect to L (or stable L -pair for short) is a pair (V, Φ) where V is a holomorphic vector bundle on X , and $\Phi: V \rightarrow V \otimes L$ is a vector bundle homomorphism such that one has $\mu(W) < \mu(V)$ (where $\mu = \frac{\text{rank}}{\text{degree}}$) for all Φ -invariant subbundles $W \subset V$. $\square$

(We say that $W \subset V$ is Φ -invariant if $\Phi(W) \subset L \otimes W$.)

Of course, stable pairs always exist by taking $\Phi \equiv 0$ and V to be a stable bundle. Thus the definition generalises the usual notion of stable bundle; moreover one can define isomorphism, families and the moduli problem in the obvious way. One then has the following result due to Nitin Nitsure.

(2) Theorem [3]: There exists a coarse moduli scheme $\mathcal{M}'$ for stable L -pairs of fixed rank n and degree d (ie. rank and degree of the

bundle). $\mathcal{K}'$ is a separated Noetherian scheme of finite type over $\mathbb{C}$. and contains a connected component $\mathcal{K} \subset \mathcal{K}'$ such that $(V, \Phi) \in \mathcal{K}$ whenever V is a semistable bundle. Moreover, $\mathcal{K}$ is a quasi-projective variety, and is smooth and irreducible if $\deg L \geq 2g-2$. $\square$

In the case $\deg L \geq 2g-2$ we shall give the dimension of $\mathcal{K}$ below (remark (11)).

One may conjecture that $\mathcal{K}$ is the only connected component, though this is not known in general. We remark in addition that one has an identical theorem for *tracefree* stable pairs (ie. requiring $\text{tr} \Phi \in H^0(X, L)$ to vanish) and we shall denote the resulting connected variety by $\mathcal{K}_0$.

We now aim to use these spaces to study the moduli space $N = N(n, d)$ of semistable bundles on X (up to S -equivalence) of rank n and degree d . Recall that N is a normal projective variety, smooth on the open set N^S of stable bundles, with dimension $n^2(g-1) + 1$. We shall consider a diagram of the following form:

$$(3) \quad \begin{array}{ccc} \mathcal{K} & \xrightarrow{h} & W = \bigoplus_{i=1}^n H^0(X, L^i) \\ \downarrow \text{fibre } H^0(L \otimes \text{End} V) & & \\ \downarrow \text{over stable } V & & \\ N(n, d) & & \end{array}$$

The vertical map, if well-defined, will be a rational map which forgets the homomorphism Φ when V is semistable. Clearly this will have fibre $H^0(X, L \otimes \text{End} V)$ when $V \in N^S$.

The horizontal map h is defined by

$$h(V, \Phi) = (\text{tr } \Phi, \text{tr } \Lambda^2 \Phi, \dots, \Lambda^n \Phi);$$

ie. the coefficients of the characteristic polynomial of Φ . More precisely, let $Z \xrightarrow{p} X$ denote the total space of the line bundle L . Then there is a tautological section $\lambda \in H^0(Z, p^*L)$, and the characteristic polynomial is

$$\det |p^*\Phi - \lambda \otimes \text{id}| \equiv \lambda^n - \lambda^{n-1} p^*a_1 + \dots + (-)^n p^*a_n$$

where $a_i = \text{tr } \Lambda^i \Phi \in H^0(X, L^i)$.

This polynomial is to be viewed as an element of $H^0(Z, p^*L^n)$, and as such has as zero-locus some curve $\tilde{X} \subset Z$, whose points represent the eigenvalues of Φ .

The above construction is due to Nigel Hitchin [1], who was interested in the case $L = K$, the canonical line bundle. We shall call the curve $\tilde{X}$ the *spectral curve* of the pair (V, Φ) ; we note that $\tilde{X} \in |\mathcal{O}_{\tilde{L}}(n)|$ where $\tilde{L}$ denotes the ruled surface $P(\tilde{L}^1 \otimes \mathcal{O}_X)$ which compactifies Z . In particular, $H^0(\tilde{L}, \mathcal{O}_{\tilde{L}}(n)) \cong H^0(X, S^n(L \otimes \mathcal{O}_X)) \cong \mathbb{C} \otimes W$ ie. W can be viewed as an affine subspace of the projective space $|\mathcal{O}_{\tilde{L}}(n)|$. Thus we shall identify the image $h(V, \Phi)$ in (3) with the spectral curve $\tilde{X}$.

We next ask for the image $h(\mathcal{M}) \subset W$.

(4) Definition: $\tilde{X} = \{\lambda^n - a_1 \lambda^{n-1} + \dots + (-)^n a_n = 0\} \in W$ will be called *regular* if it is smooth and irreducible and has only simple branching over X , in the sense that the discriminant $\Delta(\lambda^n - a_1 \lambda^{n-1} + \dots) \in H^0(X, L^{n(n-1)})$ has only simple zeros. We write $W^{\text{reg}} \subset W$ for the subset of regular $\tilde{X}$. $\square$

(5) Proposition: Suppose (i) $|L^n|$ is a base-point-free linear series on X ; and (if $n > 2$) (ii) $H^0(X, L^{n-1}) \neq 0$. Then W^{reg} is a non-empty Zariski-open subset of W . Moreover, a regular $\tilde{X}$ has genus $\tilde{g} = \frac{1}{2}n(n-1)\deg L + n(g-1) + 1$. $\square$

Next, for each stable pair (V, Φ) , the corresponding spectral curve $\tilde{X}$ comes equipped with a natural sheaf, called the *spectral sheaf*,

$$F = \ker(p^*\Phi - \lambda \text{id}) \subset p^*V;$$

the sheaf of 'eigenvectors' of Φ . F is a torsion-free sheaf of rank 1 on $\tilde{X}$ and is a line bundle when $\tilde{X}$ is smooth.

One expects to be able to reconstruct (V, Φ) locally from its eigenvalues and eigenvectors, where these are linearly independent, and analytically continue over all of X . The result is the following.

(6) Proposition: If $\tilde{X}$ is regular then $V = (p_*F^{-1})^V$, and Φ is adjoint to the homomorphism $V^V \rightarrow L \otimes V^V$ obtained by pushing down the tautological homomorphism $F^{-1} \xrightarrow{\otimes \lambda} (p^*L) \otimes F^{-1}$. $\square$

(7) Remark: The idea of the proof is to regard the line bundle F as giving a section of the projectivised bundle $P(p^*V) \rightarrow \tilde{X}$. This section has an image $\mathcal{I} \subset P(V)$ via the diagram

$$\begin{array}{ccc} P(p^*V) & \rightarrow & P(V) \\ F \uparrow \downarrow & & \downarrow q \\ \tilde{X} & \xrightarrow{p} & X \end{array}$$

$\mathcal{I}$ has a natural morphism $\pi: \mathcal{I} \rightarrow \tilde{X}$ such that π^*F is isomorphic to the restriction of the tautological line bundle $\mathcal{O}_{P(V)}(-1)$ to $\mathcal{I}$, and one checks that $p^*F^{-1} \cong q_*\pi^*F^{-1} \cong q_*\mathcal{O}_{P(V)}(1) \cong V^V$.

The statement about Φ then follows easily. $\square$

One thus has a correspondence between L-pairs (V, Φ) and spectral pairs $(\tilde{X}, F)$. The next lemma shows that this inverse construction even yields, for general F , a pair (V, Φ) for which V is a stable bundle.

(8) Lemma: Suppose $\tilde{X}$ is regular, and for each $i = 1, \dots, n-1$ let

$$S_i = \{F \in \text{Pic } \tilde{X} \mid v := (p_* F^{-1})^V \text{ has a destabilising} \\ \text{subbundle } W \subset V \text{ of rank } i\}$$

('destabilising' means $\mu(W) \geq \mu(V)$.) Then S_i is a closed subvariety of $\text{Pic } \tilde{X}$ with codimension $\geq (2 \cdot \binom{n}{i} - n - 1)(g - 1)$.

In particular V is stable away from a subvariety of codimension $\geq (n-1)(g-1)$. $\square$

(9) Theorem: If $\tilde{X}$ is regular then the fibre $h^{-1}(\tilde{X})$ of the horizontal map in the diagram (3) is isomorphic to $\text{Jac}(\tilde{X})$, and contains a closed subvariety S of codimension $\geq (n-1)(g-1)$ whose complement consists of points corresponding to stable pairs (V, Φ) for which V is a stable bundle.

The idea of the proof is of course to construct a family of L-pairs (V, Φ) on X , parametrised by $\text{Jac}(\tilde{X})$, by the prescription of proposition (6). (Note that the degree of F is fixed by the requirement that $\deg V = d$.) These pairs automatically have spectral curve $\tilde{X}$, and in particular must be stable. (This is because the curve $\mathfrak{A} \subset P(V)$ of remark (7) is irreducible and generically spans $P(V)$, so that V cannot have any Φ -invariant subbundles.) Thus we get a morphism $\text{Jac}(\tilde{X}) \rightarrow \mathcal{M}'$, and by lemma (8) the image lies in

the component $\mathcal{K}$, hence coincides with the fibre $h^{-1}(\tilde{X})$. $\square$

As a corollary one sees that $h : \mathcal{K} \rightarrow W$ is surjective on W^{reg} . We shall consider the closure $\bar{\mathcal{K}}^{\text{reg}} \subset \mathcal{K}$ of the open subvariety $\mathcal{K}^{\text{reg}} := h^{-1}(W^{\text{reg}})$, which is an irreducible component of $\mathcal{K}$. If L is sufficiently positive in the sense of proposition (5) then the diagram (3) can be replaced by:-

$$(10) \quad \begin{array}{ccc} \begin{array}{c} \text{irred} \\ \text{quasi} \\ \text{proj} \end{array} \bar{\mathcal{K}}^{\text{reg}} & \xrightarrow[\text{morphism}]{h \text{ dominant}} & W \subset |\mathcal{O}_L(n)| \\ & \downarrow \begin{array}{c} \text{fibre } H^0(L \otimes \text{End } V) \\ \text{over stable } V \end{array} & \\ & N(n, d) & \end{array}$$

The vertical map is now a well-defined rational map, and the dimension of $\bar{\mathcal{K}}^{\text{reg}}$ can be computed:

$$\begin{aligned} \dim \bar{\mathcal{K}}^{\text{reg}} &= \dim W + \tilde{g} \\ &= n^2 \deg L + 1 + \sum_{i=1}^n h^i(X, L^i). \end{aligned}$$

(11) *Remark:* Note that the positivity conditions of proposition (5) are satisfied by all L for which $\deg L \geq \frac{2g}{n}$; and in particular all L for which $\deg L \geq 2g-2$ (if $g \geq 2$, $n \geq 2$). In this case theorem (2) says $\mathcal{K} = \bar{\mathcal{K}}^{\text{reg}}$ and is smooth; thus in this case one has

$$\dim \mathcal{K} = n^2 \deg L + 1 + h^1(X, L). \quad \square$$

We shall give two examples of the above construction.

Example 1: $L = K$

K denotes the canonical line bundle on X . When $n=2$ and d is odd,

this is the case considered by Hitchin in [1]. By remark (11) $\mathcal{M}$ is smooth of dimension $2n^2(g-1) + 2 = 2 \cdot \dim N(n,d)$. Moreover, the vertical fibre in (10) over $V \in N^S$ is $H^0(X, K \otimes \text{End} V) \cong H^1(X, \text{End} V)^* \cong T_V^* N^S$, and in this way one has an inclusion $T^* N^S \subset \mathcal{M}$ as a dense open subset.

In Hitchin's original situation $\mathcal{M}$ is a moduli space for solutions of the $(\text{SU}(2)-)$ self-dual Yang-Mills equations on X , which are identified with stable K -pairs. Hitchin showed that the canonical symplectic structure on $T^* N^S$ extends over $\mathcal{M}$ and that with respect to this symplectic structure the components of the map $h : \mathcal{M} \rightarrow W$ all *Poisson-commute*. But $\dim \mathcal{M} = 2 \cdot \dim W$, so by Liouville's theorem of symplectic geometry the general fibre has to be a complex torus. The spectral curve construction then identifies these tori.

Note also that $\tilde{g} = \dim N(n,d)$. In fact, one can show [4] that the vertical projection on to $N(n,d)$ restricts to a general horizontal fibre $\text{Jac}(\tilde{X})$ as a dominant rational map $F \rightarrow (p_* F^{-1})^V$ of finite degree $1^g \cdot 2^{3g-3} \cdot 3^{5g-5} \dots n^{(2n-1)(g-1)}$.

In the case $n=2$ one can use this map to reprove a classical theorem of Nagata on ruled surfaces, as follows. For any bundle $V \rightarrow X$ of rank 2 we define its *Segre invariant* to be

$$s(V) = \deg V - 2 \cdot \max \deg W ; \text{ max taken over line subbundles } W \subset V ; \\ = \text{ minimum self-intersection of a section of } P(V).$$

(12) Theorem[2,4]: For all V , $s(V) \leq g$.

ie. every ruled surface has a cross-section of self-intersection $\leq g$.

Idea of proof: Any counterexample is necessarily stable, so it suffices to consider the filtration of $N = N(n, d)$:-

$$N_0 \subset N_1 \subset \dots ; \text{ where } N_k = \{V \mid s(V) \leq 2k\}.$$

One knows that the N_k are closed subvarieties, and one wants to show that the filtration terminates at $k = \left\lceil \frac{g}{2} \right\rceil$. One constructs a corresponding filtration on $\text{Jac}(\tilde{X}) = \tilde{J} \rightarrow N$ as follows.

Consider the diagram

$$(13) \quad \begin{array}{ccc} \tilde{J} \times \text{Pic}(X) & \xrightarrow{\mu} & \text{Pic}(\tilde{X}) \\ \downarrow \text{proj on} & & \\ \tilde{J} & & \end{array}$$

1st factor

where $\mu(F, W) = \sigma^* F \otimes p^*(F \cdot W^{-1})$; and $\sigma: \tilde{X} \rightarrow \tilde{X}$ is the sheet-interchange over X .

Claim: $H^0(\tilde{X}, \mu(F, W)) \cong H^0(X, W^{-1} \otimes V_F)$ where $V_F = (p_* F^{-1})^\vee$.

If we assume for simplicity that $d = \deg V = 0$, then noting that $\deg \mu(F, W) = 2g - 2 - 2 \cdot \deg W$, one sees that $V_F \in N_k \iff \exists W$ such that $H^0(\tilde{X}, \mu(F, W)) \neq 0$ and $\deg W \geq -k \iff F \in \pi \circ \mu^{-1}(W_{d(k)})$ where $d(k) = 2g - 2 + 2k$ and $W_{d(k)} \subset \text{Pic}(\tilde{X})$ is the subvariety of special line bundles of degree $\leq d(k)$.

Writing $\tilde{J}_k = \pi \circ \mu^{-1}(W_{d(k)})$ therefore defines a filtration of $\tilde{J}$ which matches up with that on N .

Claim: $\tilde{J}_k \rightarrow N_k$ has finite degree for each $k > 0$, and

$$\dim N_k = \dim \tilde{J}_k = \begin{cases} d(k) + g = 3g - 2 + 2k & \text{if } 2k \leq g - 1 \\ 4g - 3 = \tilde{g} & \text{if } 2k \geq g - 1. \end{cases}$$

ie. the dimension that one would expect; this involves checking certain transversality conditions in the diagram (13). $\square$

(14) *Remark:* One may note that $\tilde{J}_0 \rightarrow N_0$ does not have finite degree. This corresponds to the fact that points of N_0 represent 'positive-

dimensional' S-equivalence classes of non-stable bundles. The fibre of $\tilde{J}_0$ over such a point then consists of line bundles F which 'sweep out' the equivalence class downstairs by the prescription $F \rightarrow V_F$.

Example 2: $L^2 = K$

We shall now take for L some fixed theta-characteristic on X . Note that any such L satisfies the conditions of proposition (5).

Referring again to diagram (10) one has

$$\begin{aligned} \dim \bar{M}^{\text{reg}} &= n^2(g-1) + 1 + h^1(L) + 1 \\ &= \dim N(n,d) + h^0(L) + 1. \end{aligned}$$

But the term $h^0(L)$ comes from the first component of the vector space W , ie. from the *trace*. Taking tracefree stable pairs one finds that all of the above work goes over unchanged, $\bar{M}^{\text{reg}}$ being replaced by $\bar{M}_0^{\text{reg}}$ in (10), where now when $L^2 = K$ one has

$$\dim \bar{M}_0^{\text{reg}} = \dim N(n,d) + 1.$$

The next lemma says that in fact (if n is even) the vertical map in (10) is dominant with 1-dimensional fibres.

(15) *Lemma: Suppose n is even. Then there exists a theta-characteristic L and a Zariski-open subset $\mathcal{U} \subset N^S(n,d)$ such that if $V \in \mathcal{U}$ then $h^0(X, L \otimes \text{End}_0 V) = 1$ and for any $\Phi \in H^0(X, L \otimes \text{End}_0 V)$, the pair (V, Φ) has regular spectral curve. $\square$*

The $\mathbb{C}^*$ -action which factors out these 1-dimensional fibres over $\mathcal{U}$ is actually well-defined on $\bar{M}_0^{\text{reg}}$ by $(V, \Phi) \rightarrow (V, c\Phi)$ and has a quasi-projective quotient which we shall denote by $\mathcal{R} = \bar{M}_0^{\text{reg}}$, which is consequently *birational* to $N(n,d)$. The horizontal map in (10) is

equivariant with respect to the $\mathbb{C}^*$ -action $(a_1, \dots, a_n) \rightarrow (ca_1, c^2a_2, \dots, c^na_n)$ and one has the following result:

(16) Theorem: For each even integer n and integer $d \pmod n$ there exists an irreducible quasi-projective variety $\mathcal{R}$ and a weighted projective space $\mathbb{P}$ of dimension $\frac{1}{2}n(n-1)(g-1)$ such that:

(i) $\mathcal{R}$ is birational to $N(n, d)$;

(ii) $\exists$ dominant morphism $\mathcal{R} \rightarrow \mathbb{P}$ with generic fibre

$\text{Jac}(\tilde{X})/\pm 1$ if $n=2$; $\text{Jac}(\tilde{X})$ if $n>2$;

where $\tilde{X}$ is a (variable) smooth irreducible curve of genus

$\tilde{g} = 1 + \frac{1}{2}n(n+1)(g-1)$. $\square$.

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