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On the Finite Church-Rosser Property of Nonlinear Term Rewriting Systems * (Preliminary report)

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Abstract

This paper proves that *an infinitely nonoverlapping (possibly nonlinear) TRS is finitely Church-Rosser.* The condition *infinitely nonoverlapping* is a nonoverlapping condition under unification with infinite terms. The property *finitely Church-Rosser* is equivalent to *uniquely normalizing with respect to equality* (i.e. $x = y \Longrightarrow x \equiv y$ for any normal forms x, y), and is an intermidiate property between *Church-Rosser* and *uniquely normalizing with respect to reduction.*

1 Introduction

Equational logic has been applied in the program-specification and the other logical frameworks. A Term Rewriting System (TRS), intuitively which is a set of directed equations (deduction rules), have been adopted for an execution model of equational logic. That is, a TRS converts expressions using equations only forward, whereas equational logic permits using them both forward and backward. For these purposes, one of the important properties of a TRS is confluence-related properties, namely, *Church-Rosser property, unique normalizability*, etc. *Church-Rosser property*, which is equivalent to *confluence*, guarantees that the congruent relation (equality) among expressions will be examined without back-track. *Unique normalizability*, which is deduced from confluence, guarantees that the result of an execution is uniquely determined if terminates.

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Several criteria have been proposed for the *confluence* of a TRS and its variations [4,5,6,8,9,10]. Generally speaking, neither *confluence*, *unique-normalizability*, nor *termination* is decidable. Most of known sufficient conditions for confluence of a TRS are restricted to nonoverlapping TRSs [4,5,6].

Intuitively speaking, the nonoverlapping property means that no *reducible expressions (redexes)* overlap on one term. This property seems to provide the implicit commutativity of reductions. Thus, a *nonoverlapping* TRS would be *confluent.*

In fact, a *nonoverlapping* TRS is known to be *confluent* if either *left-linear* or *strongly-normalizing*, where a TRS is said to be *left-linear* if every variables appear at most once on left-hand side of reduction rules, and said to be *strongly-normalizing* if every reduction paths are terminating.

Nevertheless, both possibly *nonterminating* and *nonlinear* TRSs may be neither *confluent* nor *uniquely-normalizing* even if *nonoverlapping.* Typical *nonconfluent* cases are shown in the following three examples.

Example 1 [9]

$$R_1 \stackrel{\text{def}}{=} \left\{ \begin{array}{lcl} d(x,x) & \rightarrow & 0 \\ d(x,f(x)) & \rightarrow & 1 \\ 2 & \rightarrow & f(2) \end{array} \right\}$$

(Critical on $d(2,2) \rightarrow d(2,f(2))$.)

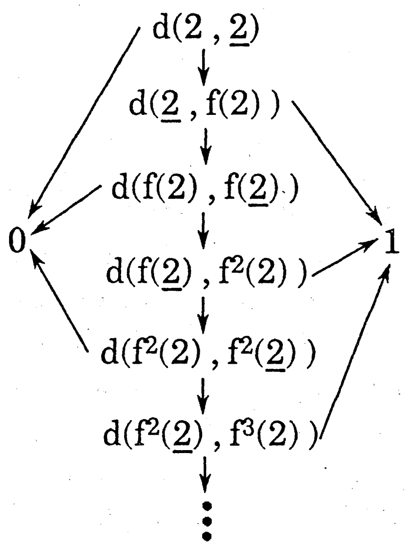

Fig.1 R overlapping example R_1

Example 2

$$R_2 \stackrel{\text{def}}{=} \left\{ \begin{array}{lcl} d(f(x),x) & \rightarrow & 0 \\ d(x,g(x)) & \rightarrow & 1 \\ 2 & \rightarrow & f(3) \\ 3 & \rightarrow & g(2) \end{array} \right\}$$

(Critical on $d(2,3) \rightarrow d(f(3),3),\ d(2,g(2))$.)

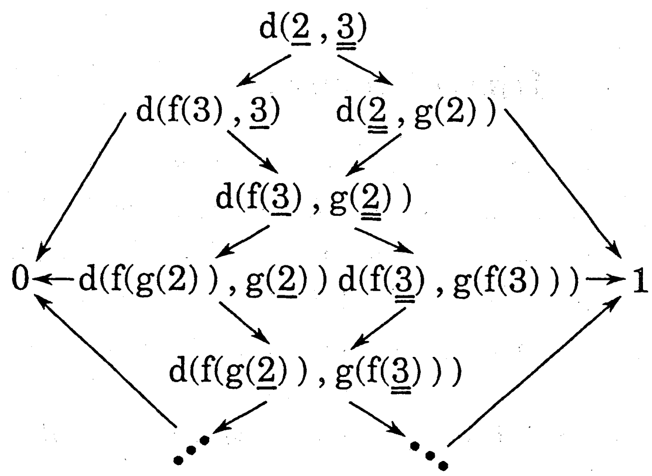

Fig.2 R overlapping example R_2

Examples 1 and 2 show the existence of unmeetable branches on nonlinear reduction paths, although no redexes overlap. For instance, in Example 1 reductions on leaves of some redex will produce another redex corresponding to a different reduction rule. In fact, a redex $d(2,2)$ of the first rule is converted to a different redex $d(2, f(2))$ of the second rule by the reduction $2 \to f(2)$ at a leave of $d(2,2)$. Further in Example 2, reductions in subterms of some non-redex term will produces different redexes. In fact, a non-redex term $d(2,3)$ is reduced to either a redex $d(f(3),3)$ of the first rule or a redex $d(2, g(2))$ of the second rule. These are said to be *R-overlapping*. This concept will be further discussed in Section 4.2.

Example 3

$$R_3 \stackrel{\text{def}}{=} \left\{ \begin{array}{lll} d(x,x) & \to & 0 \\ f(x) & \to & d(x, f(x)) \\ 1 & \to & f(1) \end{array} \right\}$$

(Critical on $f(f(1))$.)

$$\begin{array}{ll} 1 & \stackrel{*}{\to} f^n(1) = f(f^{n-1}(1)) \\ & \to d(f^{n-1}(1), f^n(1)) \\ & \to d(f^n(1), f^n(1)) \\ & \to 0 \end{array}$$

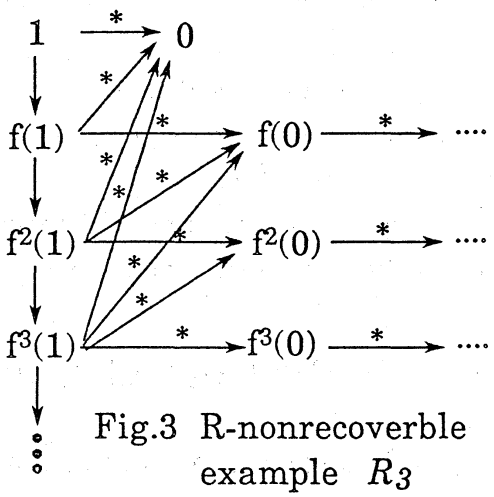

Fig.3 R-nonrecoverble example R_3

Example 3 shows the case where once some redex is modified by reductions at its leaves, then it will be never recovered as a redex. Note that such a reduction path will never terminate. This example is analogous to the nonconfluent example of λ-calculus with a nonlinear δ-reduction rule, namely $\delta_S xx \to \epsilon$ by Staples [1].

One of the previously investigated approaches is restricting reduction strategies. That is, a reduction rule is applied only when accompanied conditions are satisfied. This is called a *conditional TRS*. The main known result is that *a restricted nonlinear membership conditional TRS is confluent* [9]. Intuitively speaking, the restricted nonlinear membership conditional TRS imposes that a nonlinear reduction rule is applied only after subterms of all nonlinear occurrences reach normal forms, by analogy of λ-calculus with Church's δ [1].

This paper investigates the *finite Church-Rosser property* of an *infinitely nonoverlapping* TRS. *Finite Church-Rosser property* is *Church-Rosser property* on normalizable terms. That is, congruence between two terms is examined by syntactical comparison between their normal forms (if exist). This property implies uniqueness of the normal form. The assumption, *infinitely nonoverlapping*, is a natural extension of the *left-linear nonoverlapping*, and is decidable. The only difference between them is that the unification with infinite terms [2,3,7] is applied instead of usual unifications.

The main theorm investigated here is

An infinitely nonoverlapping TRS is finitely Church-Rosser.

For the investigation, a key concepts *R-nonoverlapping* is also proposed. A TRS R is said to be *R-nonoverlapping* iff all reduction paths have no substantially separated branches.

First, an *infinitely nonoverlapping* TRS R is shown to be *R-nonoverlapping*. Second, an *R-nonoverlapping* TRS is shown to be *finitely Church-Rosser*.

Note that Example 1 and 2 above are both *infinitely overlapping*, though *nonoverlapping*. In fact, $d(x,x)$ and $d(x,f(x))$ have an infinite unifier $x = f(f(f(\cdots)))$. Also, $d(f(x),x)$ and $d(x,g(x))$ have an infinite unifier $x = f(g(f(g(\cdots))))$.

Example 3 is *infinitely nonoverlapping* and is not *confluent*. This causes from that once a reduction path enters an unmeetable path, the reduction sequence always falls into an infinite loop, and never terminates. Thus, by restricting discussion to normalizable terms, R_3 is shown to be *finitely Church-Rosser*.

2 Unification with infinite terms

2.1 Variation of unifications

Unifications are classified into following three classes. They are,

- Unification without occur check.
- Unification with occur check.
- Unification with infinite terms (called *infinite unification*).

Unification without occur check does not care on name conflicts. Thus, even for finite terms, this is not correct for non-linear terms. For instance, $f(x,x)$ and $f(g(y),h(y))$ are unified as $\{x = g(y),\ x = h(y)\}$. In other words, consistency of binding environments is not preserved.

In contrast, unification with occur check treats name conflicts as unification *failed*. This is correct on finite terms, but not correct on infinite terms. For instance, unification between $f(x,x)$ and $f(z,g(z))$ is failed, though it can be unified with the infinite term $f(g(g(g(\cdots))),g(g(g(\cdots))))$.

There have been proposed several algorithms for unification with infinite terms [2,3,7]. The substantial difference on infinite unification is that expressions defining a binding environment can refer the environment itself recursively. Therefore, a looped infinite term such as $g(g(g(\cdots)))$ (the solution for $x = g(x)$) is permitted as a unifier. A looped infinite term can be represented by a cyclic finite graph as an internal form.

Thus, the algorithm of infinite unification terminates as same as usual unification algorithms do.

In the next section, the algorithm called $UNIFY0$ for unification with infinite terms by Martelli and Rossi is briefly introduced. For details, refer [7].

2.2 Algorithm of unification with infinite terms

The algorithm $UNIFY0$ computes the *common parts* and *frontiers* iteratively. This terminates when frontiers reach *solved forms* or *fails* during the iterative processes.

The common part of a set of terms M is a dual concept to the usual unifier. Intuitively, the common part is obtained by superposing all terms of M and by taking the part which is common to all of them starting from the root.

For instance, the common part of $M = \{f(x, g(h(a), v), y), f(h(y), g(x, b), z)\}$ is $C : f(x, g(x, v), y)$, where variables are noted by x, y, z, u, v and constants are noted by a, b, c . Notice that the common part does not exist iff two terms have different function symbols at the roots, such as $M = \{f(x, y), g(z, u)\}$.

The frontier is intuitively an environment for variables in the common part. More specifically, the frontier is a set of multiequations (which are pairs $\{S_i = M_i\}$ of a set of variables $S_i(\neq \phi)$ and a set of non-variable terms M_i), where every multiequation is associated with a leaf of the common part and consists of all subterms corresponding to that leaf.

For instance, the frontier of M above is $F : \{\{x\} = \{h(y), h(a)\}, \{v\} = b, \{y, z\} = \phi\}$. In F, $\{y, z\} = \phi$ means $y = z$, but no non-variable terms are substituted.

With definitions above, the unification algorithm $UNIFY0$ starts with a set of multiequations and repeatedly applies transformations until all multiequations become *solved forms* , or *fails* during the iterative processes. A frontier $\{S_i = M_i\}$ is said to be a *solved form* iff $S_i \cap S_j = \phi$ for $\forall i, j$ s.t. $i \neq j$ and $card(M_i) = 1$ for $\forall i$.

Transformers produce equivalent multiequations, which means a set of all unifiers is preserved. In $UNIFY0$, the following two transformations are used.

COMPACTION Given a set L containing two multiequations $S = M$ and $S' = M'$, with $S \cap S' \neq \phi$. The new set L' of multiequations is obtained by replacing these two multiequations with a multi equation $S \cup S' = M \cup M'$.

REDUCTION Given a set L containing a multiequation $S = M$, such that $M \neq \phi$ and M has a common part C and a frontier F. The new set of multiequations L' is obtained by replacing $S = M$ with the multiequation $S = \{C\}$ and with all multiequations of F. If there does not exist the common part, then stop with *failed*.

ALGORITHM : UNIFY0 Let P, Q be terms. Set L as all frontiers of a pair (P, Q), perform on L any of the following actions. If neither applies, then stop with *success*. When success, P and Q are said to be *infinitely unifiable* .

- If there are two multiequations $S = M$ and $S' = M'$ with $S \cap S' \neq \phi$, then apply *COMPACTION*.
- If there is a multiequation $S = M$ such that M includes more than two terms, then compute the common part and the frontier of M. And then if M has no common part then stop with *failure*. Else apply *REDUCTION*.

Remark Note that every right hand side of frontiers are subterms of either given terms P or Q.

Example Unify two terms $P = g(x, f(z, h(x)), x), Q = g(f(h(y), z), y, y)$. Then, the common part C of $M = \{P, Q\}$ is $\{g(x, y, x)\}$, and the frontier $F^{(0)}$ of them is

$$F^{(0)} = \{ \begin{array}{lcl} \{x\} & = & \{f(h(y), z)\}, \\ \{y\} & = & \{f(z, h(x))\}, \\ \{x, y\} & = & \phi \end{array} \}$$

Then,

Step 1a *COMPACTION* $\quad F^{(1)} = \{ \ \{x, y\} = \{f(h(y), z),\ f(z, h(x))\} \ \}$

Step 1b *REDUCTION* $\quad F^{(2)} = \{ \begin{array}{lcl} \{x, y\} & = & \{f(z, z)\} \\ \{z\} & = & \{h(y)\} \\ \{z\} & = & \{h(x)\} \end{array} \}$

Step 2a *COMPACTION* $\quad F^{(3)} = \{ \begin{array}{lcl} \{x, y\} & = & \{f(z, z)\} \\ \{z\} & = & \{h(x), h(y)\} \end{array} \}$

Step 2b *REDUCTION* $\quad F^{(4)} = \{ \begin{array}{lcl} \{x, y\} & = & \{f(z, z)\} \\ \{x, y\} & = & \phi \\ \{z\} & = & \{h(x)\} \end{array} \}$

Step 3a *COMPACTION* $\quad F^{(5)} = \{ \begin{array}{lcl} \{x, y\} & = & \{f(z, z)\} \\ \{z\} & = & \{h(x)\} \quad \text{(solved form)} \end{array} \}$

Finish *M and N are unified to* $g(f(h(f\cdots), h(f\cdots)), f(h(f\cdots), h(f\cdots)))$.

3 Basic definitions and results on confluence

A reduction system is a structure $R = \langle A, \rightarrow \rangle$ consisting of an object set A and any binary relation $\rightarrow$ on A (i.e., $\rightarrow \subseteq A \times A$), called a reduction relation. A *reduction* (starting with x_0) in R is a finite or an infinite sequence $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$. The transitive closure of $\rightarrow$ is noted as $\xrightarrow{*}$. A *less-than n-step reduction* is defined as $x \xrightarrow{n} y$ iff $\exists m \leq n\ \exists z_1, z_2, \cdots, z_{m-1}$ s.t. $x \rightarrow z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow z_{m-1} \rightarrow y$.

A *congruent relation* $\overset{*}{\leftrightarrow}$ in R is the transitive reflexive closure of the binary relation $\leftrightarrow$ where $x \leftrightarrow y$ is defined to be $x \rightarrow y \lor y \rightarrow x$. A *less-than n-step congruent relation* is defined as $x \overset{n}{\leftrightarrow} y$ iff $\exists m \leq n\ \exists z_1, z_2, \cdots, z_{m-1}$ s.t. $x \leftrightarrow z_1 \leftrightarrow z_2 \leftrightarrow \cdots \leftrightarrow z_{m-1} \leftrightarrow y$.

A set of *normal forms* of R is defined as $NF(R) \overset{\text{def}}{=} \{x \in A \mid \neg\exists y \text{ s.t. } x \rightarrow y\}$

The important properties of a reduction system $R = \langle A, \rightarrow \rangle$ are *termination-related* properties (e.g. *weakly-normalizing*, *strongly-normalizing*), and *confluence-related* properties (e.g. *confluent*, *Church-Rosser*, *uniquely-normalizing*).

Definition A reduction system $R = \langle A, \rightarrow \rangle$ is said to be *weakly-normalizing* (**WN**) iff $\forall x \in A\ \exists y \in NF(R)$ s.t. $x \xrightarrow{*} y$. A TRS $R = \langle A, \rightarrow \rangle$ is said to be *strongly normalizing* (**SN**) iff all reduction paths are terminating. i.e. $\forall x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$ $\exists n$ s.t. $x_n \in NF(R)$.

Definition $R = \langle A, \rightarrow \rangle$ is said to be *confluent* iff $\forall x, y, z \in A$ s.t. $x \xrightarrow{*} y \land x \xrightarrow{*} z \implies y \downarrow z$ (i.e. $\exists w \in A$ s.t. $y \xrightarrow{*} w$ and $z \xrightarrow{*} w$).

$R = \langle A, \rightarrow \rangle$ is said to be *Church-Rosser* (**CR**) iff $\forall x, y, z \in A$ s.t. $x \overset{*}{\leftrightarrow} y \implies x \downarrow y$.

Definition $R = \langle A, \rightarrow \rangle$ is said to be *uniquely-normalizing* (**UN**) iff $\forall x \in A\ \forall y, z \in NF(R)$ s.t. $x \rightarrow y \land x \rightarrow z \implies y \equiv z$. ($x \equiv y$ iff x and y are syntactically same.)

Definition $R = \langle A, \rightarrow \rangle$ is said to be *locally-confluent* iff $\forall x, y, z \in A$ s.t. $x \rightarrow y \land x \rightarrow z \implies y \downarrow z$.

Fact 1

$$
\begin{array}{ccccc}
\textbf{UN} \land \textbf{WN} & \Longrightarrow & \textbf{CR} & \Longrightarrow & \textbf{UN} \\
 & & \Updownarrow & & \\
\textit{locally-confluent} \land \textbf{SN} & \Longrightarrow & \textit{confluent} & \Longrightarrow & \textit{locally-confluent}
\end{array}
$$

However, the inverse of implication arrows above are not satisfied [4].

Definition An *occurrence* $occur(M, N)$ of a subterm N in a term M is defined inductively as

$$occur(M,N) \stackrel{\text{def}}{=} \begin{cases} \epsilon & \text{if } N = M \\ i \cdot u & \text{if } u = occur(N_i, N) \text{ and } M = f(N_1, \cdots, N_n) \end{cases}$$

The subterm N of M at occurrence u is noted as M/u. (That is, $u = occur(M,N)$.)

Definition The order on occurrences u, v is defined as $u \preceq v \iff \exists w$ s.t. $v = u \cdot w$. If $u \preceq v \ \wedge \ u \neq v$ then it is noted as $u \prec v$. The occurrences u, v is said to be *disjoint* and noted $u|v$ iff $u \not\preceq v$ and $v \not\preceq u$.

Notation

$$\begin{array}{lcl} V(M) & \stackrel{\text{def}}{=} & \{x \mid \text{variable } x \text{ which is contained in } M\} \\ V_{NL}(M) & \stackrel{\text{def}}{=} & \{x \mid \text{variable } x \text{ which occur more than once in } M\} \\ O(M) & \stackrel{\text{def}}{=} & \{occur(M,N) \mid \text{for } \forall N : \text{ subterm of } M\} \\ \bar{O}(M) & \stackrel{\text{def}}{=} & \{occur(M,N) \mid N \notin V(M)\} \\ O_{NL}(M,x) & \stackrel{\text{def}}{=} & \{occur(M,x) \mid x \in V_{NL}(M)\} \\ u \cdot V & \stackrel{\text{def}}{=} & \{u \cdot v \mid v \in V\} \\ U \cdot v & \stackrel{\text{def}}{=} & \{u \cdot v \mid u \in U\} \\ U \cdot V & \stackrel{\text{def}}{=} & \{u \cdot v \mid u \in U, v \in V\} \\ Min(U) & \stackrel{\text{def}}{=} & \{w \in U \mid w' \not\prec w \quad \text{for } \forall w' \in U\} \end{array}$$

where $u, v \in O(M)$ and $U, V \subseteq O(M)$ for a term M.

Definition A finite set $R = \{(\alpha_i, \beta_i)\}$ of ordered pairs of two terms is said to be a *Term Rewriting System (TRS)* iff each α_i is not a variable and $V(\alpha_i) \supseteq V(\beta_i)$ is satisfied for $\forall i$.

A *reduction* is defined on a term M as $M \to N$ at u iff there exists a substitution σ and an occurrence $u \in \bar{O}(M)$ s.t. $\sigma(\alpha_i) \equiv M/u$ and $N \equiv M[u \leftarrow \sigma(\beta_i)]$.

A *congruent relation* is defined on terms M and N as $M \leftrightarrow N$ at u iff $M \to N$ at u or $N \to M$ at u.

In the situation above, M/u is said to be a *redex*. A set of all occurrences of redexes for $\alpha_i \to \beta_i$ in M is noted as $Redex(M, \alpha_i)$, and $Redex(M) \stackrel{\text{def}}{=} \cup_i Redex(M, \alpha_i)$.

Definition A pair of reduction rules $\alpha_i \to \beta_i$ and $\alpha_j \to \beta_j$ is said to be *nonoverlapping* iff "$\exists u \in \bar{O}(\alpha_i)$ s.t. α_i/u and α_j are unifiable $\iff i = j$ and $u = \epsilon$" is satisfied. A TRS R is said to be *nonoverlapping* iff all pairs of reduction rules are *nonoverlapping*.

Remark If a TRS is nonoverlapping, then $Redex(M, \alpha_i) \cap Redex(M, \alpha_j) = \phi$ for $\forall M \ \forall i, j$ s.t. $i \neq j$.

Definition A reduction rule $\alpha_i \to \beta_i$ is said to be *left-linear* iff $\forall x \in V(\alpha_i)$ appears only once in α_i. A TRS R is said to be *left-linear* iff all reduction rules in R are *left-linear*.

Fact 2 A *nonoverlapping* TRS is *locally-confluent*.
A *left-linear nonoverlapping* TRS is *confluent*[4].

4 Finite Church-Rosser Property of a TRS

4.1 Finite Church-Rosser property of an infinitely nonoverlapping TRS

Definition A TRS R is said to be *finitely confluent* iff $\forall x, y, z$ s.t. $x \xrightarrow{*} y \ \wedge\ x \xrightarrow{*} z$ satisfy the condition

$$(\exists y', z' \in NF(R) \ s.t.\ y \xrightarrow{*} y' \ \wedge\ z \xrightarrow{*} z') \quad implies \quad y \downarrow z.$$

Remark *Finitely confluent* is equivalent to *uniquely normalizing*. Then, for a strongly-normalizing TRS R, *finitely confluent, locally confluent*, and *confluent* are equivalent.

As an analogy to the relation between *confluence* and *finite confluence*, *finite Church-Rosser property* is defined as follows.

Definition A TRS R is said to *finitely Church-Rosser* iff $\forall x, y, x'y'$ s.t. $x \xrightarrow{*} x' \wedge y \xrightarrow{*} y'$ satisfy the condition

$$(x \overset{*}{\leftrightarrow} y \ \wedge\ x', y' \in NF(R)) \quad implies \quad x' \equiv y'.$$

Note that, *Finite Church-Rosser property* is equivalent to "UN in [8]", and *finite confluence* is equivalent to "UN$^{\rightarrow}$ in [8]".

Remark *Church-Rosser* $\Longrightarrow$ *finitely Church-Rosser* $\Longrightarrow$ *finitely confluent*,
However, *Church-Rosser* $\not\Longleftarrow$ *finitely Church-Rosser* $\not\Longleftarrow$ *finitely confluent.*

For instance, the example R_3 is *finitely Church-Rosser*, but not *Church-Rosser* (See section 4.2). And, the example R_4 is *finitely confluent*, but not *finitely Church-Rosser* (See Fig.4). Note that R_4 is overlapping.

Example 4

$$R_4 \stackrel{\text{def}}{=} \left\{ \begin{array}{lcl} f(2) & \rightarrow & 0 \\ f(3) & \rightarrow & 1 \\ f(x) & \rightarrow & f(f(4)) \end{array} \right\}$$

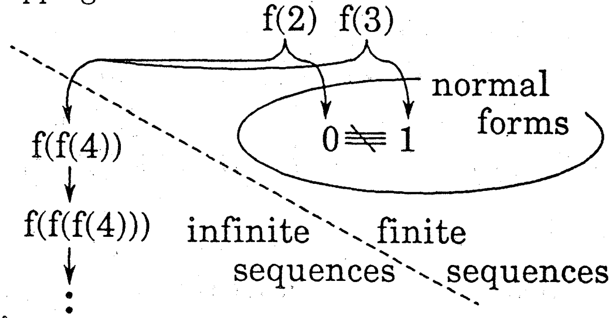

Fig.4 A finitely confluent, but not finitely Church-Rosser example R_4.

If a TRS R is *weakly-normalizing*, then

Church-Rosser $\Longleftrightarrow$ *finitely Church-Rosser* $\Longleftrightarrow$ *finitely confluent.*

Note that neither R_3 nor R_4 is *weakly-normalizing.*

Definition A pair of reduction rules $\alpha_i \to \beta_i$ and $\alpha_j \to \beta_j$ is said to be *infinitely nonoverlapping* iff " $\exists u \in \bar{O}(\alpha_i)$ s.t. α_i/u and α_j are infinitely unifiable $\Longleftrightarrow i = j$ and $u = \epsilon$" is satisfied. A TRS R is said to be *infinitely nonoverlapping* iff all pairs of reduction rules are infinitely nonoverlapping.

An infinitely nonoverlapping TRS is nonoverlapping. And in case of a left-linear TRS, infinitely nonoverlapping is equivalent to nonoverlapping. Thus, a class of infinitely nonoverlapping TRSs is a natural extension of left-linear nonoverlapping TRSs to nonlinear TRSs. Our main conjecture is the next claim.

Conjecture *An infinitely nonoverlapping TRS is finitely Church-Rosser.*

In the following sections, we will prove this **conjecture** . Among these investigation, *R-nonoverlapping* is a key concept.

Intuitively, a TRS R is said to be *R-nonoverlapping* if there do not exist branches of reduction paths in which applications of reduction rules are implicitly overlapping. They are the cases of reduction paths starting with $d(2,2)$ in **Example 1** and $d(2,3)$ in **Example 2**. (See Fig.1 and 2)

First, an *infinitely nonoverlapping* TRS R is proved to be *R-nonoverlapping.* Second, an *R-nonoverlapping* TRS R is proved to be *finitely Church-Rosser.*

4.2 Proof of conjecture

Definition Let $M \overset{*}{\leftrightarrow} N$ be $M \equiv M_0 \leftrightarrow M_1 \leftrightarrow M_2 \leftrightarrow \cdots \leftrightarrow M_n \equiv N$ where $\forall i$ s.t. $M_{i-1} \leftrightarrow M_i$ at u_i. Then, *reduced occurrence sequence* $\overline{REDEX}(M \overset{*}{\leftrightarrow} N)$, *reduced term sequence* $\overline{TERM}(M \overset{*}{\leftrightarrow} N)$, *invariant occurrences* $O_{inv}(M \overset{*}{\leftrightarrow} N)$, *boundary occurrences* $\partial O(M \overset{*}{\leftrightarrow} N)$, are defined as follows.

$$\begin{array}{lcl}
\overline{REDEX}(M \overset{*}{\leftrightarrow} N) & \overset{\text{def}}{=} & (u_1, u_2, \cdots, u_n) \\
\overline{TERM}(M \overset{*}{\leftrightarrow} N) & \overset{\text{def}}{=} & (M_0, M_1, M_2, \cdots, M_n) \\
O_{inv}(M \overset{*}{\leftrightarrow} N) & \overset{\text{def}}{=} & \{u \in \bar{O}(M) \mid u_i \not\preceq u \text{ for } \forall u_i \in \overline{REDEX}(M \overset{*}{\to} N)\} \\
\partial O(M \overset{*}{\leftrightarrow} N) & \overset{\text{def}}{=} & Min(\overline{REDEX}(M \overset{*}{\to} N))
\end{array}$$

Definition Assume $U = \{u_1, u_2, \cdots u_k\} \subseteq \bar{O}(M)$ s.t $i \neq j \implies u_i | u_j$ for $\forall u_i, u_j \in U$.

A *parallel reduction* is defined to be $M \twoheadrightarrow N$ at U iff $M \equiv M_0 \to M_1 \to M_2 \to \cdots \to M_k \equiv N$ where $\forall i$ s.t. $M_{i-1} \to M_i$ at u_i. A *less-than n-step parallel reduction* is noted as $M \xrightarrow{\text{n}}\!\!\!\!\rightarrow N$.

A *parallel congruent relation* is defined to be $M \twoheadleftarrow\!\!\!\twoheadrightarrow N$ at U iff $M \equiv M_0 \leftrightarrow M_1 \leftrightarrow M_2 \leftrightarrow \cdots \leftrightarrow M_k \equiv N$ where $\forall i$ s.t. $M_{i-1} \leftrightarrow M_i$ at u_i. A *less-than n-step parallel congruent relation* is noted as $M \overset{\text{n}}{\twoheadleftarrow\!\!\twoheadrightarrow} N$.

Definition Let R be a TRS, and M, N be a term s.t. $M \overset{\text{n}}{\twoheadleftarrow\!\!\twoheadrightarrow} N$ for $n \geq 0$. R is said to be *$\langle R,n\rangle$-nonoverlapping* at M, iff $\forall u, v \in O_{inv}(M \overset{\text{m}}{\twoheadleftarrow\!\!\twoheadrightarrow} N)$ s.t. $u \in Redex(M, \alpha_i)$, $v \in Redex(N, \alpha_j)$, $0 \leq m \leq n$ satisfies the following condition

$$(v \in u \cdot \bar{O}(\alpha_i) \vee u \in v \cdot \bar{O}(\alpha_j)) \implies (u = v \ \wedge \ i = j).$$

If R is *$\langle R,n\rangle$-nonoverlapping* for $\forall n \geq 0$, R is said to be *R-nonoverlapping*.

Proposition 1 An *infinitely nonoverlapping* TRS R is *R-nonoverlapping.*

Before entering the proof, several technical lemmas should be prepared.

Lemma 1 Let a TRS R be *$\langle R,n\text{-}1\rangle$-nonoverlapping*, and $M \overset{\text{n}}{\twoheadleftarrow\!\!\twoheadrightarrow} N$.
Assume $\exists u, v \in O_{inv}(M \overset{\text{m}}{\twoheadleftarrow\!\!\twoheadrightarrow} N)$, $\exists \alpha_i \rightarrow \beta_i, \alpha_j \rightarrow \beta_j \in R$ s.t. $u \in v \cdot \bar{O}(\alpha_j)$.
Then, $u \cdot \bar{O}(\alpha_i) \cap v \cdot \bar{O}(\alpha_j) \subseteq O_{inv}(M \overset{\text{n}}{\twoheadleftarrow\!\!\twoheadrightarrow} N)$.

Lemma 2 Let a TRS R be *$\langle R,n\rangle$-nonoverlapping.*
Assume $\exists \sigma, \sigma'\ \exists \alpha_i \rightarrow \beta_i \in R$, $0 \leq \exists m \leq n$ s.t. $\sigma(\alpha_i) \overset{\text{m}}{\twoheadleftarrow\!\!\twoheadrightarrow} \sigma'(\alpha_i)$.
Then, $\epsilon \in O_{inv}(\sigma(\alpha_i) \overset{\text{m}}{\twoheadleftarrow\!\!\twoheadrightarrow} \sigma'(\alpha_i)) \implies \sigma(\beta_i) \overset{\text{m}'}{\twoheadleftarrow\!\!\twoheadrightarrow} \sigma'(\beta_i)$ for some $m' < m$.

Lemma 3 Let a TRS R be *$\langle R,n\text{-}1\rangle$-nonoverlapping.*
Assume $M \overset{\text{m}}{\twoheadleftarrow\!\!\twoheadrightarrow} N$ for some $m \leq n$, and $\exists u \in \bar{O}(M) \cap \bar{O}(N)$ s.t. M/u and N/u have different function symbols at the roots. Then, either (a) or (b) is satisfied for some $m', n' < m$ and some $v \preceq u$ s.t. $m' + n' < m$.

(a) $\exists M', N' \in \overline{TERM}(M/v \overset{\text{m}}{\twoheadleftarrow\!\!\twoheadrightarrow}' N/v)$
 s.t. $M/v \overset{\text{m}'}{\twoheadleftarrow\!\!\twoheadrightarrow} M' \rightarrow N' \overset{\text{n}'}{\twoheadleftarrow\!\!\twoheadrightarrow} N/v$ and $M' \rightarrow N'$ at $\epsilon \in O_{inv}(M/v \overset{\text{m}'}{\twoheadleftarrow\!\!\twoheadrightarrow} M')$.

(b) $\exists M', N' \in \overline{TERM}(M/v \overset{\text{m}}{\twoheadleftarrow\!\!\twoheadrightarrow}' N/v)$
 s.t. $M/v \overset{\text{m}'}{\twoheadleftarrow\!\!\twoheadrightarrow} M' \leftarrow N' \overset{\text{n}'}{\twoheadleftarrow\!\!\twoheadrightarrow} N/v$ and $N' \rightarrow M'$ at $\epsilon \in O_{inv}(N' \overset{\text{n}'}{\twoheadleftarrow\!\!\twoheadrightarrow} N/v)$.

Furthermore, $\overline{TERM}(M/v \overset{\text{m}'}{\twoheadleftarrow\!\!\twoheadrightarrow} M' \leftrightarrow N') \subseteq \overline{TERM}(M/v \overset{\text{m}}{\twoheadleftarrow\!\!\twoheadrightarrow}' N/v)$.

Proof of lemma 3 The proof is due to the induction on m. For $m = 1$, the statement is obvious. Let the statement be satisfied when less than m.

From the assumption, $\exists! v \in \partial O(M \overset{\text{m}}{\twoheadleftarrow\!\!\twoheadrightarrow} N)$ s.t. $v \preceq u$. Then, $M/v \overset{\text{m}}{\twoheadleftarrow\!\!\twoheadrightarrow}' N/v$. Assume any subsequence of $M/v \overset{\text{m}}{\twoheadleftarrow\!\!\twoheadrightarrow}' N/v$ satisfies neither (a) nor (b).

Then, $\exists M', M'', N', N'' \in \overline{TERM}(M/v \overset{\text{m}}{\twoheadleftarrow\!\!\twoheadrightarrow}' N/v)$

s.t. $M/v \overset{m}{\twoheadleftarrow\!\!\twoheadrightarrow}' N/v \equiv M/v \overset{m_1}{\twoheadleftarrow\!\!\twoheadrightarrow} M' \leftarrow M'' \overset{m_2}{\twoheadleftarrow\!\!\twoheadrightarrow} N'' \rightarrow N' \overset{m_3}{\twoheadleftarrow\!\!\twoheadrightarrow} N/v$
for $M' \rightarrow M''$ and $N' \rightarrow N''$ both at $\epsilon \in O_{inv}(M/v \overset{m_1}{\twoheadleftarrow\!\!\twoheadrightarrow} M') \cap O_{inv}(N' \overset{m_3}{\twoheadleftarrow\!\!\twoheadrightarrow} N/v)$.

Thus, there exist a subsequence $P' \leftarrow P \overset{m'}{\twoheadleftarrow\!\!\twoheadrightarrow} Q \rightarrow Q'$ in $M' \leftarrow M'' \overset{m_2}{\twoheadleftarrow\!\!\twoheadrightarrow} N'' \rightarrow N'$ for $P \rightarrow P'$ and $Q \rightarrow Q'$ both at $\epsilon \in O_{inv}(P \overset{m'}{\twoheadleftarrow\!\!\twoheadrightarrow} Q)$.

From $\langle R, n\text{-}1\rangle$-*nonoverlapping* property and the relation $m' < m \leq n$, there exist $\exists\sigma, \sigma'\ \exists\alpha_i \rightarrow \beta_i \in R$ s.t. $P/v \equiv \sigma(\alpha_i)$ and $Q/v \equiv \sigma'(\alpha_i)$. Then, $\exists m'' < m$ s.t $M/v \overset{m''}{\twoheadleftarrow\!\!\twoheadrightarrow}{}'' N/v$ from **lemma 2**. Form the induction hypothesis, **lemma 3** is proved. (q.e.d.)

Proof of proposition 1 we will prove that R is $\langle R, n\rangle$-*nonoverlapping* by induction on n. Since $\langle R, 0\rangle$-*nonoverlapping* is equivalent to *nonoverlapping*, the statement is obvious for $n = 0$.

Assume R be $\langle R, n\text{-}1\rangle$-*nonoverlapping* as an induction hypothesis, and let R be not $\langle R, n\rangle$-*nonoverlapping*. Then, $\exists M, N$ s.t. $M \overset{n}{\twoheadleftarrow\!\!\twoheadrightarrow} N$ and $\exists u, v \in O_{inv}(M \overset{n}{\twoheadleftarrow\!\!\twoheadrightarrow} N)$ s.t. $u \in Redex(M, \alpha_i) \wedge v \in Redex(N, \alpha_j) \wedge \neg(u = v\ \wedge\ i = j) \wedge u \in v \cdot \bar{O}(\alpha_j)$ (or $v \in u \cdot \bar{O}(\alpha_i)$).

From assumption, α_i and α_j are *infinitely nonoverlapping* (except α_i overlaps with itself at the root). Thus, along the execution of the infinite unification algorithm on α_i and α_j/w s.t. $u = v \cdot w$ and $\neg(w = \epsilon\ \wedge\ i = j)$, there exist non-variable subterms P, P' of α_i or α_j/w s.t. some frontier $\{x\} = (P, P')$ failed. (That is, P and P' have different function symbols at their roots.)

There are three cases the frontier $\{x\} = (P, P')$ fails. Let $M/u \equiv \sigma(\alpha_i)$ and $N/v \equiv \sigma'(\alpha_j)$.

[case 1] $P \in \alpha_i,\ P' \in \alpha_j$.
i.e. $\exists s \in u \cdot \bar{O}(\alpha_i) \cap v \cdot \bar{O}(\alpha_j)$ s.t. $Q = \sigma(P) = M/s, Q' = \sigma'(P') = N/s$.

[case 2] $P, P' \in \alpha_i$.
i.e. $\exists s, s' \in u \cdot \bar{O}(\alpha_i),\ \exists r \in \bar{O}(\alpha_i),\ \exists t, t' \in v \cdot O_{NL}(\alpha_j, x)$

$$\text{s.t.} \quad \begin{cases} s = t \cdot r,\ s' = t' \cdot r,\ t \neq t', \\ Q = \sigma(P) = M/s,\ Q' = \sigma(P') = M/s',\ N/t \equiv N/t'. \end{cases}$$

[case 3] $P, P' \in \alpha_j$.
i.e. $\exists t, t' \in v \cdot \bar{O}(\alpha_j),\ \exists r \in \bar{O}(\alpha_j),\ \exists s, s' \in u \cdot O_{NL}(\alpha_i, x)$

$$\text{s.t.} \quad \begin{cases} t = s \cdot r,\ t' = s' \cdot r,\ s \neq s', \\ Q = \sigma'(P) = N/t,\ Q' = \sigma'(P') = N/t',\ M/s \equiv M/s'. \end{cases}$$

Then, contradiction will be deduced case-by-case from the fact that Q and Q' have different function symbols at their roots.

[case 1] Q and Q' have different function symbols at their roots. Then, $u \prec \exists t \preceq s$

s.t. $t \notin O_{inv}(M \overset{\mathrm{n}}{\twoheadleftarrow\!\!\twoheadrightarrow} N)$. However, this contradicts to **lemma 1** from the induction hypothesis.

[case 2] Q and Q' have different function symbols at their roots and $S = S'$. Then, from **lemma 1**, there exists $W \equiv M/p$ and $W' \equiv M/p'$ s.t. $t \preceq p \preceq s$, $t' \preceq p' \preceq s'$ and $W \overset{\mathrm{m}}{\twoheadleftarrow\!\!\twoheadrightarrow} W'$ for $0 < m \leq n$.

From **Lemma 3**, there exist $W'' \in \overline{TERM}(W \overset{\mathrm{m}}{\twoheadleftarrow\!\!\twoheadrightarrow} W')$ s.t. $W'' \overset{\mathrm{m'}}{\twoheadleftarrow\!\!\twoheadrightarrow} W$ (or W') and $\exists r' \in Redex(W'', \alpha_k) \cap O_{inv}(W'' \overset{\mathrm{m'}}{\twoheadleftarrow\!\!\twoheadrightarrow} W)$ ((or W') where $r' \preceq r$, $0 \leq m' \leq m$. Then, α_i and α_k are $\langle R, m' \rangle$-overlapping at $s \cdot r'$ (or $s' \cdot r'$). This leads a contradiction.

[case 3] Same as in [case 2]. (q.e.d)

Proposition 2 An *R-nonoverlapping* TRS R is *finitely Church-Rosser.*

Proof Let $M, N \in NF(R)$ s.t. $M \overset{\mathrm{n}}{\leftrightarrow} N$. We will prove $M \equiv N$ by induction on n. Then, R is proved to be *finitely Church-Rosser.*

As an initial induction step, $M \equiv N$ is obvious for $n = 0$.

As an induction hypothesis, let $M \equiv N'$ hold for $\forall m < n$ $\forall N'$ s.t. $M \overset{\mathrm{m}}{\leftrightarrow} N'$ and $M, N' \in NF(R)$.

Assume $M \overset{\mathrm{n}}{\leftrightarrow} N$ and $M \not\equiv N$ where $M, N \in NF(R)$. From **lemma 3**, $\exists m < n$ $\exists u \in \partial O(M \overset{\mathrm{n}}{\leftrightarrow} N)$ s.t. $M/u \overset{\mathrm{m}}{\leftrightarrow} M' \to N'$ and $M' \to N'$ at $\epsilon \in O_{inv}(M/u \overset{\mathrm{m}}{\leftrightarrow} M')$.

Let $M' \to N'$ at ϵ be by the rule $\alpha_i \to \beta_i$. If $\alpha_i \to \beta_i$ is a *left-linear* reduction rule, then *R-nonoverlapping* property and $\epsilon \in O_{inv}(M/u \overset{\mathrm{m}}{\leftrightarrow} M')$ implies $\epsilon \in Redex(M/u, \alpha_i)$. This contradicts to the assumption $M/u \in NF(R)$.

Then, $\alpha_i \to \beta_i$ must be a nonlinear rule. And from *R-nonoverlapping* property and $M/u \in NF(R)$,

$$\exists x \in V(\alpha_i) \quad \exists v, v' \in O_{NL}(\alpha_i, x) \quad \exists w, w' \in \partial O(M/v \overset{\mathrm{m}}{\leftrightarrow} M')$$
$$\text{s.t. } v \preceq w,\ v' \preceq w',\ v \neq v', \text{ and } M/u \cdot w \not\equiv M/u \cdot w'.$$

Note that $M'/v \equiv M'/v'$. Then, $\exists p, q$ s.t. $M/u \cdot v \overset{\mathrm{p}}{\leftrightarrow} M'/v \equiv M'/v' \overset{\mathrm{q}}{\leftrightarrow} M/u \cdot v'$ and $p + q \leq m < n$. This contradicts to the induction hypothesis. (q.e.d.)

Theorm An *infinitely nonoverlapping* TRS is *finitely Church-Rosser.*

Corollary 1 An *infinitely nonoverlapping* TRS R is *uniquely-normalizing.*

Corollary 2 If an *infinitely nonoverlapping* TRS R is *weakly-normalizing*, then R is *confluent.*

5 Conclusion

In this paper, the *finite Church-Rosser* property of a *nonlinear* TRS was investigated. Main result was

An infinitely nonoverlapping TRS is finitely Church-Rosser.

Finite Church-Rosser property guarantees that congruence between two terms is examined by syntactical comparison between their normal forms (if exists). The condition *infinitely nonoverlapping* is a natural extension of *left-linear nonoverlapping*. The difference between *infinitely nonoverlapping* and *nonoverlapping* is that the unification with infinite terms [2,3,7] is applied instead of a usual unification with occur check.

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