

Explicit formulae for sums of products of Cauchy numbers including poly-Cauchy numbers

By

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Abstract

Recently, K. Kamano studied sums of products of Bernoulli numbers including poly-Bernoulli numbers. A relation among these sums was given, and an explicit expression of sums of two products was also given, reduced to the famous Euler's formula. The concept of poly-Cauchy numbers is given by the author as a generalization of the classical Cauchy number and an analogue of poly-Bernoulli number. In this paper, we investigate sums of products of Cauchy numbers including poly-Cauchy numbers in order to give explicit expressions in any m products.

§ 1. Introduction

Cauchy numbers of the first kind c_n ($n = 0, 1, 2, \dots$) are defined by the integral of the falling factorial:

$$c_n = \int_0^1 x(x-1) \cdots (x-n+1) dx$$

and the generating function of c_n is given by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}$$

([4, 16]). On the other hand, Bernoulli numbers B_n ($n = 0, 1, 2, \dots$) are defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

The following identity on sums of two products of Bernoulli numbers is known as Euler's formula:

$$\sum_{i=0}^n \binom{n}{i} B_i B_{n-i} = -n B_{n-1} - (n-1) B_n \quad (n \geq 0).$$

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Many generalizations of Euler's formula have been studied. For example, a formula of sums of N products of Bernoulli numbers ([5]) and a formula of sums of N products of Bernoulli polynomials ([3]) have been considered. Other types of sums of products have been also studied (e.g. [1, 2, 6, 15, 17, 18]). M. Kaneko ([11]) introduced the poly-Bernoulli numbers $B_n^{(k)}$ by

$$\frac{\text{Li}_k(1 - e^{-x})}{1 - e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!},$$

where

$$\text{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}$$

is the k -th polylogarithm function. If $k = 1$, $B_n = (-1)^n B_n^{(1)}$ is the n -th classical Bernoulli number. Kamano ([8]) considered the sums of products of Bernoulli numbers including poly-Bernoulli numbers, $S_m^{(k)}(n)$ for $m \geq 1$, $k \geq 1$, $n \geq 0$

$$S_m^{(k)}(n) := \sum_{\substack{i_1 + \dots + i_m = n \\ i_1, \dots, i_m \geq 0}} \binom{n}{i_1, \dots, i_m} \underbrace{B_{i_1} \dots B_{i_{m-1}}}_{m-1} B_{i_m}^{(k)},$$

where $\binom{n}{i_1, \dots, i_m}$ is the multinomial coefficient defined by

$$\binom{n}{i_1, \dots, i_m} = \frac{n!}{i_1! \dots i_m!}.$$

The following identity holds ([8, Theorem 1]).

Proposition 1.1. For $k \in \mathbb{Z}$ and $m \geq 0$, we have

$$\begin{aligned} \sum_{l=0}^m (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} S_{m+1}^{(k-l)}(n) \\ = \begin{cases} n(n-1) \dots (n-m+1) \sum_{l=0}^m \left\{ \begin{matrix} m \\ l \end{matrix} \right\} B_{n-m+l}^{(k)} & (n \geq m); \\ 0 & (0 \leq n \leq m-1). \end{cases} \end{aligned}$$

Kamano also gave explicit formulae of $S_2^{(k)}(n)$ and $S_3^{(k)}(n)$.

Recently, the author ([12]) introduced *poly-Cauchy numbers* of the first kind $c_n^{(k)}$ ($n \geq 0$, $k \geq 1$) by

$$\underbrace{\int_0^1 \dots \int_0^1}_k (x_1 x_2 \dots x_k) (x_1 x_2 \dots x_k - 1) \dots (x_1 x_2 \dots x_k - n + 1) dx_1 dx_2 \dots dx_k.$$

The generating function of $c_n^{(k)}$ is given by

$$(1.1) \quad \text{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!},$$

where

$$\text{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k}$$

is the k -th *polylogarithm factorial function*, which is also introduced in [12]. When $k = 1$, $c_n = c_n^{(1)}$ is the n -th classical Cauchy number. Note that $c_n^{(k)}$ for $k \leq 0$ are also defined by the generating function (1.1). In [14] the author investigated the sums of products of Cauchy numbers, including poly-Cauchy numbers, $T_m^{(k)}(n)$ for $m \geq 1$, $k \geq 1$, $n \geq 0$

$$T_m^{(k)}(n) := \sum_{\substack{i_1 + \dots + i_m = n \\ i_1, \dots, i_m \geq 0}} \binom{n}{i_1, \dots, i_m} \underbrace{c_{i_1} \dots c_{i_{m-1}}}_{m-1} c_{i_m}^{(k)}.$$

$T_m^{(k)}(n)$ is an analogue of Kamano's $S_m^{(k)}(n)$. When $k = 1$, $T_m^{(1)}(n)$ was studied by Zhao ([20]). The following identity holds ([14, Theorem 1]).

Proposition 1.2. For $k \in \mathbb{Z}$ and $m \geq 0$, we have

$$\begin{aligned} \sum_{l=0}^m (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} T_{m+1}^{(k-l)}(n) \\ = \begin{cases} \sum_{l=0}^m \sum_{i=0}^{n-m} \frac{n!}{i!} \binom{l}{n-m-i} \left\{ \begin{matrix} m \\ l \end{matrix} \right\} c_{l+i}^{(k)} & (n \geq m); \\ 0 & (0 \leq n \leq m-1). \end{cases} \end{aligned}$$

The author ([14]) also gave explicit formulae of $T_2^{(k)}(n)$ and $T_3^{(k)}(n)$. Kamano ([8]) mentioned that explicit formulae of $S_m^{(k)}(n)$ for $m \geq 4$ might be obtained but seemed to be complicated to describe. The author failed to obtain explicit formulae of $T_m^{(k)}(n)$ as well as $S_m^{(k)}(n)$ for $m \geq 4$ by Kamano's original method. Finally, we ([9]) are successful to obtain explicit formulae of $S_m^{(k)}(n)$ for any general $m \geq 2$ by a different method. In this paper, we give explicit formulae of $T_m^{(k)}(n)$ for any integer $m \geq 2$.

§ 2. Explicit formulae for $T_m^{(k)}(n)$

The generating function of $T_m^{(k)}(n)$ is given by

$$\left(\frac{x}{\ln(1+x)} \right)^{m-1} \text{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} T_m^{(k)}(n) \frac{x^n}{n!}.$$

Put

$$G_k(x) := \text{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}.$$

Since

$$\text{Lif}_1(z) = \frac{e^z - 1}{z}, \quad \text{Lif}_0(z) = e^z \quad \text{and} \quad \text{Lif}_{-1}(z) = (z+1)e^z,$$

we have

$$(2.1) \quad G_1(x) = \frac{x}{\ln(1+x)}, \quad G_0(x) = 1+x \quad \text{and} \quad G_{-1}(x) = (1+x)(\ln(1+x)+1).$$

Since

$$(2.2) \quad x^m \frac{d^l}{dx^l} G_k(x) = \sum_{i=0}^{\infty} c_{l+i}^{(k)} \frac{x^{m+i}}{i!} \quad (m, l \geq 0, k \geq 1),$$

the coefficient of x^n in

$$x^m \frac{d^l}{dx^l} G_k(x)$$

is equal to

$$\begin{cases} \frac{c_{n-m+l}^{(k)}}{(n-m)!} & (n \geq m); \\ 0 & (0 \leq n \leq m-1). \end{cases}$$

We need the following Lemma ([14, Lemma 1]).

Lemma 2.1. *For an integer k and a positive integer m , we have*

$$\begin{aligned} & \left(\left\{ \begin{matrix} m \\ m \end{matrix} \right\} \frac{d^m}{dx^m} + \left\{ \begin{matrix} m \\ m-1 \end{matrix} \right\} \frac{1}{1+x} \frac{d^{m-1}}{dx^{m-1}} + \cdots + \left\{ \begin{matrix} m \\ 1 \end{matrix} \right\} \frac{1}{(1+x)^{m-1}} \frac{d}{dx} \right) G_k(x) \\ &= \frac{1}{(1+x)^m (\ln(1+x))^m} \sum_{l=0}^m (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} G_{k-l}(x). \end{aligned}$$

In [14] explicit formulae of $T_2^{(k)}(n)$ and $T_3^{(k)}(n)$ were given. By using a different method, it is possible to express $T_m^{(k)}(n)$ for $m = 4, 5, \dots$. However, we can say much more as shown in the next general theorem. In general, we can obtain the following explicit expression of $T_m^{(k)}(n)$ for any general $m \geq 2$.

Theorem 2.2. *For $n \geq 0$ and $k > 0$ we have*

$$(2.3) \quad T_m^{(0)}(n) = T_{m-1}^{(1)}(n) + n T_{m-1}^{(1)}(n-1),$$

$$(2.4) \quad \begin{aligned} T_m^{(k)}(n) &= \sum_{r=0}^{m-2} (-1)^r \binom{n}{r} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{(i+1)^k} T_{m-r}^{(0)}(n-r) \\ &+ \frac{(-1)^{m-1} n!}{(n-m+1)!} \sum_{\substack{j_1+j_2+\cdots+j_{m-1}=k+m-2 \\ j_1, j_2, \dots, j_{m-1} \geq 1}} 2^{-j_2} 3^{-j_3} \cdots (m-1)^{-j_{m-1}} \sum_{j=1}^{j_1} \sum_{\kappa=0}^{m-1} P_{m,\kappa}(n) c_{n-\kappa}^{(j)}, \end{aligned}$$

$$(2.5) \quad \begin{aligned} T_m^{(-k)}(n) &= \sum_{r=0}^{m-2} (-1)^r \binom{n}{r} \sum_{i=0}^r \binom{r}{i} (-1)^i (i+1)^k T_{m-r}^{(0)}(n-r) \\ &+ \frac{n!}{(n-m+1)!} \sum_{\substack{j_1+j_2+\cdots+j_{m-1}=k-m+1 \\ j_1, j_2, \dots, j_{m-1} \geq 0}} 2^{j_2} 3^{j_3} \cdots (m-1)^{j_{m-1}} \sum_{j=0}^{j_1} \sum_{\kappa=0}^{m-1} P_{m,\kappa}(n) c_{n-\kappa}^{(-j)}. \end{aligned}$$

where

$$P_{m,\kappa}(n) = \sum_{t=0}^{\kappa} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \binom{m-t-1}{m-\kappa-1} \frac{(n-m+1)!}{(n-m-\kappa+t+1)!} \quad (\kappa = 0, 1, \dots, m-2)$$

and

$$P_{m,m-1}(n) = \sum_{t=0}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \frac{(n-m+1)!}{(n-2m+t+2)!} = (n-m+1)^{m-1}.$$

Remark. There are some alternative versions to express $T_m^{(k)}(n)$ explicitly. Since for non-negative integers r and k

$$\left\{ \begin{matrix} k+1 \\ r+1 \end{matrix} \right\} = \frac{1}{r!} \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} (i+1)^k$$

([19]) and

$$\left\{ \begin{matrix} k \\ m-1 \end{matrix} \right\} = \sum_{\substack{j_1+\dots+j_{m-1}=k-m+1 \\ j_1, \dots, j_{m-1} \geq 0}} 1^{j_1} 2^{j_2} \dots (m-1)^{j_{m-1}}$$

([4, p.207]), the identities in above Theorem can be also written as

$$\begin{aligned} T_m^{(k)}(n) &= \sum_{r=0}^{m-2} (-1)^r \binom{n}{r} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{(i+1)^k} T_{m-r}^{(0)}(n-r) \\ &\quad + \frac{(-1)^n n!}{(n-m+1)!} \sum_{\substack{j_1+j_2+\dots+j_{m-1}=k-m+1 \\ j_1, j_2, \dots, j_{m-1} \geq 1}} 2^{-j_2} 3^{-j_3} \dots (m-1)^{-j_{m-1}} \sum_{j=1}^{j_1} \\ &\quad \times \sum_{\kappa=0}^{n-m+1} \left[\begin{matrix} n-m+1 \\ \kappa \end{matrix} \right] \frac{(-1)^\kappa}{(m+\kappa)^j}, \\ T_m^{(-k)}(n) &= \sum_{r=0}^{m-2} (-1)^r \binom{n}{r} \sum_{i=0}^r \binom{r}{i} (-1)^i (i+1)^k T_{m-r}^{(0)}(n-r) \\ &\quad + \frac{(-1)^{n-m+1} n!}{(n-m+1)!} \sum_{\substack{j_1+j_2+\dots+j_{m-1}=k-m+1 \\ j_1, j_2, \dots, j_{m-1} \geq 0}} 2^{j_2} 3^{j_3} \dots (m-1)^{j_{m-1}} \sum_{j=0}^{j_1} \\ &\quad \times \sum_{\kappa=0}^{n-m+1} \left[\begin{matrix} n-m+1 \\ \kappa \end{matrix} \right] (-1)^\kappa (m+\kappa)^j \\ &= \sum_{r=0}^{m-2} \frac{n!}{(n-r)!} \left\{ \begin{matrix} k+1 \\ r+1 \end{matrix} \right\} T_{m-r}^{(0)}(n-r) \\ &\quad + \frac{(-1)^{n-m+1} n!}{(n-m+1)!} \sum_{j=0}^{k-m+1} \left\{ \begin{matrix} k-j \\ m-1 \end{matrix} \right\} \sum_{\kappa=0}^{n-m+1} \left[\begin{matrix} n-m+1 \\ \kappa \end{matrix} \right] (-1)^\kappa (m+\kappa)^j. \end{aligned}$$

§ 2.1. Proof of Theorem 2.2

Lemma 2.3. For $r \geq 0$ we have

$$\sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{i+1-y} = \frac{r!}{(1-y)(2-y)\cdots(r+1-y)}.$$

Proof. The proof is done by induction on r . See also e.g. [7, (5.41)]. □

Lemma 2.4. For $n \geq 0$ and $m \geq 1$ we have

$$\frac{1}{n+1-y} = \sum_{r=0}^{m-1} (-1)^r \binom{n}{r} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{i+1-y} + \frac{(-1)^m \binom{n}{m} m!}{(1-y)\cdots(m-y)(n+1-y)}.$$

Proof. Put

$$f_n(m) = \sum_{r=0}^{m-1} \binom{n}{r} \frac{(-1)^r r!}{(1-y)\cdots(r+1-y)} + \frac{(-1)^m \binom{n}{m} m!}{(1-y)\cdots(m-y)(n+1-y)}.$$

When $m = 1$, it is easy to see that $f_n(1) = 1/(n+1-y)$. For $m \geq 1$

$$\begin{aligned} f_n(m+1) - f_n(m) &= \frac{(-1)^m \binom{n}{m} m!}{(1-y)\cdots(m+1-y)} + \frac{(-1)^{m+1} \binom{n}{m+1} (m+1)!}{(1-y)\cdots(m+1-y)(n+1-y)} \\ &\quad - \frac{(-1)^m \binom{n}{m} m!}{(1-y)\cdots(m-y)(n+1-y)} \\ &= \frac{(-1)^m \binom{n}{m} m!}{(1-y)\cdots(m+1-y)(n+1-y)} \left((n+1-y) - (m+1) \frac{n-m}{m+1} - (m+1-y) \right) \\ &= 0. \end{aligned}$$

Hence, for any integer $m \geq 1$, $f_n(m) = 1/(n+1-y)$. Together with Lemma 2.3, the proof is done. □

Proposition 2.5. For $k \geq 1$ and $m \geq 2$ we have

$$\begin{aligned} (2.6) \quad G_k(x) &= \sum_{r=0}^{m-2} (1+x) \frac{(-\ln(1+x))^r}{r!} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{(i+1)^k} \\ &\quad + \sum_{\substack{j_1+\cdots+j_{m-1}=k+m-2 \\ j_1, \dots, j_{m-1} \geq 1}} 2^{-j_2} 3^{-j_3} \cdots (m-1)^{-j_{m-1}} \sum_{j=1}^{j_1} \sum_{l=0}^{m-1} (-1)^l \left[\begin{matrix} m \\ l+1 \end{matrix} \right] G_{j-l}(x). \end{aligned}$$

Proof. We shall prove

$$(2.7) \quad \frac{1}{(n+1)^k} = \sum_{r=0}^{m-1} (-1)^r \binom{n}{r} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{(i+1)^k} + \frac{1}{n+1} \sum_{\substack{j_1+\cdots+j_m \leq k-1 \\ j_1, \dots, j_m \geq 0}} \frac{(-1)^m \binom{n}{m}}{2^{j_2} \cdots m^{j_m} (n+1)^{j_1}}.$$

Taking the infinite sum $\sum_{k=1}^{\infty} y^{k-1}$ for small $y > 0$, the left-hand side is equal to $1/(n+1-y)$. The right-hand side equals

$$\begin{aligned}
& \sum_{r=0}^{m-1} (-1)^r \binom{n}{r} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{i+1} \sum_{k=1}^{\infty} \left(\frac{y}{i+1} \right)^{k-1} \\
& + \frac{1}{n+1} \sum_{j_1, \dots, j_m \geq 0} \frac{(-1)^m \binom{n}{m}}{2^{j_2} \dots m^{j_m} (n+1)^{j_1}} \sum_{l=j_1+\dots+j_m} y^l \\
& = \sum_{r=0}^{m-1} (-1)^r \binom{n}{r} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{i+1-y} \\
& + \frac{1}{n+1} \sum_{j_1, \dots, j_m \geq 0} \frac{(-1)^m \binom{n}{m}}{2^{j_2} \dots m^{j_m} (n+1)^{j_1}} \frac{y^{j_1+\dots+j_m}}{1-y} \\
& = \sum_{r=0}^{m-1} (-1)^r \binom{n}{r} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{i+1-y} \\
& + \frac{(-1)^m \binom{n}{m} m!}{(1-y) \dots (m-y) \cdot (n+1-y)}.
\end{aligned}$$

By Lemma 2.4, the identity (2.7) holds.

By taking the summation $\sum_{n=0}^{\infty} (\ln(1+x))^n/n!$ on both sides of (2.7), we have

$$\begin{aligned}
(2.8) \quad \sum_{n=0}^{\infty} \frac{(\ln(1+x))^n}{n!(n+1)^k} &= \sum_{r=0}^{m-1} \frac{(1+x)(-\ln(1+x))^r}{r!} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{(i+1)^k} \\
&+ \sum_{n=0}^{\infty} \sum_{\substack{j_1+\dots+j_m \leq k-1 \\ j_1, \dots, j_m \geq 0}} \frac{(-1)^m \binom{n}{m}}{2^{j_2} \dots m^{j_m} (n+1)^{j_1}} \frac{(\ln(1+x))^n}{(n+1)!}.
\end{aligned}$$

Note that $n+r$ is replaced by n and

$$1+x = \sum_{n=0}^{\infty} \frac{(\ln(1+x))^n}{n!}.$$

Since

$$G_k(x) = \sum_{n=0}^{\infty} \frac{(\ln(1+x))^n}{n!(n+1)^k}$$

and

$$\sum_{l=0}^m (-1)^l \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} (n+1)^l = (-1)^m \binom{n}{m} m!$$

(see e.g. [7, (6.13)]), by replacing m by $m-1$, we get the result. $\square$

Proof of Theorem 2.2. By (2.1) we have

$$\left(\frac{x}{\ln(1+x)} \right)^{m-1} G_0(x) = \left(\frac{x}{\ln(1+x)} \right)^{m-2} G_1(x) + x \left(\frac{x}{\ln(1+x)} \right)^{m-2} G_1(x).$$

Hence, the identity (2.3) holds. We shall prove the identity (2.4). The identity (2.5) can be proven similarly.

Replacing m by $m-1$ and k by j in Lemma 2.1, and multiplying both sides by $(1+x)^{m-1}x^{m-1}$, we have

$$\begin{aligned}
& \left(\frac{x}{\ln(1+x)} \right)^{m-1} \sum_{l=0}^{m-1} (-1)^{m-l-1} \begin{bmatrix} m \\ l+1 \end{bmatrix} G_{j-l}(x) \\
&= (1+x)^{m-1} x^{m-1} \sum_{t=0}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \frac{1}{(1+x)^t} \frac{d^{m-t-1}}{dx^{m-t-1}} G_j(x) \\
&= (1+x)^{m-1} x^{m-1} \sum_{t=0}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \frac{1}{(1+x)^t} \sum_{n=0}^{\infty} c_{m+n-t-1}^{(j)} \frac{x^n}{n!} \\
&= \sum_{t=0}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} (1+x)^{m-t-1} \sum_{n=0}^{\infty} \frac{n!}{(n-m+1)!} c_{n-t}^{(j)} \frac{x^n}{n!} \\
&= \sum_{t=0}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \sum_{r=0}^{m-t-1} \binom{m-t-1}{r} x^r \sum_{n=0}^{\infty} \frac{n!}{(n-m+1)!} c_{n-t}^{(j)} \frac{x^n}{n!} \\
&= \sum_{t=0}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \sum_{r=0}^{m-t-1} \binom{m-t-1}{r} \sum_{n=0}^{\infty} \frac{n!}{(n-r-m+1)!} c_{n-r-t}^{(j)} \frac{x^n}{n!} \\
&= \sum_{t=0}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \sum_{\kappa=0}^{m-1} \binom{m-t-1}{m-\kappa-1} \sum_{n=0}^{\infty} \frac{n!}{(n-\kappa+t-m+1)!} c_{n-\kappa}^{(j)} \frac{x^n}{n!} \quad (\kappa = r+t) \\
&= \sum_{n=0}^{\infty} \frac{n!}{(n-m+1)!} \sum_{\kappa=0}^{m-1} P_{m,\kappa}(n) c_{n-\kappa}^{(j)} \frac{x^n}{n!}.
\end{aligned}$$

Thus, by Proposition 2.5 we obtain

$$\begin{aligned}
& \left(\frac{x}{\ln(1+x)} \right)^{m-1} G_k(x) \\
&= \sum_{r=0}^{m-2} \frac{(-x)^r}{r!} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{(i+1)^k} \sum_{n=r}^{\infty} T_{m-r}^{(0)}(n-r) \frac{x^{n-r}}{(n-r)!} \\
&\quad + (-1)^{m-1} \left(\frac{x}{\ln(1+x)} \right)^{m-1} \sum_{\substack{j_1+\dots+j_{m-1}=k+m-2 \\ j_1, \dots, j_{m-1} \geq 1}} 2^{-j_2} 3^{-j_3} \dots (m-1)^{-j_{m-1}} \sum_{j=1}^{j_1} \\
&\quad \times \sum_{l=0}^{m-1} (-1)^{m-l-1} \begin{bmatrix} m \\ l+1 \end{bmatrix} G_{j-l}(x).
\end{aligned}$$

Comparing the coefficients of $x^n/n!$, we have the identity (2.4). $\square$

§ 2.2. Explicit formulae for $T_4^{(k)}(n)$

By using a little different method from those in [8] and [14], we may continue to obtain the explicit expressions of $T_m^{(k)}(n)$ for $m = 4, 5, \dots$. However, as special cases of Theorem 2.2,

immediately we can get explicit expressions of $T_m^{(k)}(n)$ for any integer $m \geq 2$. Expressions of $T_2^{(k)}(n)$ and $T_3^{(3)}(n)$ are already known ([14]). Here are explicit formulae for $T_4^{(k)}(n)$.

Corollary 2.6. *For $n \geq 0$ and $k > 0$ we have*

$$(2.9) \quad T_4^{(0)}(n) = T_3^{(1)}(n) + nT_3^{(1)}(n-1),$$

$$T_4^{(k)}(n) = T_4^{(0)}(n) - \left(1 - \frac{1}{2^k}\right)nT_3^{(0)}(n-1) + \left(\frac{1}{2} - \frac{1}{2^k} + \frac{1}{2 \cdot 3^k}\right)n(n-1)T_2^{(0)}(n-2)$$

$$- n(n-1)(n-2) \sum_{\mu=1}^k 3^{\mu-k-1} \sum_{\nu=1}^{\mu} 2^{\nu-\mu-1}$$

$$(2.10) \quad \times \sum_{j=1}^{\nu} ((n-3)^3 c_{n-3}^{(j)} + (3n^2 - 15n + 19)c_{n-2}^{(j)} + 3(n-2)c_{n-1}^{(j)} + c_n^{(j)})$$

$$T_4^{(-k)}(n) = T_4^{(0)}(n) - \left(1 - 2^k\right)nT_3^{(0)}(n-1) + \left(\frac{1}{2} - 2^k + \frac{3^k}{2}\right)n(n-1)T_2^{(0)}(n-2)$$

$$+ n(n-1)(n-2) \sum_{\mu=0}^{k-3} 2^{k-\mu-3} \sum_{\nu=0}^{\mu} 3^{\mu-\nu}$$

$$(2.11) \quad \times \sum_{j=0}^{\nu} ((n-3)^3 c_{n-3}^{(-j)} + (3n^2 - 15n + 19)c_{n-2}^{(-j)} + 3(n-2)c_{n-1}^{(-j)} + c_n^{(-j)}).$$

§ 3. Poly-Cauchy numbers of the second kind

Similarly to $T_m^{(k)}(n)$, define $\hat{T}_m^{(k)}(n)$ by

$$\hat{T}_m^{(k)}(n) := \sum_{\substack{i_1 + \dots + i_m = n \\ i_1, \dots, i_m \geq 0}} \binom{n}{i_1, \dots, i_m} \underbrace{\hat{c}_{i_1} \dots \hat{c}_{i_{m-1}}}_{m-1} \hat{c}_{i_m}^{(k)} \quad (m \geq 1, n \geq 0),$$

where $\hat{c}_n^{(k)}$ are poly-Cauchy numbers of the second kind ([12]), defined by

$$\hat{c}_n^{(k)} = \underbrace{\int_0^1 \dots \int_0^1}_{k} (-x_1 x_2 \dots x_k) (-x_1 x_2 \dots x_k - 1) \dots (-x_1 x_2 \dots x_k - n + 1) dx_1 dx_2 \dots dx_k.$$

The generating function of $\hat{c}_n^{(k)}$ is given by

$$\hat{G}_k(x) := \text{Lif}_k(-\ln(1+x)) = \sum_{n=0}^{\infty} \hat{c}_n^{(k)} \frac{x^n}{n!}.$$

Note that $\hat{c}_n^{(k)}$ for $k \leq 0$ are also defined by this generating function.

§ 3.1. Explicit formulae for $\hat{T}_m^{(k)}(n)$

Theorem 3.1. For $n \geq 0$ and $k > 0$ we have

(3.1)

$$\hat{T}_m^{(0)}(n) = (-1)^n \sum_{l=0}^n (-1)^l \frac{n!}{l!} \hat{T}_{m-1}^{(1)}(l),$$

(3.2)

$$\begin{aligned} \hat{T}_m^{(k)}(n) &= \hat{T}_m^{(0)}(n) + \sum_{r=1}^{m-2} \frac{1}{r!} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^{i+n-r}}{(i+1)^k} \sum_{l=0}^{n-r} \binom{n-l-1}{r-1} (-1)^l \frac{n!}{l!} \hat{T}_{m-r}^{(0)}(l) \\ &\quad + (-1)^{m-1} n! \sum_{\substack{j_1+j_2+\dots+j_{m-1}=k+m-2 \\ j_1, j_2, \dots, j_{m-1} \geq 1}} 2^{-j_2} 3^{-j_3} \dots (m-1)^{-j_{m-1}} \sum_{j=1}^{j_1} \\ &\quad \times \left(\frac{\hat{c}_n^{(j)}}{(n-m+1)!} + \sum_{t=1}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \sum_{l=0}^{n-m+1} (-1)^{n-m-l+1} \binom{n+t-m-l}{t-1} \frac{\hat{c}_{m+l-t-1}^{(j)}}{l!} \right), \end{aligned}$$

(3.3)

$$\begin{aligned} \hat{T}_m^{(-k)}(n) &= \hat{T}_m^{(0)}(n) + \sum_{r=1}^{m-2} \frac{1}{r!} \sum_{i=0}^r \binom{r}{i} (-1)^{i+n-r} (i+1)^k \sum_{l=0}^{n-r} \binom{n-l-1}{r-1} (-1)^l \frac{n!}{l!} \hat{T}_{m-r}^{(0)}(l) \\ &\quad + n! \sum_{\substack{j_1+j_2+\dots+j_{m-1}=k-m+1 \\ j_1, j_2, \dots, j_{m-1} \geq 0}} 2^{j_2} 3^{j_3} \dots (m-1)^{j_{m-1}} \sum_{j=0}^{j_1} \\ &\quad \times \left(\frac{\hat{c}_n^{(-j)}}{(n-m+1)!} + \sum_{t=1}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \sum_{l=0}^{n-m+1} (-1)^{n-m-l+1} \binom{n+t-m-l}{t-1} \frac{\hat{c}_{m+l-t-1}^{(-j)}}{l!} \right). \end{aligned}$$

Remark. The identities in above Theorem can be also written as

$$\begin{aligned} \hat{T}_m^{(0)}(n) &= (-1)^n \sum_{l=0}^n (-1)^l \frac{n!}{l!} \hat{T}_{m-1}^{(1)}(l), \\ \hat{T}_m^{(k)}(n) &= \hat{T}_m^{(0)}(n) + \sum_{r=1}^{m-2} \frac{1}{r!} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{(i+1)^k} (-1)^{n-r} \sum_{l=0}^{n-r} \binom{n-l-1}{r-1} (-1)^l \frac{n!}{l!} \hat{T}_{m-r}^{(0)}(l) \\ &\quad + \frac{(-1)^{n-m+1} n!}{(n-m+1)!} \sum_{\substack{j_1+j_2+\dots+j_{m-1}=k+m-2 \\ j_1, j_2, \dots, j_{m-1} \geq 1}} 2^{-j_2} 3^{-j_3} \dots (m-1)^{-j_{m-1}} \sum_{j=1}^{j_1} \sum_{\lambda=0}^{n-m+1} \frac{1}{(m+\lambda)^j} \\ &\quad \times \sum_{\mu=0}^{n-m-\lambda+1} \binom{\lambda+\mu}{\lambda} \begin{bmatrix} n-m+1 \\ \lambda+\mu \end{bmatrix} (m-1)^\mu, \end{aligned}$$

$$\begin{aligned}
\hat{T}_m^{(-k)}(n) &= \hat{T}_m^{(0)}(n) + \sum_{r=1}^{m-2} (-1)^n \left\{ \begin{matrix} k+1 \\ r+1 \end{matrix} \right\} \sum_{l=0}^{n-r} \binom{n-l-1}{r-1} (-1)^l \frac{n!}{l!} \hat{T}_{m-r}^{(0)}(l) \\
&+ \frac{(-1)^n n!}{(n-m+1)!} \sum_{j=0}^{k-m+1} \left\{ \begin{matrix} k-j \\ m-1 \end{matrix} \right\} \sum_{\lambda=0}^{n-m+1} (m+\lambda)^j \\
&\times \sum_{\mu=0}^{n-m-\lambda+1} \binom{\lambda+\mu}{\lambda} \left[\begin{matrix} n-m+1 \\ \lambda+\mu \end{matrix} \right] (m-1)^\mu.
\end{aligned}$$

Proof. Notice that

$$\begin{aligned}
\frac{x^{m-1}}{(1+x)^m (\ln(1+x))^{m-r-1}} &= \left(\frac{x}{1+x} \right)^r \left(\frac{x}{(1+x) \ln(1+x)} \right)^{m-r-1} \frac{1}{1+x} \\
&= \left(\frac{x}{1+x} \right)^r \sum_{n=0}^{\infty} T_{m-r}^{(0)}(n) \frac{x^n}{n!} \\
&= x^r \left(\sum_{\mu=0}^{\infty} (-1)^\mu \binom{\mu+r-1}{\mu} x^\mu \right) \left(\sum_{l=0}^{\infty} T_{m-r}^{(0)}(l) \frac{x^l}{l!} \right) \\
&= x^r \sum_{n=0}^{\infty} \sum_{l=0}^n (-1)^{n-l} \binom{n+r-l-1}{r-1} \frac{T_{m-r}^{(0)}(l)}{l!} x^n \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^n (-1)^{n-l} \binom{n+r-l-1}{r-1} (n+r)! \frac{T_{m-r}^{(0)}(l)}{l!} \frac{x^{n+r}}{(n+r)!} \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^{n-r} (-1)^{n-r-l} \binom{n-l-1}{r-1} \frac{n!}{l!} T_{m-r}^{(0)}(l) \frac{x^n}{n!}.
\end{aligned}$$

Hence, similarly to the identity (2.6), for $k \geq 1$ and $m \geq 2$ we have

$$\begin{aligned}
(3.4) \quad \hat{G}_k(x) &= \hat{G}_0(x) + \sum_{r=1}^{m-2} \frac{(\ln(1+x))^r}{(1+x)r!} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{(i+1)^k} \\
&+ \sum_{\substack{j_1+\dots+j_{m-1}=k+m-2 \\ j_1, \dots, j_{m-1} \geq 1}} 2^{-j_2} 3^{-j_3} \dots (m-1)^{-j_{m-1}} \sum_{j=1}^{j_1} \sum_{l=0}^{m-1} (-1)^l \left[\begin{matrix} m \\ l+1 \end{matrix} \right] \hat{G}_{j-l}(x).
\end{aligned}$$

Now, we have

$$\begin{aligned}
& \left(\frac{x}{(1+x)\ln(1+x)} \right)^{m-1} \sum_{l=0}^{m-1} (-1)^{m-l-1} \begin{bmatrix} m \\ l+1 \end{bmatrix} \hat{G}_{j-l}(x) \\
&= x^{m-1} \sum_{t=0}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \frac{1}{(1+x)^t} \frac{d^{m-t-1}}{dx^{m-t-1}} \hat{G}_j(x) \\
&= x^{m-1} \sum_{t=0}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \frac{1}{(1+x)^t} \sum_{n=0}^{\infty} \hat{c}_{m+n-t-1}^{(j)} \frac{x^n}{n!} \\
&= \sum_{t=0}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \frac{1}{(1+x)^t} \sum_{n=0}^{\infty} \frac{n!}{(n-m+1)!} \hat{c}_{n-t}^{(j)} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{n!}{(n-m+1)!} \hat{c}_n^{(j)} \frac{x^n}{n!} \\
&\quad + \sum_{t=1}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \left(\sum_{\mu=0}^{\infty} (-1)^{\mu} \binom{t+\mu-1}{\mu} x^{\mu} \right) \left(\sum_{\nu=0}^{\infty} \frac{\nu!}{(\nu-m+1)!} \hat{c}_{\nu-t}^{(j)} \frac{x^{\nu}}{\nu!} \right) \\
&= \sum_{n=0}^{\infty} \frac{n!}{(n-m+1)!} \hat{c}_n^{(j)} \frac{x^n}{n!} \\
&\quad + \sum_{t=1}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \sum_{n=0}^{\infty} \sum_{\nu=0}^n (-1)^{n-\nu} \binom{t+n-\nu-1}{t-1} \frac{n!}{(\nu-m+1)!} \hat{c}_{\nu-t}^{(j)} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} n! \left(\frac{\hat{c}_n^{(j)}}{(n-m+1)!} \right. \\
&\quad \left. + \sum_{t=1}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \sum_{l=0}^{n-m+1} (-1)^{n-m-l+1} \binom{n+t-m-l}{t-1} \frac{\hat{c}_{m+l-t-1}^{(j)}}{l!} \right) \frac{x^n}{n!}.
\end{aligned}$$

By multiplying $(x/(1+x)\ln(1+x))^{m-1}$ on both sides of (3.5) and comparing the coefficients of $x^n/n!$, we obtain the identity (3.2). The identity (3.3) is similarly proven. The identity (3.1) is clear. $\square$

For Remark, we can insist that

$$\begin{aligned}
& \frac{\hat{c}_n^{(j)}}{(n-m+1)!} + \sum_{t=1}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \sum_{l=0}^{n-m+1} (-1)^{n-m-l+1} \binom{n+t-m-l}{t-1} \frac{\hat{c}_{m+l-t-1}^{(j)}}{l!} \\
&= \frac{(-1)^n}{(n-m+1)!} \sum_{\lambda=0}^{n-m+1} \frac{1}{(m+\lambda)^j} \sum_{\mu=0}^{n-m-\lambda+1} \binom{\lambda+\mu}{\lambda} \begin{bmatrix} n-m+1 \\ \lambda+\mu \end{bmatrix} (m-1)^{\mu}.
\end{aligned}$$

By putting $m = 4$ in Theorem 3.1, we get explicit formulae for $\hat{T}_4^{(k)}(n)$.

Corollary 3.2. *For $n \geq 0$ and $k > 0$ we have*

$$(3.5) \quad \hat{T}_4^{(0)}(n) = (-1)^n \sum_{l=0}^n (-1)^l \frac{n!}{l!} \hat{T}_3^{(1)}(l),$$

$$(3.6) \quad \begin{aligned} \hat{T}_4^{(k)}(n) = & \hat{T}_4^{(0)}(n) - \left(1 - \frac{1}{2^k}\right) (-1)^{n-1} \sum_{l=0}^{n-1} (-1)^l \frac{n!}{l!} \hat{T}_3^{(0)}(l) \\ & + \left(\frac{1}{2} - \frac{1}{2^k} + \frac{1}{2 \cdot 3^k}\right) (-1)^{n-2} \sum_{l=0}^{n-2} (n-l-1) (-1)^l \frac{n!}{l!} \hat{T}_2^{(0)}(l) \\ & - n(n-1)(n-2) \sum_{\mu=1}^k 3^{\mu-k-1} \sum_{\nu=1}^{\mu} 2^{\nu-\mu-1} \sum_{j=1}^{\nu} \left(\hat{c}_n^{(j)} \right. \\ & \left. + 3(-1)^{n-3} (n-3)! \sum_{l=0}^{n-3} (-1)^l \frac{\hat{c}_{l+2}^{(j)}}{l!} + (-1)^{n-3} (n-3)! \sum_{l=0}^{n-3} (n-l-2) (-1)^l \frac{\hat{c}_{l+1}^{(j)}}{l!} \right), \end{aligned}$$

$$(3.7) \quad \begin{aligned} \hat{T}_4^{(-k)}(n) = & \hat{T}_4^{(0)}(n) + \left(1 - 2^k\right) (-1)^{n-1} \sum_{l=0}^{n-1} (-1)^l \frac{n!}{l!} \hat{T}_3^{(0)}(l) \\ & + \left(\frac{1}{2} - 2^k + \frac{3^k}{2}\right) (-1)^{n-2} \sum_{l=0}^{n-2} (n-l-1) (-1)^l \frac{n!}{l!} \hat{T}_2^{(0)}(l) \\ & + n(n-1)(n-2) \sum_{\mu=0}^{k-3} 2^{k-\mu-3} \sum_{\nu=0}^{\mu} 3^{\mu-\nu} \sum_{j=0}^{\nu} \left(\hat{c}_n^{(-j)} \right. \\ & \left. + 3(-1)^{n-3} (n-3)! \sum_{l=0}^{n-3} (-1)^l \frac{\hat{c}_{l+2}^{(-j)}}{l!} + (-1)^{n-3} (n-3)! \sum_{l=0}^{n-3} (n-l-2) (-1)^l \frac{\hat{c}_{l+1}^{(-j)}}{l!} \right). \end{aligned}$$

§ 4. Two kinds of poly-Cauchy numbers

Define $U_m^{(k)}(n)$ by

$$U_m^{(k)}(n) := \sum_{\substack{i_1 + \dots + i_m = n \\ i_1, \dots, i_m \geq 0}} \binom{n}{i_1, \dots, i_m} \underbrace{c_{i_1} \dots c_{i_{m-1}}}_{m-1} \hat{c}_{i_m}^{(k)} \quad (m \geq 1, n \geq 0).$$

Then we obtain the following ([14, Theorem 7]).

Proposition 4.1. *For an integer k and a non-negative integer m , we have*

$$\begin{aligned} \sum_{l=0}^m (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} U_{m+1}^{(k-l)}(n) \\ = \begin{cases} \sum_{l=0}^m \sum_{i=0}^{n-m} \frac{n!}{i!} \binom{l}{n-m-i} \left\{ \begin{matrix} m \\ l \end{matrix} \right\} \hat{c}_{l+i}^{(k)} & (n \geq m); \\ 0 & (0 \leq n \leq m-1). \end{cases} \end{aligned}$$

Define $V_m^{(k)}(n)$ by

$$V_m^{(k)}(n) := \sum_{\substack{i_1 + \dots + i_m = n \\ i_1, \dots, i_m \geq 0}} \binom{n}{i_1, \dots, i_m} \underbrace{\hat{c}_{i_1} \cdots \hat{c}_{i_{m-1}}}_{m-1} c_{i_m}^{(k)} \quad (m \geq 1, n \geq 0).$$

Then we obtain the following ([14, Theorem 10]).

Proposition 4.2. *For an integer k and a non-negative integer m , we have*

$$\begin{aligned} & \sum_{l=0}^m (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} V_{m+1}^{(k-l)}(n) \\ &= \begin{cases} (-1)^{n-m} \sum_{l=0}^{m-1} \sum_{i=0}^{n-m} (-1)^i \frac{n!}{i!} \binom{n-l-i-1}{n-m-i} \left\{ \begin{matrix} m \\ l \end{matrix} \right\} c_{l+i}^{(k)} + \frac{n!}{(n-m)!} c_n^{(k)} & (n \geq m); \\ 0 & (0 \leq n \leq m-1). \end{cases} \end{aligned}$$

§ 4.1. Explicit formulae for $U_m^{(k)}(n)$

We can obtain explicit formulae for $U_m^{(k)}(n)$. The proof is similar to the case of $T_m^{(k)}(n)$ and is omitted.

Theorem 4.3. *For $n \geq 0$ and $k > 0$ we have*

$$(4.1) \quad U_m^{(0)}(n) = U_{m-1}^{(1)}(n),$$

$$\begin{aligned} (4.2) \quad U_m^{(k)}(n) &= \sum_{r=0}^{m-2} \binom{n}{r} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{(i+1)^k} U_{m-r}^{(0)}(n-r) \\ &+ \frac{(-1)^{m-1} n!}{(n-m+1)!} \sum_{\substack{j_1 + j_2 + \dots + j_{m-1} = k+m-2 \\ j_1, j_2, \dots, j_{m-1} \geq 1}} 2^{-j_2} 3^{-j_3} \cdots (m-1)^{-j_{m-1}} \sum_{j=1}^{j_1} \sum_{\kappa=0}^{m-1} P_{m,\kappa}(n) \hat{c}_{n-\kappa}^{(j)}, \end{aligned}$$

$$\begin{aligned} (4.3) \quad U_m^{(-k)}(n) &= \sum_{r=0}^{m-2} \binom{n}{r} \sum_{i=0}^r \binom{r}{i} (-1)^i (i+1)^k U_{m-r}^{(0)}(n-r) \\ &+ \frac{n!}{(n-m+1)!} \sum_{\substack{j_1 + j_2 + \dots + j_{m-1} = k-m+1 \\ j_1, j_2, \dots, j_{m-1} \geq 0}} 2^{j_2} 3^{j_3} \cdots (m-1)^{j_{m-1}} \sum_{j=0}^{j_1} \sum_{\kappa=0}^{m-1} P_{m,\kappa}(n) \hat{c}_{n-\kappa}^{(-j)}. \end{aligned}$$

where $P_{m,\kappa}(n)$ ($\kappa = 0, 1, \dots, m-2, m-1$) are given as in Theorem 2.2.

Remark. The identities in above Theorem can be also written as

$$\begin{aligned}
U_m^{(k)}(n) &= \sum_{r=0}^{m-2} \binom{n}{r} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{(i+1)^k} U_{m-r}^{(0)}(n-r) \\
&\quad + \frac{(-1)^{n-m+1} n!}{(n-m+1)!} \sum_{\substack{j_1+j_2+\dots+j_{m-1}=k+m-2 \\ j_1, j_2, \dots, j_{m-1} \geq 1}} 2^{-j_2} 3^{-j_3} \dots (m-1)^{-j_{m-1}} \sum_{j=1}^{j_1} \\
&\quad \times \sum_{\kappa=0}^{n-m+1} \begin{bmatrix} n-m+1 \\ \kappa \end{bmatrix} \frac{1}{(m+\kappa)^j}, \\
U_m^{(-k)}(n) &= \sum_{r=0}^{m-2} \frac{(-1)^r n!}{(n-r)!} \left\{ \begin{matrix} k+1 \\ r+1 \end{matrix} \right\} U_{m-r}^{(0)}(n-r) \\
&\quad + \frac{(-1)^{n-m} n!}{(n-m+1)!} \sum_{j=0}^{k-m+1} \left\{ \begin{matrix} k-j \\ m-1 \end{matrix} \right\} \sum_{\kappa=0}^{n-m+1} \begin{bmatrix} n-m+1 \\ \kappa \end{bmatrix} (m+\kappa)^j.
\end{aligned}$$

§ 4.2. Explicit formulae for $V_m^{(k)}(n)$

We can obtain explicit formulae for $V_m^{(k)}(n)$. The proof is similar to the case of $\hat{T}_m^{(k)}(n)$ and is omitted.

Theorem 4.4. For $n \geq 0$ and $k > 0$ we have

(4.4)

$$V_m^{(0)}(n) = V_{m-1}^{(1)}(n),$$

(4.5)

$$\begin{aligned}
V_m^{(k)}(n) &= V_m^{(0)}(n) + \sum_{r=1}^{m-2} \frac{1}{r!} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^{i+n}}{(i+1)^k} \sum_{l=0}^{n-r} \binom{n-l-1}{r-1} (-1)^l \frac{n!}{l!} V_{m-r}^{(0)}(l) \\
&\quad + (-1)^{m-1} n! \sum_{\substack{j_1+j_2+\dots+j_{m-1}=k+m-2 \\ j_1, j_2, \dots, j_{m-1} \geq 1}} 2^{-j_2} 3^{-j_3} \dots (m-1)^{-j_{m-1}} \sum_{j=1}^{j_1} \\
&\quad \times \left(\frac{\hat{c}_n^{(j)}}{(n-m+1)!} + \sum_{t=1}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \sum_{l=0}^{n-m+1} (-1)^{n-m-l+1} \binom{n+t-m-l}{t-1} \frac{\hat{c}_{m+l-t-1}^{(j)}}{l!} \right),
\end{aligned}$$

(4.6)

$$\begin{aligned}
V_m^{(-k)}(n) &= V_m^{(0)}(n) + \sum_{r=1}^{m-2} \frac{1}{r!} \sum_{i=0}^r \binom{r}{i} (-1)^{i+n} (i+1)^k \sum_{l=0}^{n-r} \binom{n-l-1}{r-1} (-1)^l \frac{n!}{l!} V_{m-r}^{(0)}(l) \\
&\quad + n! \sum_{\substack{j_1+j_2+\dots+j_{m-1}=k-m+1 \\ j_1, j_2, \dots, j_{m-1} \geq 0}} 2^{j_2} 3^{j_3} \dots (m-1)^{j_{m-1}} \sum_{j=0}^{j_1} \\
&\quad \times \left(\frac{\hat{c}_n^{(-j)}}{(n-m+1)!} + \sum_{t=1}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \sum_{l=0}^{n-m+1} (-1)^{n-m-l+1} \binom{n+t-m-l}{t-1} \frac{\hat{c}_{m+l-t-1}^{(-j)}}{l!} \right).
\end{aligned}$$

Remark. The identities in above Theorem can be also written as

$$\begin{aligned}
 V_m^{(k)}(n) &= V_m^{(0)}(n) + \sum_{r=1}^{m-2} \frac{1}{r!} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{(i+1)^k} (-1)^n \sum_{l=0}^{n-r} \binom{n-l-1}{r-1} (-1)^l \frac{n!}{l!} V_{m-r}^{(0)}(l) \\
 &\quad + \frac{(-1)^{n-m+1} n!}{(n-m+1)!} \sum_{\substack{j_1+j_2+\dots+j_{m-1}=k+m-2 \\ j_1, j_2, \dots, j_{m-1} \geq 1}} 2^{-j_2} 3^{-j_3} \dots (m-1)^{-j_{m-1}} \sum_{j=1}^{j_1} \\
 &\quad \times \sum_{\lambda=0}^{n-m+1} \frac{(-1)^{\lambda+1}}{(m+\lambda)^j} \sum_{\mu=0}^{n-m-\lambda+1} \binom{\lambda+\mu}{\lambda} \begin{bmatrix} n-m+1 \\ \lambda+\mu \end{bmatrix} (m-1)^\mu, \\
 V_m^{(-k)}(n) &= V_m^{(0)}(n) + \sum_{r=1}^{m-2} (-1)^{n-r} \left\{ \begin{matrix} k+1 \\ r+1 \end{matrix} \right\} \sum_{l=0}^{n-r} \binom{n-l-1}{r-1} (-1)^l \frac{n!}{l!} V_{m-r}^{(0)}(l) \\
 &\quad + \frac{(-1)^n n!}{(n-m+1)!} \sum_{j=0}^{k-m+1} \left\{ \begin{matrix} k-j \\ m-1 \end{matrix} \right\} \sum_{\lambda=0}^{n-m+1} (m+\lambda)^j (-1)^{\lambda+1} \\
 &\quad \times \sum_{\mu=0}^{n-m-\lambda+1} \binom{\lambda+\mu}{\lambda} \begin{bmatrix} n-m+1 \\ \lambda+\mu \end{bmatrix} (m-1)^\mu.
 \end{aligned}$$

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