

UNKNOTTING NUMBERS OF DIAGRAMS OF A GIVEN NONTRIVIAL KNOT ARE UNBOUNDED

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概要

任意の非自明結び目 K と任意の自然数 n に対して、 K のあるダイアグラム D が存在して D の結び目解消数は n 以上となる。 K の結び目解消数の 2 倍が K の最小交点数から 1 引いたもの以下であることはよく知られている。ここで等式が成り立つのは K が $(2, p)$ -トーラス結び目であるときに限る。

Let L be a link in the 3-sphere S^3 and D a diagram of L on the 2-sphere S^2 . It is well known that by changing over/under information at some crossings of D we have a diagram of a trivial link. See for example [3]. Let $u(D)$ be the minimal number of such crossing changes. Namely, there are some $u(D)$ crossings of D such that changing them yields a trivial link diagram, and changing less than $u(D)$ crossings never yields a trivial link diagram. We call $u(D)$ the *unlinking number* of D . In the case that D is a diagram of a knot $u(D)$ is called the *unknotting number* of D . The *unlinking number* $u(L)$ of L is defined by the minimum of $u(D)$ where D varies over all diagrams of L . Namely we have the following equality.

$$u(L) = \min\{u(D) \mid D \text{ is a diagram of } L\}.$$

For a knot K $u(K)$ is called the *unknotting number* of K . Then it is natural to ask whether or not the set $\{u(D) \mid D \text{ is a diagram of } L\}$ is bounded above. In [1] Nakanishi showed that an unknotting number one knot 6_2 has an unknotting number two diagram. Then he showed the following theorem in [2].

Theorem 1 [2]. *Let K be a nontrivial knot. Then K has a diagram D with $u(D) \geq 2$.*

As an extension of Theorem 1, we have the following theorem.

Theorem 2. *Let L be a nontrivial link. Then for any natural number n there exists a diagram D of L with $u(D) \geq n$. That is, the set $\{u(D) \mid D \text{ is a diagram of } L\}$ is unbounded above.*

We note that Theorem 2 is an immediate consequence of the following proposition.

Proposition 3. *Let L be a nontrivial link and D a diagram of L . Then there exists a diagram D' of L with $u(D') = u(D) + 2$.*

The proof of Proposition 3 is done by using a modification of diagram illustrated in Figure 1 that is essentially the same as that used in [2]. See [4] for the detail.

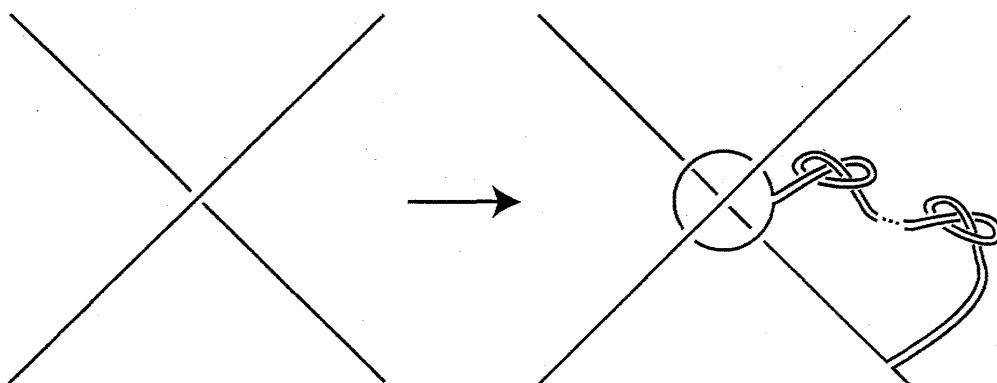


Figure 1

As an immediate consequence of Proposition 3 we have the following corollary.

Corollary 4. *Let L be a nontrivial link. Then the set $\{u(D) \mid D \text{ is a diagram of } L\}$ contains a set $\{u(L) + 2m \mid m \text{ is a non-negative integer}\}$.*

Question 5. *Let L be a nontrivial link. Is the set $\{u(D) \mid D \text{ is a diagram of } L\}$ equals the set $\{u(L) + m \mid m \text{ is a non-negative integer}\}$?*

The following proposition is a partial answer to Question 5.

Proposition 6. *Let L be an alternating link with $u(L) = 1$. Suppose that L has an alternating diagram D_0 with $u(D_0) = 1$. Then the set $\{u(D) \mid D \text{ is a diagram of } L\}$ equals the set of natural numbers $\{u(L) + m \mid m \text{ is a non-negative integer}\}$.*

Let $c(D)$ be the number of crossings in D . We call $c(D)$ the *crossing number* of D . Then the *crossing number* $c(L)$ of L is defined by the minimum of $c(D)$ where D varies over all diagrams of L . It is natural to ask the relation between $u(D)$ and $c(D)$, or $u(L)$ and $c(L)$. For a diagram D of a knot K other than a trivial diagram the following inequality is well-known. See for example [3].

$$u(K) \leq u(D) \leq \frac{c(D) - 1}{2}.$$

In particular this inequality holds for a minimal crossing diagram D of K where $c(D) = c(K)$. Thus for any nontrivial knot K we have the following inequality.

$$u(K) \leq \frac{c(K) - 1}{2}.$$

It is also well known that the equality holds for $(2, p)$ -torus knots. Conversely we have the following theorem.

Theorem 7. (1) *Let D be a diagram of a knot that satisfies the equality*

$$u(D) = \frac{c(D) - 1}{2}.$$

Then D is one of the diagrams illustrated in Figure 2. Namely D is a reduced alternating diagram of some $(2, p)$ -torus knot, or D is a diagram with just one crossing.

(2) Let K be a nontrivial knot that satisfies the equality

$$u(K) = \frac{c(K) - 1}{2}.$$

Then K is a $(2, p)$ -torus knot for some odd number $p \neq \pm 1$. Namely only 2-braid knots satisfy the equality.

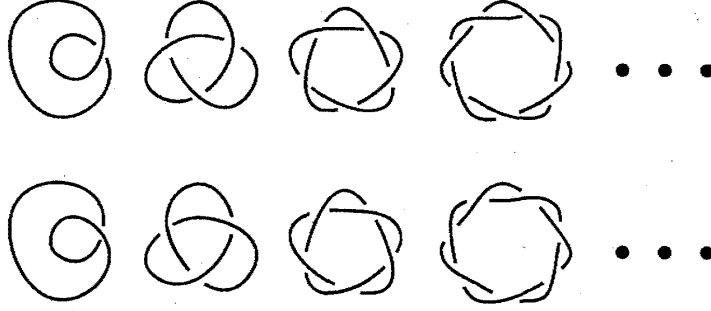


Figure 2

For links the situation is somewhat different. Let D be a diagram of a link. Then the following inequality is well-known.

$$u(L) \leq u(D) \leq \frac{c(D)}{2}.$$

Thus for any link L we have the following inequality.

$$u(L) \leq \frac{c(L)}{2}.$$

The following theorem shows that not only $(2, p)$ -torus links but some other links satisfy the equality.

Theorem 8. (1) Let $D = \gamma_1 \cup \cdots \cup \gamma_\mu$ be a diagram of a μ -component link that satisfies the equality

$$u(D) = \frac{c(D)}{2}.$$

Then each γ_i is a simple closed curve on $\mathbb{S}^2$ and for each pair i, j , the subdiagram $\gamma_i \cup \gamma_j$ is an alternating diagram or a diagram without crossings.

(2) Let L be a μ -component link that satisfies the equality

$$u(L) = \frac{c(L)}{2}.$$

Then L has a diagram $D = \gamma_1 \cup \cdots \cup \gamma_\mu$ such that each γ_i is a simple closed curve on $\mathbb{S}^2$ and for each pair i, j , the subdiagram $\gamma_i \cup \gamma_j$ is an alternating diagram or a diagram without crossings.

Two examples of such links are illustrated in Figure 3. We note that for a link described in Theorem 8 the unlinking number equals the sum of the absolute values of all pairwise linking numbers.

The detail will appear in [4].

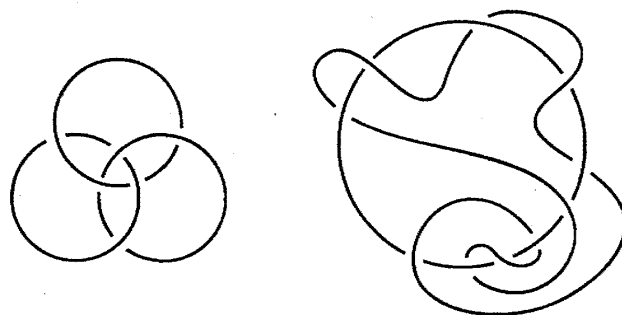


Figure 3

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