

Alternative Randomization for Valuing American Options*

北海道大学・経済学研究科

木村 俊一 (Toshikazu Kimura)

Graduate School of Economics

Hokkaido University

1 Introduction

European-style options, which can only be exercised at its maturity, have closed-form formulas for their values in the standard model pioneered by Black and Scholes [1] and Merton [8]. Although a vast majority of traded options are of American-style optimally exercised before the maturity, there are no closed-form formulas for their values even in the standard model. The principal difficulty in analyzing American options may be the absence of an explicit expression for the *early exercise boundary*, which is an optimal level of critical asset value where early exercise occurs.

Due to the lack of closed-form formulas for American option values, many approximate and/or numerical solutions have been developed so far. Broadie and Detemple [2] numerically evaluated recent methods for computing American option values. From those numerical experiments, it comes out that a numerical procedure developed by Carr [3] is fast and accurate among existing methods. Carr's procedure is based on valuing an American option with a randomized maturity, so that it is called the *randomization* approach. The purpose of this paper is to improve Carr's randomization approach by introducing alternative randomization based on an order statistic from an exponential population.

2 Free Boundary Problem

Let $(S_t)_{t \geq 0}$ be the stock price governed by the risk-neutralized diffusion process

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sigma dW_t, \quad t \geq 0, \quad (2.1)$$

where $r > 0$ is the risk-free interest rate, $\delta \geq 0$ is a continuous dividend rate, $\sigma > 0$ is a volatility of the asset returns, and $(W_t)_{t \geq 0}$ is a standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. We consider an American *put* option written on $(S_t)_{t \geq 0}$, which has maturity date T and strike price K . Let

$$P \equiv P(t, S_t) = P(t, S_t; K, r, \delta), \quad 0 \leq t \leq T,$$

denote the value of the American put option at time t . See Remark 1 below for the call value.

*This paper is an abbreviated version of Kimura [5].

McKean [7] showed that the alive American put value P and an early exercise boundary $(B_t)_{t \in [0, T]}$ can be jointly obtained by solving a *free boundary problem*, which is specified by the Black-Scholes-Merton PDE

$$\frac{1}{2}\sigma^2 S^2 P_{SS} + (r - \delta)SP_S - rP + P_t = 0, \quad S > B_t, \quad (2.2)$$

together with the boundary conditions

$$\lim_{S \uparrow \infty} P(t, S) = 0, \quad (2.3)$$

$$\lim_{S \downarrow B_t} P(t, S) = K - B_t, \quad (2.4)$$

$$\lim_{S \downarrow B_t} P_S(t, S) = -1, \quad (2.5)$$

and the terminal condition

$$P(T, S) = (K - S)^+. \quad (2.6)$$

The condition (2.4) is often called the *value-matching condition* and (2.5) is called the *smooth-pasting* or *high-contact condition*.

It is sometimes convenient to work with the equations where the current time t is replaced by the remaining time until maturity $s \equiv T - t$. From (2.2)–(2.6), the put price for the reversed process $\hat{P}(s, S_s) \equiv P(T - s, S_{T-s})$ satisfies the PDE

$$\frac{1}{2}\sigma^2 S^2 \hat{P}_{SS} + (r - \delta)S\hat{P}_S - r\hat{P} - \hat{P}_s = 0, \quad S > \hat{B}_s, \quad (2.7)$$

with the boundary conditions

$$\lim_{S \uparrow \infty} \hat{P}(s, S) = 0, \quad (2.8)$$

$$\lim_{S \downarrow \hat{B}_s} \hat{P}(s, S) = K - \hat{B}_s, \quad (2.9)$$

$$\lim_{S \downarrow \hat{B}_s} \hat{P}_S(s, S) = -1, \quad (2.10)$$

and the initial condition

$$\hat{P}(0, S) = (K - S)^+. \quad (2.11)$$

Remark 1 Let $C(0, S; K, r, \delta)$ denote the initial value of the associated American call option with the same parameters as those in the put option. McDonald and Schroder [6] proved the *parity relation*

$$C(0, S; K, r, \delta) = P(0, K; S, \delta, r). \quad (2.12)$$

Let $B_t^P \equiv B_t^P(K, r, \delta)$ and $B_t^C \equiv B_t^C(K, r, \delta)$ denote the early exercise boundaries of the American put and call options, respectively. Carr and Chesney [4] showed *symmetry relation*

$$B_t^C(K, r, \delta) = \frac{K^2}{B_t^P(K, \delta, r)}. \quad (2.13)$$

Due to the parity/symmetry relations, the results for the American call option can be derived from the associated put option.

3 Randomization Approach

Carr [3] developed a valuing method for the American put. Carr's randomization approach consists of the following steps:

1. Randomize the maturity date by an *exponentially* distributed random variable $\tilde{T}$ with mean $E[\tilde{T}] = \lambda^{-1} = T$ in order to value the so-called *Canadian option*.
2. Extend the result to the case that $\tilde{T}$ is distributed as the *n-stage Erlangian* distribution with the same mean $E[\tilde{T}] = T$.
3. Take the limit of the randomized option value by letting $n \rightarrow \infty$ to obtain the underlying American option value.

To understand the meaning of the step 3 above, Figure 1 illustrates the convergence of the *n-stage Erlangian* distribution to Dirac's delta function concentrated at the mean $\lambda^{-1} = 1$.

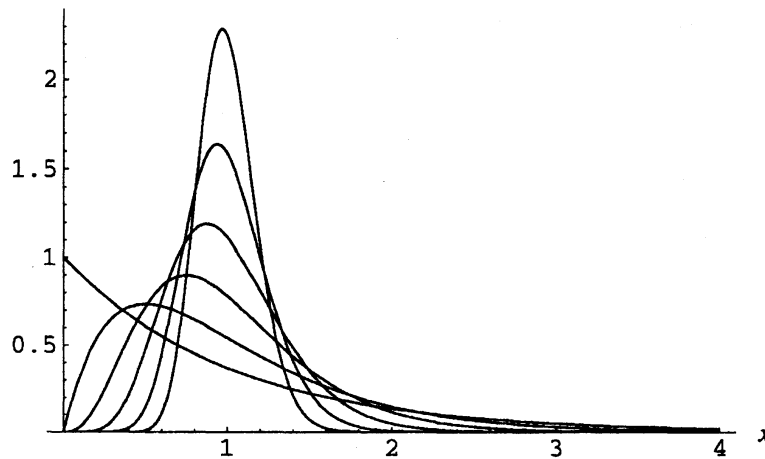


Figure 1: *n*-stage Erlangian pdf ($\lambda^{-1} = 1$, $n = 1, 2, 4, 8, 16, 32$)

Actually, the idea of Carr's randomization is *not* new. In the theory of integral transforms, this idea goes by the name of the Post-Widder inversion formula [9]: For a continuous function $g(t)$ ($t \geq 0$), define

$$g_n^*(T) = \int_0^\infty g(t) \frac{(nt/T)^{n-1} n}{(n-1)! T} e^{-nt/T} dt. \quad (3.1)$$

Then, we have

$$\lim_{n \rightarrow \infty} g_n^*(T) = g(T), \quad (3.2)$$

which is the essential point of Carr's randomization method.

For $\lambda > 0$, let

$$P^* \equiv P^*(\lambda, S) = \int_0^\infty \lambda e^{-\lambda s} \hat{P}(s, S) ds \quad (3.3)$$

be the Laplace-Carson transform (LCT) of $\hat{P}(s, S)$. Then, from (2.7)–(2.11), $P^*(\lambda, S)$ satisfies the ODE

$$\frac{1}{2}\sigma^2 S^2 P_{SS}^* + (r - \delta) S P_S^* - (\lambda + r) P^* + \lambda(K - S)^+ = 0, \quad S > L^*, \quad (3.4)$$

together with the boundary conditions

$$\lim_{S \uparrow \infty} P^*(\lambda, S) = 0, \quad (3.5)$$

$$\lim_{S \downarrow L^*} P^*(\lambda, S) = K - L^*, \quad (3.6)$$

$$\lim_{S \downarrow L^*} P_S^*(\lambda, S) = -1. \quad (3.7)$$

The early exercise boundary $L^* \equiv L^*(\lambda)$ is given by the LCT of $\hat{B}_s = B_{T-s}$

$$L^*(\lambda) = \int_0^\infty \lambda e^{-\lambda s} \hat{B}_s ds, \quad (3.8)$$

which is a *constant* due to the *memoryless property* of the exponential distribution.

Theorem 1

$$P^*(\lambda, S) = \begin{cases} K - S, & S \leq L^* \\ \frac{\lambda}{\lambda + r} K - \frac{\lambda}{\lambda + \delta} S + c(S) + b(S) + d(S), & L^* < S < K \\ p(S) + b(S) + d(S), & S \geq K, \end{cases} \quad (3.9)$$

where

$$c(S) = \frac{1}{\theta_+ - \theta_-} \frac{\lambda}{\lambda + \delta} \left(1 - \frac{r - \delta}{\lambda + r} \theta_- \right) K \left(\frac{S}{K} \right)^{\theta_+}, \quad (3.10)$$

$$p(S) = \frac{1}{\theta_+ - \theta_-} \frac{\lambda}{\lambda + \delta} \left(1 - \frac{r - \delta}{\lambda + r} \theta_+ \right) K \left(\frac{S}{K} \right)^{\theta_-}, \quad (3.11)$$

$$b(S) = -\frac{\theta_+}{\theta_-} c(L^*) \left(\frac{S}{L^*} \right)^{\theta_-}, \quad (3.12)$$

$$d(S) = -\frac{1}{\theta_-} \frac{\delta}{\lambda + \delta} L^* \left(\frac{S}{L^*} \right)^{\theta_-}, \quad (3.13)$$

and the parameters $\theta_{\pm}$ are two roots of the quadratic equation $\frac{1}{2}\sigma^2\theta^2 + (r - \delta - \frac{1}{2}\sigma^2)\theta - (\lambda + r) = 0$, i.e.,

$$\theta_{\pm} = \frac{1}{\sigma^2} \left\{ -(r - \delta - \frac{1}{2}\sigma^2) \pm \sqrt{(r - \delta - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda + r)} \right\}. \quad (3.14)$$

Proof. See Kimura [5]. □

Remark 2 The function $c(S)$ ($p(S)$) appeared in (3.9) can be interpreted as the randomized value of a European call (put) paying $(S - K)^+$ ($(K - S)^+$). Also, the function $b(S)$ ($d(S)$) can be interpreted as the present value of interest (dividends) received below the early exercise boundary L^* .

Remark 3 Carr's result for $b(S)$ (when $\delta = 0$) is *invalid*. The correct one is

$$b^{(1)}(S) = \left(\frac{S}{\underline{S}_1}\right)^{\gamma-\varepsilon} qK \left(RrT + \frac{1}{2\varepsilon p}\right) \left(\frac{\underline{S}_1}{K}\right)^{\gamma+\varepsilon}, \quad (3.15)$$

in terms of his notation; cf. [3, Equation (15)].

Theorem 2

(i) The early exercise boundary L^* of the Canadian-American put option satisfies the equation

$$\lambda \left(\frac{L^*}{K}\right)^{\theta_+} = r(\theta_+ - 1) - \delta\theta_+ \frac{L^*}{K}. \quad (3.16)$$

(ii) For the limiting case $\lambda \rightarrow 0$, we have

$$L^*(0) = \lim_{s \rightarrow \infty} \hat{B}_s = \frac{r(\theta_+^\circ - 1)}{\delta\theta_+^\circ} K = \frac{\theta_-^\circ}{\theta_-^\circ - 1} K, \quad (3.17)$$

where $\theta_\pm^\circ = \lim_{\lambda \rightarrow 0} \theta_\pm$. In particular, if $\delta = 0$, then

$$L^*(0) = \lim_{s \rightarrow \infty} \hat{B}_s = \frac{K}{1 + \frac{\sigma^2}{2r}}. \quad (3.18)$$

(iii) For the limiting case $\lambda \rightarrow \infty$, we have

$$\lim_{\lambda \rightarrow \infty} L^*(\lambda) = \hat{B}_0 = B_T = \min\left(\frac{r}{\delta}, 1\right) K. \quad (3.19)$$

Proof. See Kimura [5]. □

4 New Randomization Based on an Order Statistic

Let $X_1, \dots, X_{n+m}$ be independent and exponentially distributed random variables with parameter α (> 0), and let $X_{(i)}$ denote the i -th smallest of these random variables ($i = 1, \dots, n+m$). Then, the probability density function (pdf) of $X_{(n+1)}$ is

$$f(t) = \frac{(n+m)!}{n!(m-1)!} (1 - e^{-\alpha t})^n \alpha e^{-\alpha t}, \quad t \geq 0. \quad (4.1)$$

The mean and variance of $X_{(n+1)}$ are given by

$$E[X_{(n+1)}] = \frac{1}{\alpha} \sum_{i=0}^n \frac{1}{m+i} \approx \frac{1}{\alpha} \ln \frac{2n+2m+1}{2m-1}, \quad (4.2)$$

$$V[X_{(n+1)}] = \frac{1}{\alpha^2} \sum_{i=0}^n \frac{1}{(m+i)^2} \approx \frac{1}{\alpha^2} \ln \frac{2n+2}{(2m-1)(2n+2m+1)}. \quad (4.3)$$

In addition, the modal value of $X_{(n+1)}$ is

$$M[X_{(n+1)}] \equiv \arg \max_t f(t) = \frac{1}{\alpha} \ln \frac{n+m}{m}. \quad (4.4)$$

If we let either $E[X_{(n+1)}] = T$ or $M[X_{(n+1)}] = T$, then $X_{(n+1)}$ can be another candidate for the random maturity $\tilde{T}$, because $\lim_{n,m \rightarrow \infty} V[X_{(n+1)}] = 0$. For computational convenience, we adopt the mode matching $M[X_{(n+1)}] = T$, so that α can be determined as

$$\alpha = \frac{1}{T} \ln \frac{n+m}{m}. \quad (4.5)$$

Figure 2 shows the differences between the mean and mode matchings in the order-statistic-based randomization. From the figures (a) and (b), we find almost no differences between these matchings for large values of n .

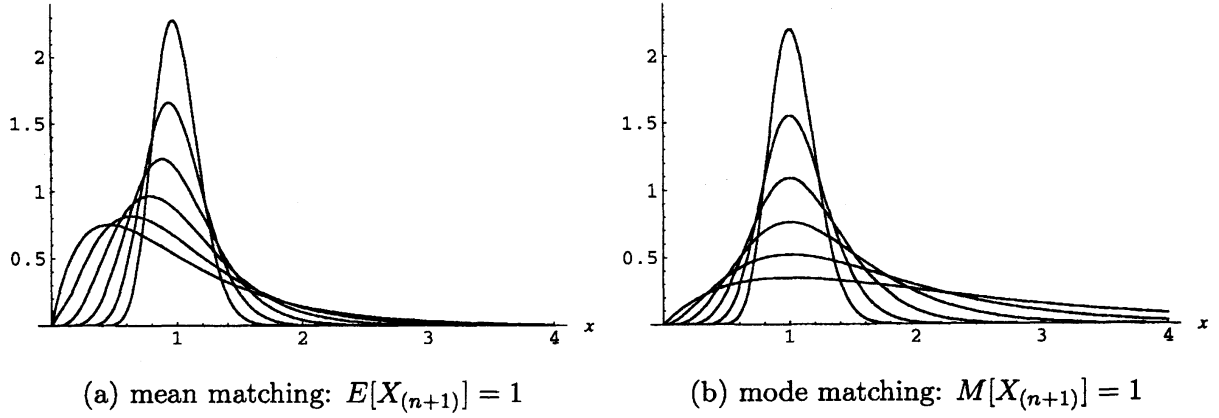


Figure 2: The pdf of the order statistic $X_{(n+1)}$ ($n = m = 1, 2, 4, 8, 16, 32$)

For a continuous function $g(t)$ ($t \geq 0$), define

$$g_{n,m}^*(T) = \frac{(n+m)!}{n!(m-1)!} \int_0^\infty g(t) (1 - e^{-\alpha t})^n \alpha e^{-m\alpha t} dt. \quad (4.6)$$

Then, we have

$$\lim_{n,m \rightarrow \infty} g_{n,m}^*(T) = g(T). \quad (4.7)$$

Theorem 3 The sequence $(g_{n,m}^*)_{n,m \geq 1}$ satisfies the recursion

$$\begin{aligned} g_{0,m}^*(T) &= \int_0^\infty m\alpha e^{-m\alpha t} g(t) dt \\ g_{n,m}^*(T) &= \frac{n+m}{n} g_{n-1,m}^*(T) - \frac{m}{n} g_{n-1,m+1}^*(T), \quad n \geq 1. \end{aligned} \quad (4.8)$$

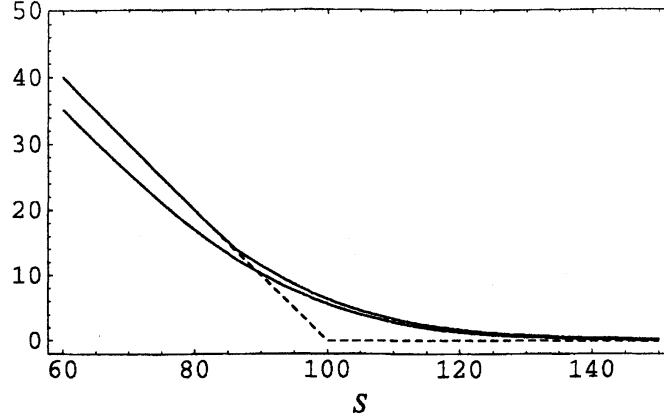


Figure 3: American & European put values
($t = 0, K = 100, T = 1, r = 0.05, \delta = 0, \sigma = 0.2$)

Proof. See Kimura [5]. □

A simple and practical setting for the parameters n and m is $n = m$. For a set of the parameters $\{t, S, K, T, r, \delta, \sigma\}$, if we have a functional program for computing $P^*(\lambda, S)$ for any $\lambda \geq 0$, then the N -th randomized approximation $\pi_N \equiv g_{N,N}^* \approx P(t, S)$ ($N \geq 1$) can be obtained by the following algorithm:

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 $\alpha = \frac{1}{T-t} \ln 2$ 
for  $m = N$  to  $2N$  do
   $g_{0,m}^* = P^*(m\alpha, S)$ 
next  $m$ 
for  $n = 1$  to  $N$  do
  for  $m = N$  to  $2N - n$  do
     $g_{n,m}^* = \frac{n+m}{n} g_{n-1,m}^* - \frac{m}{n} g_{n-1,m+1}^*$ 
  next  $m$ 
next  $n$ 
 $\pi_N = g_{N,N}^*$ 

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In order to speed up the convergence of N -th randomized approximation π_N , Carr [3] suggested using the Richardson extrapolation scheme. In this paper, however, we use another extrapolation scheme defined below, from the error analysis of the N -th approximation π_N ; see Kimura [5] for details.

$$\begin{cases} \pi_N^{(0)} = \pi_N, & N = 2^0, 2^1, 2^2, \dots \\ \pi_N^{(k)} = \frac{1}{2^k - 1} \left\{ 2^k \pi_N^{(k-1)} - \pi_{\frac{N}{2}}^{(k-1)} \right\}, & N = 2^k, 2^{k+1}, \dots, \quad k \geq 1. \end{cases} \quad (4.9)$$

Figure 3 illustrates the curve of an American put and the associated European put values as a function of the present asset value S .

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