

Endomorphisms of a Module over a Local Ring¹

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The matrix A of an endomorphism σ of a module M over a ring R is completely determined by the choice of a basis X for M , where A is called the matrix of σ relative to X .

Therefore, it will be natural to seek X giving a simple A , which is our primitive motivation.

Now, let R be a field. Then we have a good example of such A expressed in a nice form for a suitable X . Indeed, we know the following fact (see Lang[7,p557, Theorem 2.1] or Herstein[4,p307, Theorem 6.7.1]):

Theorem. Let R be a field. Then there are m elements $\{x_1, x_2, \dots, x_m\}$ in M and m polynomials $\{g_1(t), g_2(t), \dots, g_m(t)\}$ in the polynomial ring $R[t]$ over R in one indeterminate t such that A is a direct sum of m companion matrices of $\{g_1(t), g_2(t), \dots, g_m(t)\}$.

What can we say about this result, if R is a local ring? Is it possible to get a concise form of A as above? To analyze this problem is the purpose of this note.

So, let R be a local ring with the identity 1 and the unique maximal ideal $\mathfrak{m}$, M a free module of rank n over R , and $\text{End}_R M$ the endomorphism ring of M .

Then we have two canonical maps

$$\pi_R : R \rightarrow \overline{R} = R/\mathfrak{m} \quad \text{defined by} \quad a \mapsto \bar{a} = a + \mathfrak{m}$$

and

$$\pi_M : M \rightarrow \overline{M} = M/\mathfrak{m}M \quad \text{defined by} \quad x \mapsto \bar{x} = x + \mathfrak{m}M.$$

¹This is an abstract and the details will be published elsewhere.

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Since $\bar{R}$ is a field, $\bar{M}$ is a vector space over $\bar{R}$ by the scalar multiplication $\bar{a}\bar{x} = \overline{ax}$ for $a \in R$ and $x \in M$. Clearly the ring homomorphism π_R is an R -module homomorphism if we define $a\bar{b} = \overline{ab}$ for $a, b \in R$. Also π_M is an R -module homomorphism.

Further, for $x \in M$ and $\sigma \in \text{End}_R M$, if we define $\bar{\sigma}\bar{x} = \overline{\sigma x}$, we obtain an endomorphism $\bar{\sigma}$ of $\bar{M}$, that is, $\bar{\sigma} \in \text{End}_{\bar{R}} \bar{M}$. Thus we have the third canonical map

$$\pi_E : \text{End}_R M \rightarrow \text{End}_{\bar{R}} \bar{M} \quad \text{by} \quad \sigma \mapsto \bar{\sigma},$$

which is a ring homomorphism.

An element $\rho \in \text{End}_R M$ is called a permutation if it is a permutation on some basis for M . Also $\delta \in \text{End}_R M$ is diagonal if the matrix of δ is diagonal relative to some basis for M .

Also we denote the ring of $r \times s$ matrices over R by $M_{r,s}(R)$, and by $M_r(R)$ if $r = s$. Then, our results are as follows:

Theorem A. For any $\sigma \in \text{End}_R M$ there is a new basis X and a permutation ρ on X such that the matrix of $\rho^{-1}\sigma$ relative to X is expressed as

$$\begin{pmatrix} I_{n-m} & O_{n-m,m} \\ B_{m,n-m} & D_m \end{pmatrix},$$

where

- (i) m is the number of the invariant factors of $\bar{\sigma}$,
- (ii) $I_{n-m} \in M_{n-m}(R)$ is the identity matrix,
- (iii) $O_{n-m,m} \in M_{n-m,m}(R)$ is the zero matrix,
- (iv) $D_m \in (d_{ij}) \in M_m(R)$ is a matrix with $d_{ij} \equiv 0 \pmod{m}$ if $i \neq j$, i.e., diagonal modulo m ,

and

- (v) $B_{m,n-m} = (b_{ij}) \in M_{m,n-m}(R)$ is a matrix such that for any $i = 1, 2, \dots, m$ we have

$$b_{ij} \equiv 0 \pmod{m}$$

for $j \leq \prod_{\lambda=1}^{i-1} (n_\lambda - 1)$ or $\prod_{\mu=1}^i (n_\mu - 1) < j$.

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