

G-FUNCTIONS

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$$g(z) = \sum_{n=0}^{\infty} g_n z^n$$

where the g_n satisfy the following conditions,

- The g_n all belong to an algebraic numberfield k .
- $g(z)$ satisfies a linear differential equation with polynomial coefficients.
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$$\sum_v \max(|g_0|_v, |g_1|_v, \dots, |g_n|_v) = O(n)$$

as $n \rightarrow \infty$, where the sum is over all normalised valuations $|\cdot|_v$ of k .

EXAMPLES of G-FUNCTIONS

- Functions, algebraic over $k(z)$.
- Polylogarithms

$$L_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

- Hypergeometric functions

$$F(a, b, c|z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ and $a, b, c \in \mathbb{Q}$.

QUESTION

What are G-functions?

(EXAMPLE) Consider

$$(1.3) \quad z(z^2 - 1)f'' + (3z^2 - 1)f' + (z + B)f = 0$$

where B is an arbitrary rational number. For any solution

$$g(z) = \sum_{n=0}^{\infty} g_n z^n$$

we have the recurrence relation

$$n^2 f_n = B f_{n-1} + (n-1)^2 f_{n-2} \quad f_0 = 1, f_1 = B.$$

When $B = 0$, we have $f_{2n+1} = 0, f_{2n} = \binom{2n}{n}^2 / 16^n$. Are there any other B for which $g(z)$ is a G-function?

A conjecture of E. Bombieri and B. Dwork states that differential equations having G-function solutions 'arise' from Gauss-Manin systems corresponding to 1-parameter families of algebraic varieties.

QUESTION

Let $g(z)$ be a non-algebraic G-function, let $\xi \in \overline{\mathbb{Q}}, \xi \neq 0$ and let $|\cdot|_v$ be a valuation such that $g(\xi)$ converges v -adically. Do we have $g(\xi) \in \overline{\mathbb{Q}}$ or not?

(THEOREM). Let j be the modular j -invariant of an elliptic curve with complex multiplication and endomorphism algebra $\mathbb{Q}(\sqrt{-1})$. Suppose that $\xi = 1 - 12^3/j$ has complex absolute value less than 1. Then,

$$F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2} \middle| \xi\right) \in \overline{\mathbb{Q}}.$$

(EXAMPLE). Take $j = j(2i)$. Then $1 - 12^3/j = 1323/1331$ and

$$F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2} \middle| \frac{1323}{1331}\right) = \frac{3}{4} \sqrt[4]{11}.$$

(OBSERVATION). We have 7-adically,

$$F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2} \middle| \frac{1323}{1331}\right)_7 = \frac{1}{4} \sqrt[4]{11}.$$

PICARD-FUCHS EQUATIONS

Let K be an algebraic numberfield and consider the family of elliptic curves $\mathcal{E}_t$ given by

$$y^2 = x^3 + A(t)x + B(t), \quad A(t), B(t) \in K[t].$$

Assume $\Delta(0) \neq 0$, where $\Delta(t) = \Delta(\mathcal{E}_t) = 4A(t)^3 + 27B(t)^2$ and assume that $j(t) = j(\mathcal{E}_t)$ is non-constant.

$$\Omega(t) = \frac{dx}{2y}, \quad N(t) = x \frac{dx}{2y} + \gamma \frac{dx}{2y}$$

on $\mathcal{E}_t$, where $\gamma \in K$ is to be specified later if necessary.

There exists $\mathcal{A} \in M_2(K(t))$ such that

$$\frac{d}{dt} \begin{pmatrix} \Omega(t) \\ N(t) \end{pmatrix} \equiv \mathcal{A} \begin{pmatrix} \Omega(t) \\ N(t) \end{pmatrix} \text{ (Modulo exact forms).}$$

(PICARD-FUCHS EQUATION)

$$\frac{d}{dt} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathcal{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

(FUNDAMENTAL SOLUTION MATRIX) $M(t) \in M_2(K[[t]])$ such that

$$\frac{d}{dt} M(t) = \mathcal{A}(t)M(t) \text{ and } M(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(PROPERTIES)

- $\det(M(t)) = 1$
- The elements of $M(t)$ are G-functions.
- The elements of $M(t)$ have transcendence degree 3 over $\mathbb{C}(t)$.

Let $\mathcal{E}_t$: $y^2 = x^3 - x - t$ and $\Omega(t) = dx/2y$, $N(t) = xdx/2y$. Then $\Delta(t) = 27t^2 - 4$,

$$\mathcal{A}(t) = \frac{1}{2(27t^2 - 4)} \begin{pmatrix} -9t & -6 \\ 2 & 9t \end{pmatrix}$$

and

$$M(t) = \begin{pmatrix} F(\frac{1}{12}, \frac{5}{12}, \frac{1}{2} | \frac{27}{4}t^2) & \frac{3}{4}tF(\frac{7}{12}, \frac{11}{12}, \frac{3}{2} | \frac{27}{4}t^2) \\ -\frac{1}{4}tF(\frac{5}{12}, \frac{13}{12}, \frac{3}{2} | \frac{27}{4}t^2) & F(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2} | \frac{27}{4}t^2) \end{pmatrix}.$$

Let $\mathcal{E}_t$: $y^2 = x^3 + tx + 1$ and $\Omega(t) = dx/2y$, $N(t) = xdx/2y$. Then $\Delta(t) = 4t^3 + 27$,

$$\mathcal{A}(t) = \frac{1}{2(4t^3 + 27)} \begin{pmatrix} -2t^2 & 9 \\ 3t & 2t^2 \end{pmatrix}$$

and

$$M(t) = \begin{pmatrix} F(\frac{1}{12}, \frac{7}{12}, \frac{2}{3} | -\frac{4}{27}t^3) & \frac{1}{6}tF(\frac{5}{12}, \frac{11}{12}, \frac{4}{3} | -\frac{4}{27}t^3) \\ \frac{1}{36}t^2F(\frac{7}{12}, \frac{13}{12}, \frac{5}{3} | -\frac{4}{27}t^3) & F(-\frac{1}{12}, \frac{5}{12}, \frac{1}{3} | -\frac{4}{27}t^3) \end{pmatrix}.$$

(THEOREM) Suppose E_0 is isogenous over K to E_a via $\phi : E_0 \rightarrow E_a$. Let $N = \deg \phi$. Let $\alpha, \beta \in K$ be defined by

$$\phi^* \omega_a = \alpha \omega_0 \quad \phi^* \eta_a = \frac{N}{\alpha} \eta_0 + \beta \omega_0.$$

Let v be a valuation of K such that $|a|_v < \max(1, \rho_v)$. When v is finite, also suppose that $|A(0)|_v, |B(0)|_v \leq 1$ and $|6\Delta(0)|_v = |-2A(0)B'(0) + 3B(0)A'(0)|_v = 1$. Then,

$$\text{Tr} \left(\begin{pmatrix} N/\alpha & 0 \\ -\beta & \alpha \end{pmatrix} M(a)_v \right) \in \mathbb{Z}$$

where $M(a)_v$ denotes the v -adic evaluation of $M(a)$. Moreover, when v is finite, the trace on the left hand side of (4.1) is $\leq 2\sqrt{N}$.

(COROLLARY) Suppose in addition to the above assumptions that E_0 is a C.M. curve. Choose γ such that $\text{End}(E_0)$ acts diagonally on ω_0, η_0 . Let $-d = \text{discr}(\text{End}(E_0))$ and write

$$M(t) = \begin{pmatrix} m_{11}(t) & m_{12}(t) \\ m_{21}(t) & m_{22}(t) \end{pmatrix}.$$

Then,

$$\frac{N}{\alpha} m_{11}(a)_v, \alpha m_{22}(a)_v - \beta m_{12}(a)_v \in \frac{1}{\sqrt{-d}} \mathbb{Z} \left[\frac{d + \sqrt{-d}}{2} \right].$$

(EXAMPLE 1) Take $a = 11^{3/2}/14$. Then $j(E_a) = 66^3$ and there is an isogeny of degree 2 from E_0 to E_a . Let

$$f(z) = 2 \cdot 11^{-1/4} F \left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2} | z \right)$$

Then

$$f \left(\frac{1323}{1331} \right)_{\infty} = \frac{3}{2} \quad f \left(\frac{1323}{1331} \right)_{\gamma} = \frac{1}{2}.$$

Moreover,

$$\begin{aligned} \frac{21}{242} F \left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2} | \frac{1323}{1331} \right) + F \left(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2} | \frac{1323}{1331} \right) &= \frac{3}{2} \cdot 11^{-1/4} \\ \frac{21}{242} F \left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2} | \frac{1323}{1331} \right)_{\gamma} + F \left(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2} | \frac{1323}{1331} \right)_{\gamma} &= \frac{1}{2} \cdot 11^{-1/4}. \end{aligned}$$

(EXAMPLE 2) Take a such that $j(E_a) = (3(724 + 513\sqrt{2}))^3$. Then E_a is isogenous to E_0 of degree 4. Let

$$f(z) = 4 \cdot (91 + 60\sqrt{2})^{-1/4} F \left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2} | z \right)$$

and

$$\xi = 3^3 \cdot 7^2 \cdot 11^2 (3 - 2\sqrt{2})(5 + \sqrt{2})^3 (7 - \sqrt{2})^3 / (23 \cdot 47)^3.$$

Then

$$\begin{aligned} f(\xi)_\infty &= \frac{5}{2} & f(\xi)_{3+\sqrt{2}} &= \frac{3}{2} \\ f(\xi)_{3-\sqrt{2}} &= \frac{1+i}{2} & f(\xi)_{11} &= \frac{1+2i}{2}. \end{aligned}$$

(THEOREM) Suppose E_0 is a C.M. curve and suppose γ is chosen such that the action of $\text{End}(E_0)$ on ω_0, η_0 is diagonal. Suppose E_0 is isogenous over K to E_a via $\phi : E_0 \rightarrow E_a$. Let $N = \deg \phi$. Let $\alpha, \beta \in K$ be defined by

$$\phi^* \omega_a = \alpha \omega_0 \quad \phi^* \eta_a = \frac{N}{\alpha} \eta_0 + \beta \omega_0.$$

Let v be a finite place of K which satisfies $|A(0)|_v, |B(0)|_v \leq 1$ and

$$|6\Delta(0)|_v = |-2A(0)B'(0) + 3B(0)A'(0)|_v = 1.$$

Suppose that $\overline{E_0}$ is ordinary and $|a|_v < 1$. Let $-d = \text{discr}(\text{End}(\overline{E_0}))$. Then there exists $\epsilon \in \mathbb{Z}[(d + \sqrt{-d})/2]$ with $\epsilon \bar{\epsilon} = N$ such that

$$M(a)_v = \frac{1}{N} \begin{pmatrix} \alpha & 0 \\ \beta & N/\alpha \end{pmatrix} \begin{pmatrix} \bar{\epsilon} & 0 \\ 0 & \epsilon \end{pmatrix}.$$

(EXAMPLE) Let a be such that $j(E_a) = (6(2927 + 1323\sqrt{5}))^3$. Then E_0 is 5-isogenous to E_a . Moreover, $\overline{E_0}$ is ordinary. Let

$$\xi = 2 \cdot 3^3 \cdot 7^2 \sqrt{5} (47 - 20\sqrt{5})^2 (673 + 357\sqrt{5})^3 / (11 \cdot 59 \cdot 71)^3.$$

Then

$$\begin{aligned} F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2} \middle| \xi\right)_5 &= -\frac{1}{2+i} (161 + 120\sqrt{5})^{1/4} \\ F\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2} \middle| \xi\right)_5 &= 0 \\ F\left(\frac{5}{12}, \frac{13}{12}, \frac{3}{2} \middle| \xi\right)_5 &= -\frac{(1229 - 515\sqrt{5})(161 + 120\sqrt{5})^{1/4}}{7(47 - 20\sqrt{5})(2+i)} \\ F\left(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2} \middle| \xi\right)_5 &= -(2+i)(161 + 120\sqrt{5})^{-1/4}. \end{aligned}$$

(THEOREM) Suppose E_0 is C.M. curve and γ is chosen such that the action of $\text{End}(E_0)$ on ω_0, η_0 is diagonal. Let E_a , $a \in K$ be another C.M. curve and suppose that there exists

a valuation v on K such that $|a|_v < \max(1, \rho_v)$. When v is finite we also assume that $|A(0)|_v, |B(0)|_v \leq 1$ and $|6\Delta(0)|_v = |-2A(0)B'(0) + 3B(0)A'(0)|_v = 1$. Let δ be such that the action of $\text{End}(E_a)$ on $\omega_a, \eta_a + \delta\omega_a$ is diagonal. Then,

$$\frac{m_{11}(a)_v(m_{22}(a)_v + \delta m_{12}(a)_v)}{m_{12}(a)_v(m_{21}(a)_v + \delta m_{11}(a)_v)} \in \mathbb{Q}(\text{End}(E_0), \text{End}(E_a)).$$

(EXAMPLE) Let a be such that $j(E_a) = 20^3$. Then $\text{End}(E_a) = \mathbb{Z}[\sqrt{-2}]$ and $\delta = 1/\sqrt{30}$. Denote

$$\begin{aligned} f(z) &= F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2} \mid z\right) & g(z) &= F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2} \mid z\right) \\ h(z) &= F\left(\frac{5}{12}, \frac{13}{12}, \frac{3}{2} \mid z\right) & k(z) &= F\left(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2} \mid z\right) \end{aligned}$$

and

$$Q(z) = \frac{f(z)(2250k(z) + 105g(z))}{g(z)(-49h(z) + 105f(z))}.$$

Then

$$Q\left(\frac{98}{125}\right)_{\infty} = (\sqrt{2} + 1)^4 \quad Q\left(\frac{98}{125}\right)_7 = -\frac{(2\sqrt{2} - 1)^2}{7}.$$