

Periods of tropical K3 hypersurfaces

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1. The case of curves

- $K := \overline{\mathbb{C}\{t\}}$: the convergent Puiseux series field
- $f = \sum_i k_i x^i \in K[x_1^\pm, x_2^\pm]$
- $V(\text{trop}(f))$: the tropical hypersurface defined by $\text{trop}(f)$
- $R \in \mathbb{R}^{>0}$: sufficiently large $\leadsto f_R := f|_{t=1/R} \in \mathbb{C}[x_1^\pm, x_2^\pm]$
- $V(R) := \{f_R = 0\}$: the complex hypersurface

Theorem (Katz–Markwig–Markwig, Iwao)

Take holomorphic forms $(\Omega_i(R))_{i=1}^g$ so that $\int_{\partial \sigma_i} \Omega_j(R) = \delta_{i,j}$.
The leading term of the period is given by

$$\int_{\beta_i} \Omega_j(R) \sim \frac{-1}{2\pi\sqrt{-1}} \log R \cdot T_{i,j}, \quad (R \rightarrow +\infty)$$

where $\{T_{i,j}\}_{i,j}$ is the tropical period matrix of $V(\text{trop}(f))$.

2. Tropical K3 surfaces

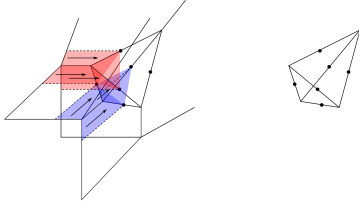
Theorem (Gross–Wilson)

Maximally degenerating families of complex K3 surfaces with Ricci-flat Kähler metrics converge to **2-spheres with integral affine structures with singularities** in the Gromov–Hausdorff limit.

The first step : To construct an integral affine 2-sphere from a tropical K3 hypersurface.

3. Contractions

- **The projection to the maximal dimensional torus orbit** induces an integral affine structure on a neighborhood of a vertex.
- Black dots are singularities.



- Similar/the same constructions have already been performed by Gross–Siebert and Kontsevich–Soibelman.

4. Radiance obstructions

- B : an integral affine manifold (with singularities)
- $\iota : B_0 \hookrightarrow B$: the smooth part
- $\mathcal{T}_B$: the sheaf of integral tangent vectors on B_0
- $\mathcal{T} := \mathcal{T}_B \otimes_{\mathbb{Z}} \mathbb{R}$
- $\{U_i\}_i$: a sufficiently fine open covering of B
- $\{s_i \in \Gamma(U_i \cap B_0, T^{\text{aff}} B_0)\}_i$

Definition (Goldman–Hirsch '84)

The **radiance obstruction** $c_B \in H^1(B, \iota_* \mathcal{T})$ is defined by

$$c_B((U_i, U_j)) := s_j - s_i$$

for each 1-simplex (U_i, U_j) of $\{U_i\}_i$.

5. Main results

- $\Delta \subset M_{\mathbb{R}}^n$: a smooth reflexive polytope
- $\tilde{\Delta} \subset N_{\mathbb{R}}^n$: the polar polytope of Δ
- $F = \max_{n \in \tilde{\Delta} \cap N} \{a(n) + n \cdot X\}$: a tropical polynomial
- $V(F)$: the tropical hypersurface defined by F
- B : an integral affine 2-sphere obtained by contracting $V(F)$
- $\Sigma \subset M_{\mathbb{R}}^n$: the normal fan of Δ
- X_{Σ} : the complex toric manifold associated with Σ
- D_{ρ} : the toric divisor on X_{Σ} corresponding to $\rho \in \Sigma(1)$
- $Y \subset X_{\Sigma}$: an anti-canonical hypersurface
- $\text{Pic}(Y)_{\text{amb}} := \text{Im} \{ \text{Pic}(X_{\Sigma}) \hookrightarrow \text{Pic}(Y) \}$
- $\cup : H^1(B, \iota_* \mathcal{T}_{\mathbb{R}}) \otimes H^1(B, \iota_* \mathcal{T}_{\mathbb{R}}) \rightarrow H^2(B, \iota_* \wedge^2 \mathcal{T}_{\mathbb{R}}) \cong \mathbb{Z}$

Theorem (Y.)

1. There is a primitive embedding

$$\psi : \text{Pic}(Y)_{\text{amb}} \hookrightarrow H^1(B, \iota_* \mathcal{T}_{\mathbb{R}})$$

that preserves the pairing.

2. The radiance obstruction c_B is given by

$$c_B = \sum_{\rho \in \Sigma(1)} \{a(n_{\rho}) - a(0)\} \psi(D_{\rho}).$$

6. Corollary

- $f = \sum_i k_i x^i \in K[x_1^\pm, x_2^\pm, x_3^\pm]$
- $R \in \mathbb{R}^{>0}$: sufficiently large $\leadsto f_R := f|_{t=1/R} \in \mathbb{C}[x_1^\pm, x_2^\pm, x_3^\pm]$
- $V(R) := \{f_R = 0\}$: the complex K3 hypersurface
- The period map

$$\mathcal{P} : (R_0, \infty) \rightarrow \left\{ [\sigma] \in \mathbb{P}((U \oplus \text{Pic}(Y)_{\text{amb}}) \otimes \mathbb{C}) \mid (\sigma, \sigma) = 0, (\sigma, \bar{\sigma}) > 0 \right\} \\ \cong \{ \sigma \in \text{Pic}(Y)_{\text{amb}} \otimes \mathbb{C} \mid (\Re \sigma, \Re \sigma) > 0 \}$$
- B : an integral affine 2-sphere obtained by contracting the tropical hypersurface $V(\text{trop}(f))$

Corollary (Y.)

1. The leading term of the period map $\mathcal{P}(R)$ is given by

$$\mathcal{P}(R) \sim \log R \cdot \psi^{-1}(c_B) \quad (R \rightarrow +\infty).$$

The element $\psi^{-1}(c_B) \in \text{Pic}(Y)_{\text{amb}} \otimes_{\mathbb{Z}} \mathbb{R}$ can be regarded as the tropical period of $B \simeq V(\text{trop}(f))$.

- 2.

$$(\psi^{-1}(c_B), \psi^{-1}(c_B)) > 0.$$

This can be regarded as the tropical version of Hodge–Riemann bilinear relation.