

Value distribution for moving targets

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1. Introduction

In 1929, R. Nevanlinna conjectured that his defect relation remains correct for distinct meromorphic functions g_j such that $T_{g_j}(r) = o(T_f(r))$ ($r \rightarrow \infty$) ($1 \leq j \leq q$):

$$\sum_{j=1}^q \delta(f, g_j) + \delta(f, \infty) \leq 2.$$

After many attempts, this defect relation was proved by Steinmetz in 1986. His proof is very simple and elegant.

Stoll considered the case of holomorphic mappings of C into $P^n(C)$. He extended Cartan's defect relation to moving targets with Ru, and I gave a simpler proof for their theorem. Also, they generalized it by Nochka's method.

I applied the above theory to the unicity theorem of Nevanlinna. This means that two meromorphic functions on the complex plane which have

the same inverse images counting multiplicities for four values are Möbius transforms of each other. I extended this theorem to moving targets.

2. Definitions

Let f be a holomorphic mapping of C into $P^n(C)$. A holomorphic mapping of $\tilde{f} = (f_0, \dots, f_n) \not\equiv \mathbf{o}$ of C into C^{n+1} is called a representation of f if $f(z) = (f_0(z) : \dots : f_n(z))$ for all $z \in C$, where $\mathbf{o}$ is the origin of C^{n+1} and $(w_0 : \dots : w_n)$ is a homogeneous coordinate system of $P^n(C)$. Moreover, if $\tilde{f}(z) \neq \mathbf{o}$ for any $z \in C$, it is said to be reduced. Take a reduced representation $\tilde{f} = (f_0, \dots, f_n)$ of f . Fix $r_0 > 0$.

Definition 1. The characteristic function of f is defined for $r > r_0$ by

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \|\tilde{f}(re^{i\theta})\| d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log \|\tilde{f}(r_0e^{i\theta})\| d\theta,$$

where $\|z\| = (\sum_{j=0}^n |z_j|^2)^{1/2}$ for $z = (z_0, \dots, z_n) \in C^{n+1}$.

Let g be a holomorphic mapping of C into $P^n(C)$ with a reduced representation $\tilde{g} = (g_0, \dots, g_n)$. We call g a moving target for f . Assume that $h := g_0f_0 + \dots + g_nf_n \not\equiv 0$.

Definition 2. The counting function of f for g is defined for $r > r_0$ by

$$N_{f,g}(r) = \frac{1}{2\pi} \int_0^{2\pi} \log |h(re^{i\theta})| d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log |h(r_0e^{i\theta})| d\theta.$$

For a meromorphic function (i.e., a holomorphic mapping into $\mathbf{P}^1(\mathbf{C})$), another counting function is defined. Let φ be a meromorphic function on $\mathbf{C}$.

Definition 3. If $\varphi \not\equiv 0$, the counting function of φ for 0 is defined by

$$N_{\varphi;0}(r) = \int_{r_0}^r \frac{n_{\varphi}(t)}{t} dt,$$

where $n_{\varphi}(t)$ is the sum of multiplicities of zeros of φ in $\{z \in \mathbf{C}; |z| \leq t\}$. For $a \in \mathbf{C}$, the counting function $N_{\varphi;a}(r) := N_{\varphi-a;0}(r)$ of φ for a is defined if $\varphi \not\equiv a$. Also, the counting function $N_{\varphi;\infty}(r) := N_{1/\varphi;0}(r)$ of φ for ∞ is defined.

It is easy to see that $T_f(r) \geq 0$ and that $T_f(r) \rightarrow \infty$ monotonically as $r \rightarrow \infty$ if f is nonconstant. Also, we can see that $N_{f,g}(r) = N_{h;0}(r)$ by the Poisson–Jensen formula. If g is constant, then it defines a hyperplane $H = \{w \in \mathbf{P}^n(\mathbf{C}); g_0 w_0 + \dots + g_n w_n = 0\}$ in $\mathbf{P}^n(\mathbf{C})$, and $h(z) = 0$ implies $f(z) \in H$. Hence, the counting function $N_{f,g}(r)$ express the growth of the inverse image of H by f .

Assume that f is nonconstant.

Definition 4. The defect of f for g is defined by

$$\delta(f, g) = \liminf_{r \rightarrow \infty} \left(1 - \frac{N_{f,g}(r)}{T_f(r) + T_g(r)} \right).$$

We can easily verify that $0 \leq \delta(f, g) \leq 1$.

Let N and q be positive integers such that $N \geq n$ and $q \geq 2N - n$ in this section and the next one. Take moving targets $g_0, \dots, g_q$ for f . Let $\tilde{g}_j = (g_{j0}, \dots, g_{jn})$ be reduced representations of g_j ($0 \leq j \leq q$).

Definition 5. If for each subset A of $\{0, 1, \dots, q\}$ such that $\#A = N+1$, there exist $j_0, \dots, j_n \in A$ such that $\det(g_{j_\mu\nu})_{0 \leq \mu, \nu \leq n} \neq 0$, then $g_0, \dots, g_q$ are said to be in N -subgeneral position. If $N = n$, they are said to be in general position.

Definition 6. Let $\mathfrak{F}$ be a field with $C \subset \mathfrak{F} \subset \mathfrak{M}$, where $\mathfrak{M}$ is the field of meromorphic functions on C . If $f_0, \dots, f_n$ are linearly independent over $\mathfrak{F}$, then f is said to be non-degenerate over $\mathfrak{F}$.

Let $\mathfrak{K}$ be the smallest field which contains C and all $g_{j\mu}/g_{j\nu}$ with $g_{j\nu} \neq 0$. If f is non-degenerate over $\mathfrak{K}$, then $g_{j0}f_0 + \dots + g_{jn}f_n \neq 0$ for any $j = 0, 1, \dots, q$. Hence, counting functions $N_{f,g_j}(r)$ and defects $\delta(f, g_j)$ can be defined.

If all g_j are constants, then each g_j defines a hyperplane $H_j = \{w \in P^n(C); g_{j0}w_0 + \dots + g_{jn}w_n = 0\}$ in $P^n(C)$. Then, if $g_0, \dots, g_q$ are in general position, $H_0, \dots, H_q$ are in general position. Also, the non-degeneracy of f over $\mathfrak{K}$ means the non-degeneracy of f over C .

In the rest of this section, we consider holomorphic mappings into $P^1(C)$ and introduce notations which are used later. Let f be a holomorphic mapping C into $P^1(C)$ with a reduced representation (f_0, f_1) . Then, we identify f with the meromorphic function f_1/f_0 if $f_0 \neq 0$. Otherwise, we identify it with the constant mapping taking the point at infinity as its value. Also, we denote by f^* the holomorphic mapping of C into $P^1(C)$ with the reduced representation $(-f_1, f_0)$.

Remark 1. We have defined two kinds of counting functions $N_{f,a}(r)$ and $N_{f,a}(r)$ for $a \in \overline{C} := C \cup \{\infty\}$ which is a constant holomorphic mapping of C into $P^1(C)$. However, if $N_{f,a}(r) - N_{f,\infty}(r) = N_{f,a^*}(r)$ for $a \in C$ and

$N_{f;a}(r) = N_{f,a^*}(r)$ for $a = \infty$.

For a subfield $\mathfrak{F}$ of $\mathfrak{M}$, put $\bar{\mathfrak{F}} = \mathfrak{F} \cup \{\infty\}$. If f is nonconstant, we define $\Gamma_f = \{h \in \mathfrak{M}; T_h(r) = o(T_f(r))(r \rightarrow \infty)\}$ which is a field. Also, if $f \neq \infty$, we define the proximity function of f for ∞ by

$$m_{f;\infty}(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

where $\log^+ x = \log(\max(1, x))$ for $x \geq 0$, and if $f \neq a$ for $a \in \mathfrak{M}$, the proximity function of f for a is defined by $m_{f;a}(r) := m_{1/(f-a);\infty}(r)$. It is easy to see that

$$T_f(r) = N_{f;a}(r) + m_{f;a}(r) + O(1) \quad (1)$$

if $f \neq a$ for $a \in \bar{C}$.

If f is nonconstant and $a \in \bar{\Gamma}_f$, then

$$\delta(f, a) = \liminf_{r \rightarrow \infty} \left(1 - \frac{N_{f,a}(r)}{T_f(r)} \right).$$

We use the notation “ $P(r) //$ ” to mean that a property $P(r)$ holds for all $r \in (r_0, \infty) - E$, where E is a subset of (r_0, ∞) of finite Lebesgue measure. We complete this section with the following which is called the lemma of the logarithmic derivative:

Lemma. *For a nonconstant meromorphic function h on C and $j = 1, 2, \dots$,*

$$m_{h^{(j)}/h;\infty}(r) = o(T_h(r)) // \quad \text{as } r \rightarrow \infty.$$

3. Defect relations

In this section, we introduce various defect relations from H. Cartan to Ru-Stoll.

Theorem A (H. Cartan). *Assume that all g_j are constants, f is non-degenerate over $\mathbb{C}$ and that $g_0, \dots, g_q$ are in general position. Then*

$$\sum_{j=0}^q \delta(f, g_j) \leq n + 1.$$

Theorem B (Nochka). *Assume that all g_j are constants, f is non-degenerate over $\mathbb{C}$ and that $g_0, \dots, g_q$ are in N -subgeneral position. Then*

$$\sum_{j=0}^q \delta(f, g_j) \leq 2N - n + 1.$$

Theorem C (Ru-Stoll). *Assume that $T_{g_j}(r) = o(T_f(r))(r \rightarrow \infty)(0 \leq j \leq q)$, f is non-degenerate over $\mathbb{R}$ and that $g_0, \dots, g_q$ are in general position. Then*

$$\sum_{j=0}^q \delta(f, g_j) \leq n + 1.$$

Theorem B and Theorem C are generalization of Theorem A and I gave a simpler proof for Theorem C in [6]. The following theorem is a generalization of the above theorems.

Theorem D (Ru-Stoll). *Assume that $T_{g_j}(r) = o(T_f(r))(r \rightarrow \infty)(0 \leq j \leq q)$, f is non-degenerate over $\mathbb{R}$ and that $g_0, \dots, g_q$ are in N -subgeneral position. Then*

$$\sum_{j=0}^q \delta(f, g_j) \leq 2N - n + 1.$$

4. Nevanlinna's unicity theorems

We say that two meromorphic functions f and g on C share the value a if the zeros of $f - a$ and $g - a$ ($1/f$ and $1/g$ if $a = \infty$) are the same. Nevanlinna [2] proved the following theorems:

Theorem E. *If two distinct nonconstant meromorphic functions f and g on C share four values $a_1, \dots, a_4$ by counting multiplicities, then g is a Möbius transformation of f , two shared values, say a_3 and a_4 , are Picard values, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

Theorem F. *If two nonconstant meromorphic functions f and g share five values, then $f \equiv g$.*

I give an extension of Theorem E by using the results of moving targets in [4] and [8]. An extension of Theorem F is conjectured, but the second main theorem for moving targets corresponding to that playing the main role in the proof of Theorem F is not proved yet.

5. Second fundamental theorem and Borel's lemma

Let f be a nonconstant holomorphic mapping of C into $P^1(C)$ with a reduced representation $\tilde{f} = (f_0, f_1)$.

Theorem G. *If $a_1, \dots, a_q \in \bar{\Gamma}_f$ are distinct, then for each $\varepsilon > 0$*

$$(q - 2 - \varepsilon)T_f(r) \leq \sum_{j=1}^q N_{f, a_j}(r) + o(T_f(r)) //.$$

Corollary. *If $a_1, \dots, a_q \in \bar{\Gamma}_f$ are distinct, then*

$$\sum_{j=1}^q \delta(f, a_j) \leq 2.$$

This is an extension of Nevanlinna's defect relation and was obtained by Steinmetz [8]. The following theorem called Borel's lemma is useful for the proof of the extension of Theorem E:

Theorem 1. *Let $N \geq 2$ be an integer, $F_1, \dots, F_N$ nonvanishing entire functions, and $a_1, \dots, a_N$ meromorphic functions such that $a_j \not\equiv 0$ and*

$$T_{a_j}(r) = o(T(r)) \quad // \quad \text{as } r \rightarrow \infty \quad (1)$$

($1 \leq j \leq N$), where $T(r) = \sum_{j=1}^N T_{F_j}(r)$. Assume that

$$a_1 F_1 + \dots + a_N F_N \equiv 1. \quad (2)$$

Then, $a_1 F_1, \dots, a_N F_N$ are linearly dependent over $\mathbb{C}$.

6. Unicity Theorem

We extend Theorem E by dividing it into two parts.

Let f and g be distinct nonconstant meromorphic functions with reduced representations (f_0, f_1) and (g_0, g_1) , respectively. Let a_j be distinct elements

of $\bar{\Gamma}_f$ with reduced representations (a_{j0}, a_{j1}) ($1 \leq j \leq 4$). We define entire functions by $F_j = a_{j0}f_0 + a_{j1}f_1$ and $G_j = a_{j0}g_0 + a_{j1}g_1$. Then $F_j \neq 0$. Also, we define meromorphic functions ψ_j by

$$G_j = \psi_j F_j. \quad (1)$$

Theorem 2. *If all ψ_j are nonvanishing entire functions, then there exist $A, B, C, D \in \Gamma_f$ such that $AD - BC \neq 0$ and*

$$g = \frac{Af + B}{Cf + D}. \quad (2)$$

Proof. By (1), we get

$$\begin{pmatrix} a_{10} & a_{11} & -a_{10}\psi_1 & -a_{11}\psi_1 \\ a_{20} & a_{21} & -a_{20}\psi_2 & -a_{21}\psi_2 \\ a_{30} & a_{31} & -a_{30}\psi_3 & -a_{31}\psi_3 \\ a_{40} & a_{41} & -a_{40}\psi_4 & -a_{41}\psi_4 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ f_0 \\ f_1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since $(g_0, g_1, f_0, f_1) \neq (0, 0, 0, 0)$, the determinant of the 4×4 matrix above is identically equal to zero. By expanding it, we have

$$b_{12}\psi_1\psi_2 + b_{34}\psi_3\psi_4 + b_{13}\psi_1\psi_3 + b_{24}\psi_2\psi_4 + b_{14}\psi_1\psi_4 + b_{23}\psi_2\psi_3 \equiv 0, \quad (3)$$

where

$$\begin{aligned} b_{12} &= b_{34} = (a_{10}a_{21} - a_{11}a_{20})(a_{30}a_{41} - a_{31}a_{40}) \\ b_{13} &= b_{24} = -(a_{10}a_{31} - a_{11}a_{30})(a_{20}a_{41} - a_{21}a_{40}) \\ b_{14} &= b_{23} = (a_{10}a_{41} - a_{11}a_{40})(a_{20}a_{31} - a_{21}a_{30}). \end{aligned}$$

For distinct j and k , we have

$$\frac{\psi_j}{\psi_k} - 1 = \frac{(a_{j1}a_{k0} - a_{j0}a_{k1})(f_0g_1 - f_1g_0)}{F_jG_k}. \quad (4)$$

Since $F_l(z) = G_l(z) = 0$ implies $f_0(z)g_1(z) - f_1(z)g_0(z) = 0$,

$$N_{\psi_j/\psi_k;1}(r) \geq \sum_{l \neq j,k} N_{f,a_l}(r) + o(T_f(r)).$$

Hence, if $\#\{j, k, \mu, \nu\} \geq 3$, by (??) and Theorem G

$$\begin{aligned} T_{\psi_j/\psi_k}(r) + T_{\psi_\mu/\psi_\nu}(r) &\geq N_{\psi_j/\psi_k;1}(r) + N_{\psi_\mu/\psi_\nu;1}(r) + O(1) \\ &\geq \sum_{l \neq j,k} N_{f,a_l}(r) + \sum_{l \neq \mu,\nu} N_{f,a_l}(r) + o(T_f(r)) \\ &\geq \frac{1}{2}T_f(r) + o(T_f(r)) //. \end{aligned}$$

Applying Theorem 1 to the identity obtained from (3)

$$\frac{b_{12}\psi_1}{b_{23}\psi_3} + \frac{b_{34}\psi_4}{b_{23}\psi_2} + \frac{b_{13}\psi_1}{b_{23}\psi_2} + \frac{b_{24}\psi_4}{b_{23}\psi_3} + \frac{b_{14}\psi_1\psi_4}{b_{23}\psi_2\psi_3} \equiv -1,$$

we have a shorter identity

$$\begin{aligned} \alpha_{12}b_{12}\psi_1\psi_2 + \alpha_{34}b_{34}\psi_3\psi_4 + \alpha_{13}b_{13}\psi_1\psi_3 \\ + \alpha_{24}b_{24}\psi_2\psi_4 + \alpha_{14}b_{14}\psi_1\psi_4 \equiv 0, \end{aligned}$$

where α_{jk} are constants not all zero. By applying Theorem 3.3 successively, we deduce that some $(b_{jk}\psi_k)/(b_{jl}\psi_l)$ are nonzero constants, where $b_{jk} = b_{kj}$ if $j > k$. The conclusion of the theorem follows from this. Q.E.D.

Remark 2. In fact, A, B, C and D are rational functions of $a_1, \dots, a_4$. Hence, if $a_1, \dots, a_4 \in \overline{\mathcal{C}}$, then A, B, C and D are constants, and f and g are Möbius transforms of each other.

We state the second part of our extension of Theorem A. Let $A, B, C, D \in \mathfrak{M}$ such that $AD - BC \not\equiv 0$. We define the mapping $S : \overline{\mathfrak{M}} \rightarrow \overline{\mathfrak{M}}$ by

$$S(F) = \begin{cases} (AF + B)/(CF + D) & (F \in \mathcal{M}) \\ A/C & (F \equiv \infty). \end{cases}$$

For a nonconstant meromorphic function f , we define the condition $P(f)$ by

$$P(f) \quad N_{h;0}(r) + N_{h;\infty}(r) = o(T_f(r)) \quad (r \rightarrow \infty)$$

for $h \in \mathfrak{M}$.

Remark 3. The conclusion of Theorem 2 is true under the weaker assumption that all ψ_j satisfy the condition $P(f)$.

Theorem 3. Assume that $A, B, C, D \in \Gamma_f$ and that

$$g = S(f). \quad (5)$$

Moreover, assume that all ψ_j satisfy the condition $P(f)$. Then, for two j , say $j = 3, 4$, F_j satisfy the condition $P(f)$, and the meromorphic function of cross ratio $(a_1^*, a_2^*, a_3^*, a_4^*)$ is identically equal to -1 .

Remark 4. Under the assumption above, the two conditions $P(f)$ and $P(g)$ are equivalent.

Remark 5. If $a_1, \dots, a_4 \in \overline{\mathcal{C}}$ and $A, B, C, D \in \mathcal{C}$, then it is easy to deduce the conclusion of the theorem as a Möbius transform which is not the identity has at most two fixed points.

Proof. It follows from (5) that

$$\frac{\psi_j}{\psi_k} = \frac{(Ba_{j1} + Da_{j0})f_0 + (Aa_{j1} + Ca_{j0})f_1}{F_j} \times \frac{F_k}{(Ba_{k1} + Da_{k0})f_0 + (Aa_{k1} + Ca_{k0})f_1}. \quad (6)$$

For distinct j and k , the common zeros of F_j and F_k are the zeros of $a_{j0}a_{k1} - a_{j1}a_{k0} (\neq 0)$ which satisfies $P(f)$, and also, the common zeros of F_j and

$(Ba_{j1} + Da_{j0})f_0 + (Aa_{j1} + Ca_{j0})f_1$ are the zeros of $(Ba_{j1} + Da_{j0})a_{j1} - (Aa_{j1} + Ca_{j0})a_{j0}$. Unless

$$(Ba_{j1} + Da_{j0})a_{j1} - (Aa_{j1} + Ca_{j0})a_{j0} \equiv 0, \quad (7)$$

it satisfies $P(f)$. Therefore, in this case, since ψ_j/ψ_k satisfies $P(f)$,

$$N_{F_j;0}(r) = o(T_f(r)) \quad \text{as } r \rightarrow \infty. \quad (8)$$

We conclude that at least one condition among (7) and (8) holds for each $j = 1, \dots, 4$. However, the number of j 's which satisfy (8) and (7), respectively, is at most two. Therefore, we may assume that for $j = 1, 2$, (7) holds, but (8) does not, and that for $j = 3, 4$, (8) holds, but (7) does not. In (6), we consider the case $j = 3, k = 1$. Then, we deduce that $(Ba_{31} + Da_{30})f_0 + (Aa_{31} + Ca_{30})f_1$ satisfies $P(f)$. However, (7) does not hold for $j = 3$. It follows from these and Theorem G that

$$(Ba_{31} + Da_{30})a_{41} - (Aa_{31} + Ca_{30})a_{40} \equiv 0.$$

Similarly, we have

$$(Ba_{41} + Da_{40})a_{31} - (Aa_{41} + Ca_{40})a_{30} \equiv 0.$$

We obtain from these two identities

$$S(a_4^*) = a_3^*, \quad S(a_3^*) = a_4^*. \quad (9)$$

Also, we have

$$S(a_j^*) = a_j^* \quad (j = 1, 2) \quad (10)$$

by (7). From (9) and (10), the identity $(a_1^*, a_2^*, a_3^*, a_4^*) \equiv -1$ is deduced.

Q.E.D.

We give an analogue of Theorem F.

Corollary 4. *Let f and g be nonconstant meromorphic functions with reduced representations (f_0, f_1) and (g_0, g_1) , respectively, and $a_j \in \bar{\Gamma}_f$ distinct with reduced representations (a_{j0}, a_{j1}) ($1 \leq j \leq 5$). Assume that all ψ_j defined by (1) are entire functions without zeros. Then, $f \equiv g$.*

Proof. Assume that $f \not\equiv g$. Then, it follows from Theorems 4.1 and 4.2 that for two j in $\{1, 2, 3, 4\}$, say $j = 3, 4$, F_j satisfy the condition $P(f)$. In the same way, F_j satisfy the condition $P(f)$ for two j in $\{1, 2, 3, 5\}$. Hence, the number of j in $\{1, 2, 3, 4, 5\}$ such that F_j satisfy the condition $P(f)$ is three or four, a contradiction to Theorem 3.1. Q.E.D.

In Corollary 4, F_j and G_j are required to have the same zeros counting multiplicities. However, Theorem F does not count the multiplicities. The following should be a complete extension of Theorem F:

Conjecture. We have $f \equiv g$, if F_j and G_j have the same zeros for each $j = 1, \dots, 5$ (not counting multiplicities).

If the number five is replaced by seven, this conjecture was proved by Toda[10], recently.

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