

6

REALIZATIONS OF LIE ALGEBRAS

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This is an expository talk on realizations of Lie algebras.

1. The classical invariant theory

Let us choose a generic polynomial

$$f(\xi|z) = \sum_{\ell=0}^n \binom{n}{\ell} \xi^{(\ell)} z^{(n-\ell)},$$

on which $SL_2(K)$ acts as follows

$$(1) \quad f\left(\begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix} \xi | z\right) = (\gamma z + \delta)^m f\left(\xi \middle| \frac{\alpha z + \beta}{\gamma z + \delta}\right),$$

i.e.,

$$(2) \quad \left(\begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix} \xi\right)^{(n)} = \sum_{p,q} \binom{n}{p} \binom{n}{q} \xi^{(n-p+q)} \alpha^{p-q} \beta^p \gamma^q \delta^{n-p-q}.$$

The corresponding realization of $sl_2(K)$ is given by

$$\begin{cases} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto \mathcal{D} = \sum \ell \xi^{(\ell-1)} \frac{\partial}{\partial \xi^{(\ell)}}, \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \Delta = \sum (n-\ell) \xi^{(\ell+1)} \frac{\partial}{\partial \xi^{(\ell)}}, \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \mathcal{H} = \sum (n-2\ell) \xi^{(\ell)} \frac{\partial}{\partial \xi^{(\ell)}}. \end{cases}$$

Definition 1.

$$\rho^{[m]} = \{\text{covariants of index } m\}$$

$$= \left\{ F(\xi, z) = \sum \binom{m}{\ell} c_{\ell}(\xi) z^{\ell} \mid \begin{array}{l} c_{\ell}(z) \in K[\xi], \\ F\left(\begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix} \xi, z\right) = (\gamma z + \delta)^m F\left(\xi, \frac{\alpha z + \beta}{\gamma z + \delta}\right) \end{array} \right\}.$$

Definition 2.

$$\begin{aligned} \mathbb{G}^{[m]} &= \{\text{semi-invariants of index } m\} \\ &= \{\varphi \in K[\xi] \mid \mathcal{D}\varphi = 0, \mathcal{M}\varphi = m\varphi\}. \end{aligned}$$

Problem. To seek all covariants of index m .

Solution. (Robert's theorem)

$$\rho^{[m]} = \{\exp(z\Delta)\varphi(\xi) \mid \varphi(\xi) \in \mathbb{G}^{[m]}\}.$$

Remark. This solution $\exp(z\Delta)\varphi(\xi)$ is a typical explicit solution of mathematical problems.

2. Automorphic forms

Let us choose a formal power series

$$f(\xi|z) = \sum_{\ell=0}^{\infty} \frac{(-2k)_{\ell}}{\ell!} \xi^{(\ell)} z^{\ell}$$

with variable coefficients, where

$$(-2k)_{\ell} = (-2k)(-2k-1)\cdots(-2k-\ell+1).$$

Denoting

$$\begin{cases} \mathcal{D} = \sum \ell \xi^{(\ell-1)} \frac{\partial}{\partial \xi^{(\ell)}}, \\ \Delta = \sum (-2k-\ell) \xi^{(\ell+1)} \frac{\partial}{\partial \xi^{(\ell)}}, \\ \mathcal{M} = \sum (-2k-2\ell) \xi^{(\ell)} \frac{\partial}{\partial \xi^{(\ell)}}, \end{cases}$$

we have a realization of $\mathfrak{sl}_2(\mathbb{C})$. Denote

$$\mathcal{G}^{[-2m]} = \{\varphi \in K[\xi] \mid \partial\varphi = 0, \Delta\varphi = -2m\varphi\}.$$

Problem. Let $h(z)$ be an automorphic form of dimension $-2k$. To seek all automorphic forms of dimension $-2m$ which are differential polynomials of $h(z)$.

Solution. Assume that the Zariski closure of the automorphic group coincides with $\mathrm{PSL}_2(\mathbb{C})$. And denote

$$h(z) = \sum_{\ell=0}^{\infty} \frac{(-2k)_{\ell}}{\ell!} \alpha^{(\ell)} z^{\ell}.$$

Then

$$\begin{aligned} & \{\exp(z\Delta)\varphi(\xi) \mid_{\xi=\alpha} \mid \varphi(\xi) \in \mathcal{G}^{[-2m]}\} \\ &= \left\{ \begin{array}{l} \text{automorphic forms of dimension } -2m \\ \text{which are differential polynomials of } h(z) \end{array} \right\}. \end{aligned}$$

Reference

Hisasi Morikawa, Invariant theory, Kinokuniya.