

Thom's transversality theorem in d-minimal structures

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概要

Ta Lê Loi proved a definable version of Thom's transversality theorem in o-minimal structures without restrictions on the differentiability class and dimensions of manifolds involved though some restrictions are required in the general case. Loi's version of Thom's transversality theorem also holds in d-minimal structures.

1 Introduction

Loi's versions of Thom's transversality theorem [13, Theorem 2] in o-minimal structures [1] holds even in d-minimal structures. Another version is found in [17, Theorem II.5.4]. Throughout, a d-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \dots)$ is fixed. The structure $\mathcal{F}$ is *d-minimal* if every definable subset X of F has a supremum and an infimum in $F \cup \{\pm\infty\}$ and it is the union of an open set and finitely many discrete sets, where the number of discrete sets does not depend on the parameters of definition of X [15, 4]. We also fix positive integers p and r with $p > 1$ and $r < p$. Throughout, 'definable' means 'definable with parameters'. We abbreviate 'definable $\mathcal{C}^p$ ' by $\mathcal{D}^p$. For instance, a $\mathcal{D}^p$ map is a definable $\mathcal{C}^p$ map.

The following is our main theorem:

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Theorem 1.1. *Let N and M be $\mathcal{D}^p$ submanifolds. Let $\mathcal{A}$ be a finite collection of $\mathcal{D}^1$ submanifolds of $J^r(N, M)$. Then the set*

$$\tau_r(\mathcal{A}) = \{f \in \mathcal{D}^p(N, M) \mid j^r f \text{ is transverse to each member of } \mathcal{A}\}$$

is dense in $\mathcal{D}^p(N, M)$. Moreover, if $\mathcal{A}$ is a stratification of a closed subset and satisfies Whitney's condition (a), then $\tau_r(\mathcal{A})$ is an open subset of $\mathcal{D}^p(N, M)$.

As a corollary, the theorem implies that the set of $\mathcal{D}^2$ functions not having degenerate critical points (they are called definable nondegenerate functions in [8]) is open and dense in $\mathcal{D}^2(N)$. This corollary was already proven in [9] when N is definably compact.

The terms and notations used in the theorem are found in several previous studies, and Section 2 introduces the references in which these are found. Loi's proof for definable Thom's transversality theorem does not use o-minimality directly, but it uses several differential geometric facts holding in o-minimal structures. Once we verify that these facts also hold in d-minimal structures, Loi's proof also works in our case. The author found d-minimal counterparts to almost all tools used in Loi's proof in the previous studies. Only one exception is the existence of $\mathcal{D}^p$ tubular neighborhood. We prove it in Section 3. We quickly look over Loi's proof and confirm that his proof is valid for d-minimal structures in Section 4.

2 Definitions

We want to recall the terms and notations used in Theorem 1.1, but several pages are required if we duplicate them here. Therefore, we only give short explanations herein.

We define a $\mathcal{D}^p$ submanifold of F^n in the same way as the classical definition of submanifolds except that the sets and maps involved in the definition are required to be definable. We often omit the ambient space F^n . We define $\mathcal{D}^p$ maps between $\mathcal{D}^p$ submanifolds similarly. A fiber bundle (E, B, π, F) is called an $\mathcal{D}^p$ fiber bundle if E , B , F and π are $\mathcal{D}^p$ and there exists a local trivialization $\{(U_i, \varphi_i)\}$ such that U_i and φ_i are also $\mathcal{D}^p$. We define $\mathcal{D}^p$ sections of the bundle similarly.

Note that the tangent bundle $\pi : TN \rightarrow N$ of a $\mathcal{D}^p$ submanifold N is an $\mathcal{D}^{p-1}$ fiber bundle. A $\mathcal{D}^{p-1}$ section of $\pi : TN \rightarrow N$ is called a $\mathcal{D}^{p-1}$ vector field on N . By the

definition, Df is a $\mathcal{D}^{p-1}$ function for every $\mathcal{D}^p$ function $f : N \rightarrow F$ and every $\mathcal{D}^{p-1}$ vector field D . We denote the set of $\mathcal{D}^p$ maps from a $\mathcal{D}^p$ submanifold N into a $\mathcal{D}^p$ submanifold M by $\mathcal{D}^p(N, M)$. We simply write $\mathcal{D}^p(N)$ in place of $\mathcal{D}^p(N, F)$. The topology on $\mathcal{D}^p(N, M)$ called the $\mathcal{D}^p$ topology is found in [2].

Definitions of the *jet bundle* $J^r(N, M)$ and the *p-jet* $j^r f$ of a $\mathcal{C}^p$ map $f : N \rightarrow M$ are found in several books on differential geometry such as [11, Section 12]. $J^r(N, M)$ is a $\mathcal{D}^{p-r}$ submanifold, and this fact is proven in the same manner as [17, p.176-177]. In addition, the natural projections $\pi_N : J^r(N, M) \rightarrow N$, $\pi_M : J^r(N, M) \rightarrow M$ and $\pi_{N \times M} : J^r(N, M) \rightarrow N \times M$ are $\mathcal{D}^{p-r}$ fiber bundles.

A partition of a definable set X into finitely many definable submanifolds $X = \bigcup_{i=1}^k C_i$ is called a *stratification* of X if, for each i , the frontier ∂C_i of C_i is a union of some of the C_j . In some definition of stratification, each C_i is required to be connected, but we do not employ this requirement in this paper. In $\mathcal{F}$, there always exists a stratification of a definable set satisfying Whitney's condition (a) by [4, Proposition 5.44].

3 Existence of $\mathcal{D}^p$ tubular neighborhoods

We prove the existence of $\mathcal{D}^p$ tubular neighborhoods. First, we introduce technical notions.

Definition 3.1. Let $\pi : F^n \rightarrow F^d$ be the projection onto the first d coordinates. A definable subset M of F^n is called a π -regular $\mathcal{D}^p$ submanifold of F^n if, for any $x \in M$, there exists a definable open box U containing the point x such that $X \cap U$ is the graph of a definable $\mathcal{C}^r$ map defined on $\pi(U)$. When $\pi : F^n \rightarrow F^d$ is a general coordinate projection, M is π -regular $\mathcal{D}^p$ submanifold of F^n if it is $\pi \circ \sigma$ -regular $\mathcal{D}^p$ submanifold, where $\sigma : F^n \rightarrow F^n$ is a permutation of coordinates such that $\pi \circ \sigma$ is the projection onto the first d coordinates. A *regular $\mathcal{D}^p$ submanifold* of F^n is a π -regular $\mathcal{D}^p$ submanifold of F^n for some coordinate projection π .

Let M be a π -regular $\mathcal{D}^p$ submanifold. Let $\sigma : F^n \rightarrow F^n$ be a permutation of coordinates such that $\pi \circ \sigma$ is the projection onto the first d coordinates. The set $\{y \in F^{n-d} \mid \sigma^{-1}(u, y) \in A\}$ is denoted by A_u^π for every $u \in F^d$ and every subset A of F^n . We set $\mathcal{B}_m(x, t) = \{y \in F^m \mid \|y - x\| < t\}$ for $x \in F^m$ and $t > 0$. The tuple

(T, η, ρ) is a *standard $\mathcal{D}^p$ tubular neighborhood* of M if

- (a) T is a definable open neighborhood of M in F^n ;
- (b) $\eta : M \rightarrow F$ is a positive definable $\mathcal{C}^p$ function such that, for all $u \in \pi(M)$, we have

$$T_u^\pi = \bigcup_{x \in M_u^\pi} \mathcal{B}_{n-d}(\pi(x), \eta(\sigma^{-1}(u, x)))$$

and

$$\mathcal{B}_{n-d}(\pi(x_1), \eta(\sigma^{-1}(u, x_1))) \cap \mathcal{B}_{n-d}(\pi(x_2), \eta(\sigma^{-1}(u, x_2))) = \emptyset$$

for all $x_1, x_2 \in M_u^\pi$ with $x_1 \neq x_2$;

- (c) $\rho : T \rightarrow M$ is a $\mathcal{D}^p$ retraction such that, for any $u \in \pi(M)$, we have $\rho(\pi^{-1}(u) \cap T) \subseteq \pi^{-1}(u) \cap M$ and $\rho(\sigma^{-1}(u, y)) = \sigma^{-1}(u, x)$ for all $x \in M_u^\pi$ and $y \in \mathcal{B}_{n-d}(x, \eta(\sigma^{-1}(u, x)))$.

Lemma 3.2. *Let $\pi : F^n \rightarrow F^d$ be a coordinate projection. A π -regular $\mathcal{D}^p$ submanifold of F^n has a standard $\mathcal{D}^p$ tubular neighborhood.*

Proof. Let M be a π -regular $\mathcal{D}^p$ submanifold of F^n . We assume that π is the projection onto the first d coordinates for simplicity. The fiber M_t^π is discrete for every $t \in F^d$ by the definition of π -regular definable $\mathcal{C}^p$ submanifolds. Consider the definable function $\rho : M \rightarrow F$ given by $\eta_0(x) = \min\{1, \inf\{\|x - y\|^2/2 \mid x \neq y \in M, \pi(y) = \pi(x)\}\}$. We have $\eta_0(x) > 0$ for every $x \in M$ because $M_{\pi(x)}^\pi$ is discrete. Furthermore, η_0 is locally bounded from below by positive constants by the definition of π -regular $\mathcal{D}^p$ submanifolds. We can find a positive $\mathcal{D}^p$ function $\eta : M \rightarrow F$ satisfying the inequality $\eta < \eta_0$ on M by [8, Lemma 3.11].

Set $T = \{y \in F^n \mid \exists x \in M, \pi(y) = \pi(x) \text{ and } \|y - x\| < \eta(x)\}$. It is easy to check that condition (b) in Definition 3.1 is satisfied. The proof is left to readers. We next define a definable map $\rho : T \rightarrow M$. For every $y \in T$, we can find a unique $x \in M$ such that $\pi(y) = \pi(x)$ and $\|y - x\|^2 < \varepsilon(x)$ by the definition of T and ε . We define $\rho(y)$ as the unique point $x \in M$ satisfying the above conditions. It is obvious that ρ satisfies condition (c) in Definition 3.1 other than the condition that ρ is of class $\mathcal{C}^r$.

We show that T is open and the definable map ρ is of class $\mathcal{C}^p$. Fix an arbitrary point $y_0 \in T$. Set $x_0 = \rho(y_0)$. By the definition of π -regular $\mathcal{D}^p$ submanifolds, we can take an open box U containing the point x_0 and a definable $\mathcal{C}^p$ map $f : \pi(U) \rightarrow F^{n-d}$ such

that $M \cap U$ is the graph of f . Set $S = \{(t, u) \in \pi(U) \times F^{n-d} \mid \|u - f(t)\|^2 < \varepsilon(t, f(t))\}$. The definable set S is open because the definable maps f and $\varepsilon(\cdot, f(\cdot))$ are continuous on $\pi(U)$. It is obvious that $y_0 \in S$ and T contains S . This implies that T is open. The restriction of ρ on S coincides with the $\mathcal{D}^p$ map given by $y \mapsto (\pi(y), f(\pi(y)))$. This implies that ρ is of class $\mathcal{C}^p$.

The pair (T, η, ρ) is a standard $\mathcal{D}^p$ tubular neighborhood of M . $\square$

Proposition 3.3. *Let X be a definable set and $f : X \rightarrow F$ be a definable map. There exists a partition $X = \bigcup_{i=1}^m C_i$ into regular $\mathcal{D}^p$ submanifolds such that the restriction $f|_{C_i}$ is of class $\mathcal{C}^p$ for every $1 \leq i \leq m$.*

Proof. For every definable subset A of F^n , we define $d(A), r(A), p(A)$ as follows: We set $d(A) = \dim A$. Let $\Pi(A)$ be the set of the coordinate projections $\pi : F^n \rightarrow F^{d(A)}$ such that $\pi(A)$ has a nonempty interior. Let $r(A, \pi)$ be a largest positive integers r such that there exists a nonempty definable open subset U of F^d contained in $\pi(A)$ with $\dim A_t^\pi = 0$ and $\text{rank}(A_t^\pi) = r$. Such $r(A, \pi)$ exists by d-minimality. We define $r(A) = \max\{r(A, \pi) \mid \pi \in \Pi(A)\}$ and $p(A) = \#\{\pi \in \Pi(A) \mid r(A) = r(A, \pi)\}$.

Fix a coordinate projection π so that $r(X) = r(X, \pi)$. We may assume that π is the coordinate projection onto the first $d := d(X)$ coordinates for simplicity. Let W be the interior of the set of points $t \in \pi(X)$ with $\dim X_t^\pi = 0$ and $\text{rank } X_t^\pi = r(X)$. Consider the set $Y := \{x \in X \cap \pi^{-1}(W) \mid x \text{ is isolated in } X_{\pi(x)}^\pi\}$. Let U_π be the set of points x in $X \cap \pi^{-1}(W)$ at which there exist an open box U in F^n containing the point x and a $\mathcal{D}^p$ map $\tau : \pi(U) \rightarrow F^n$ such that $M \cap U = \tau(\pi(U))$ and $\pi \circ \tau$ is the identity map defined on $\pi(U)$. We obviously have $U_\pi \subseteq Y$. Let $V_\pi := \{x \in U_\pi \mid f \text{ is of class } \mathcal{C}^p \text{ around } x\}$.

We can prove the following by reduction to the absurd using [4, Lemma 3.14] and [14]. We omit the proofs.

- $\pi(Y \setminus U_\pi)$ has an empty interior;
- $\pi(U_\pi \setminus V_\pi)$ has an empty interior.

It is obvious that V_π is a π -regular $\mathcal{D}^p$ submanifold and $(d(X \setminus V_\pi), r(X \setminus V_\pi), p(X \setminus V_\pi)) < (d(X), r(X), p(X))$ under the lexicographic order. The proposition follows by induction on $(d(X), r(X), p(X))$. $\square$

Lemma 3.4 (Partition of unity). *Let M be a $\mathcal{D}^p$ submanifold and $\{U_i\}_{i=1}^m$ be a finite definable open cover of M . There exists a $\mathcal{D}^p$ partition of unity subordinated to $\{U_i\}_{i=1}^m$.*

Proof. A $\mathcal{D}^p$ partition of unity is constructed in a standard way when every definable closed set is the zero set of a $\mathcal{D}^p$ function. See [5, Lemma 2.2-Lemma 2.6] and their proofs for the construction. Every definable closed set is the zero set of a $\mathcal{D}^p$ function by [16]. $\square$

Proposition 3.5. *Let M be a $\mathcal{D}^p$ submanifold and $f, \varepsilon : M \rightarrow F$ be a definable continuous functions such that $\varepsilon > 0$ on M . There exists a $\mathcal{D}^p$ function $g : M \rightarrow F$ such that $|g - f| < \varepsilon$ on M .*

Proof. We can decompose M into finitely many regular $\mathcal{D}^p$ submanifolds $C_1, \dots, C_m$ so that $f|_{C_i}$ are of class $\mathcal{C}^p$ by Proposition 3.3. Let (T_i, η_i, ρ_i) be standard $\mathcal{D}^p$ tubular neighborhood of C_i for each $1 \leq i \leq m$. Set $U_i := M \cap T_i$ and let $h_i : U_i \rightarrow F$ be the $\mathcal{D}^p$ function given by $h_i(x) = f \circ \rho_i(x)$. We extend the domain of h_i by putting $h_i(x) = 0$ off U_i . Set $V_i := \{x \in U_i \mid |h_i(x) - f(x)| < \varepsilon(x)\}$. $\{V_i\}_{i=1}^m$ is a definable open cover of M . Let $\{\phi_i : M \rightarrow F\}_{i=1}^m$ be a $\mathcal{D}^p$ partition of unity subordinated to $\{V_i\}_{i=1}^m$. The function $g : M \rightarrow F$ defined by $g(x) = \sum_{i=1}^m \phi_i(x)h_i(x)$ is a desired function. $\square$

Corollary 3.6 (Existence of a $\mathcal{D}^p$ tubular neighborhood). *Every $\mathcal{D}^p$ submanifold has a $\mathcal{D}^p$ tubular neighborhood.*

Proof. We can prove the corollary in the same manner as [2, Theorem 1.9] using Proposition 3.5. $\square$

4 A quick look at the proof of Theorem 1.1

In [13], Loi first introduced the following basic facts on $\mathcal{D}^p(N, M)$ given in [2, Proposition 1.2 and Proposition 1.3].

- Let X be a $\mathcal{D}^p$ submanifold of F^n and let $Y \subseteq X$ be a $\mathcal{D}^p$ submanifold of X .

The restriction mapping

$$\text{res} : \mathcal{D}^p(X) \ni f \mapsto f|_Y \in \mathcal{D}^p(Y)$$

is continuous.

- Let X, Y, Z be $\mathcal{D}^p$ submanifolds and $h : Y \rightarrow Z$ be a $\mathcal{D}^p$ map. Then

$$h_* : \mathcal{D}^p(X, Y) \ni f \mapsto h_*(f) = h \circ f \in \mathcal{D}^p(X, Z)$$

is continuous.

The first one is proven in the same manner as [2, Proposition 1.2]. The second one follows from the definition of $\mathcal{D}^p$ topology and chain rules.

After recalling several basic facts on definable jets and definable stratification in Section 3 and Section 4 of [13], definable Sard's theorem was introduced in Section 5 of [13]. Sard's theorem also holds in d-minimal structures. See [8, Proposition 3.5].

Three key lemmas are used in Loi's proof. The first one is the existence of a 'small' definable function, whose counterpart is found in [8, Lemma 3.12]. The second key lemma is tricky equalities obtained by using Leibnitz's rule and basic linear algebra. They hold in our cases. The third key lemma is as follows:

Lemma 4.1. *Let N, J and T be $\mathcal{D}^p$ submanifolds and $\Phi : T \times N \rightarrow J$ be a $\mathcal{D}^p$ map. Let $\mathcal{A}$ be a finite collection of $\mathcal{D}^1$ submanifolds of J . If Φ is submersive, then the set*

$$\tau(\Phi, \mathcal{A}) = \{t \in T \mid \Phi(t, \cdot) \text{ is transverse to each member of } \mathcal{A}\}$$

is a definable set and $\dim(T \setminus \tau(\Phi, \mathcal{A})) < \dim T$.

Proof. Definability of $\tau(\Phi, \mathcal{A})$ is obvious. The remaining task is to prove the inequality.

We can reduce to the case in which $\mathcal{A} = \{A\}$ by using [4, Lemma 4.5(Dim 2)]. Observe that the assumption of [4, Lemma 4.5(Dim 2)] holds in our case thanks to [14]. The inverse image $S := \Phi^{-1}(A)$ is a $\mathcal{D}^1$ submanifold of $T \times N$ by [8, Lemma 2.2]. Let $\pi : T \times N \rightarrow T$ be the projection. For every $t \in T$, t is a regular point of $\pi|_S$ if and only if $\Phi(t, \cdot)$ is transversal to A . We can prove this in a similar manner to the proof of [8, Lemma 3.7]. The lemma follows from the definable Sard's theorem [8, Proposition 3.5]. $\square$

We explain the main body of Loi's proof. First, he reduced to the case where N is a definable open set and $M = F^m$. This reduction was made using [2, Proposition 1.2 and Proposition 1.3] and the existence of $\mathcal{D}^p$ tubular neighborhoods. Both tools are available in our setting, so we can reduce to the same situation in the same manner as Loi's proof. The density of $\tau_r(\mathcal{A})$ was proved using three key lemmas. The same proof works in our case.

For openness of $\tau_r(\mathcal{A})$, Loi mainly referred the proof of [3, Proposition 3.6]. In this proof, it was demonstrated that the complement $K := T \setminus \tau_r(\mathcal{A})$ is closed. We should be careful about the difference of the assumption made by Loi with ours. Loi assumed that the underlying space is $\mathbb{R}$, but we do not. A slight modification of the proof is required. When the underlying space is $\mathbb{R}$, we can prove closedness of K by verifying the limit of every convergent sequence in K belongs to K . This criterion was used in the proof of [3, Proposition 3.6]. However, we cannot use this criterion in our case. We use the notion of pseudo-curves in [6, Definition 3.1] and a criteria for closedness in [6, Proposition 3.5] instead.

参考文献

- [1] L. van den Dries, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, Vol. 248. Cambridge University Press, Cambridge, 1998.
- [2] J. Escribano, *Approximation theorem in o-minimal structures*, Illinois J. Math., **46** (2002), 111-128.
- [3] E. A. Feldman, *The geometry of immersion. I*, Trans. Amer. Math. Soc., **120** (1965), 185-224.
- [4] A. Fornasiero, *D-minimal structures version 20*, preprint (2021), arXiv:2107.04293.
- [5] M. Fujita, *Definable C^r vector bundles and bilinear spaces in an o-minimal structure and their homotopy theorems*, preprint (2020), arXiv:2002.03081v1.
- [6] M. Fujita, *Definable quotients in d-minimal structures*, preprint (2023), arXiv:2311.08699v1.
- [7] M. Fujita, *Definable C^r structures on definable topological groups in d-minimal structures*, preprint (2024), arXiv:2404.15647v1.

- [8] M. Fujita, *Morse functions definable in d -minimal structures*, preprint (2024).
- [9] M. Fujita and T. Kawakami, *Definable Morse functions on definably compact manifolds in d -minimal structures*, Suurikaisekikokyuroku (2024).
- [10] M. Fujita, T. Kawakami and W. Komine, *Tameness of definably complete locally o -minimal structures and definable bounded multiplication*, Math. Log. Quart., 68 (2022), 496-515.
- [11] I.Kolář, J.Slovák and P.W. Michor, *Natural operations in differential geometry*, Springer-Verlag, Berlin, 1993.
- [12] T. L. Loi, *Density of Morse functions on sets definable in o -minimal structures*, Ann. Polo. Math., **89** (2006), 289-299.
- [13] T. L. Loi, *Transversailty theorem in o -minimal structures*, Compositio Math., **144** (2008), 1227–1234.
- [14] C. Miller, *Definable choice in d -minimal expansions of ordered groups*, unpublished note, <https://people.math.osu.edu/miller.1987/eidmin.pdf> (2006).
- [15] C. Miller, *Tameness in expansions of the real field*, In M. Baaz, S. -D. Friedman and J. Krajíček eds., Logic Colloquium '01, Cambridge University Press (2005), 281-316.
- [16] C. Miller and A. Thamrongthanyalak, *D -minimal expansions of the real field have the zero set property*, Proc. Amer. Math. Soc., **146** (2018), 5169–5179.
- [17] M. Shiota, *Geometry of subanalytic and semialgebraic sets*, Progress in Mathematics, vol. 150. Springer Science and Business Media, LLC, New York, 1997.

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