

Noncausal Stochastic Calculus Revisited

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1 Introduction

The causal theory of stochastic calculus originated by K.Itô in 1942 is founded on the hypothesis of Causality, saying that; "Every function $f(t, \omega)$, ($t \geq 0, \omega \in \Omega$) should be *adapted to the increasing family of σ -fields*, generated by the underlying basic process $Z_t(\omega)$, which is a square integrable semi-martingale like the Brownian motion" The hypothesis seems well fit to the principle of causality in physics, where the variable " t " appears as time parameter. Moreover it endows the theory a remarkable feature of being in natural concordance with the notion of martingale which plays indeed an essential role in Itô's Calculus.

However it is also true that the hypothesis of Causality has imposed a significant restriction on the applicability of the causal theory of stochastic calculus. Let us take for example the case of such SDEs driven by general stochastic processes, like the fractional Brownian motion, that do not have the martingale property ([8]), or the case of usual SDEs but under noncausal situations, that is the SDE with noncausal coefficients and/or noncausal initial values. Let us take moreover the case where the parameter " t " stands for the *space* parameter, or the case where " t " is multi-dimensional (ie., "a stochastic calculus" for the random field, [14]). The notion of Causality loses its sound meaning in such cases because of the lack of natural sense of *time direction*. Even in the case of physical problems where " t " appears as *time parameter*, we can find various situations of noncausal nature, such as the Cauchy problem in the theory of Brownian particle equations [13], noncausal version of the SDEs, the White noise analysis [23] etc.

Guided by these motivations the author had presented the noncausal theory of stochastic calculus in 1979 [23], by introducing the noncausal stochastic integral which is now refereed by author's name. In this note we will give a unified sketch of that non-causal stochastic calculus. We will show recent development of the theory (cf.[10],[13])

and we will also refer to some typical applications of the theory to mathematical sciences including mathematical finance.

2 Noncausal problems in stochastic analysis

To motivate our study we like to show in this paragraph some typical examples of non-causal nature in stochastic analysis. Hereafter we fix once for all a probability space $(\Omega, \mathcal{F}, P)$ on which all random quantities are defined. Among these we understand by $W_t(\omega), t \geq 0$ the standard Brownian motion and denote by $\{\mathcal{F}_t\}$ the natural filtration generated by the W ., namely the family of σ fields $\mathcal{F}_t = \sigma\{W_s; 0 \leq s \leq t\} \subset \mathcal{F}$.

By the random functions we understand those real valued functions $f(t, \omega)$ which are measurable in (t, ω) with respect to the field $B_{[0, T]} \times \mathcal{F}$ and satisfy the condition,

$$P\left\{\int_0^T f^2(t, x, \omega) dt < \infty\right\} = 1$$

We will denote by $\mathbf{H}$ the set of all such random functions. A random function $f(t, \omega) \in \mathbf{H}$ is called *causal* (or non-anticipative) if, for each $t \in [0, T]$ it is adapted to the σ field $\mathcal{F}_t$. We will denote by $\mathbf{M}$ the totality of causal random functions. For the random function f of this class, we have the Itô integral which we will denote by $\int f dW_t$ throughout the paper, while for the B-differentiable function $f \in \mathbf{M}$ the integral of symmetric type (or noncausal type) is denoted by the symbol, $\int f d_* W_t$ (see, Paragraph 3).

Example 1 – Stochastic conservation law

Suppose given two stochastic processes, X_t, Y_t ($t \geq 0$) which obey a conservation law as follows;

$$E(X_t, Y_t, t, \alpha, \beta, \gamma, \dots) = Const,$$

where α, β, γ etc are causal constants, that is the deterministic or at least the random variables which are independent of the W .. Now suppose that the process X_t is unknown but another one is known to be generated by the following mechanism,

$$dY_t = a(Y_t)dt + b(Y_t)dW_t, \quad Y_0 = y_0 \in R^1.$$

Given these we are to derive the equation of dynamics for the process X_t . We are already familiar with such problem especially in mathematical physics, but also in the mathematical finance we face to this. For example some authors like M.Schweizer, E.Platen this idea for the modelling of the price process in some specific situation, and we, S.Ogawa and M.Mancino [11], followed the same way to derive the SDE model for

price process in such market that admits a feedback of the future information about the delta hedging strategy.

For the derivation of the desired SDE, we are only to apply the Itô formula to the conservation equation. But an essential problem can arise when the parameters α, γ etc. are allowed to arbitrary random variables dependent of the W , for in such case we can no longer suppose that the process X_t is causal and we can not apply the Itô theory.

Let us take an example again from the mathematical finance.

Example 2 – Black-Sholes model in noncausal situation

The SDE in Black-Sholes model is as follows;

$$dS_t = rS_t dt + \sigma^2 S_t dW_t, \quad (1)$$

which can be interpreted in the following form of the SDE with the symmetric stochastic integrals,

$$dS_t = \left(r - \frac{\sigma^2}{2}\right) S_t dt + \sigma S_t d_* W_t, \quad (2)$$

where the term $\int d_* W_t$ represents the symmetric integral ([?]), that is; for the B-differentiable causal function $f(t, \omega)$,

$$\int f(t, \omega) d_* W_t = \int f dW_t + \frac{1}{2} \int \frac{\partial}{\partial W_t} f dt,$$

$\frac{\partial}{\partial W_t} f$ is the B-derivative of the f . The symmetric integral is a special case of the noncausal integral that we are to study in the next paragraph.

To this simple model there correspond the following two variations of noncausal nature;

1. *Case 1* ; The case where the constants r, σ in the B-S equation are replaced by arbitrary random variables.
2. *Case 2* ; The case where the Brownian motion W in the equation is replaced by such random process which does not have a martingale property, for example the Fractional Brownian motion.

The first modification can arise when we need to study the B-S model in such situation of admitting the insider trading, and the latter case has been already discussed as the fractional B-S model by many authors, like A.Shiryaev [6], P.Chiridito [1] for

example, but without giving a justification of such noncausal type "SDEs" in terms of the noncausal calculus.

For the study of these modified models, first of all we need the introduction of the noncausal stochastic calculus and then we are to give precise meaning to such noncausal SDEs as follows;

$$\begin{aligned} dX_t &= a(t, X_t, \eta(\omega))dt + b(t, X_t, \zeta(\omega))dW_t, \\ X_0(\omega) &= \xi(\omega). \end{aligned} \quad (3)$$

In view of the application to the SDEs of the cases 1 and 2, we are concerned with the study of the noncausal SDEs, with a special interest on the validity of a *noncausal Itô formula*. We will study these subjects in the paragraphs 3,4 and 5 in a much more general situation.

Example 3 – SIE of Fredholm type

Let us consider the boundary value problem for the second order SDE as follows;

$$\begin{aligned} \left\{ \frac{d}{dt}p(t, \omega) \frac{d}{dt} + q(t, \omega) \right\} X(t) &= X(t) \frac{dZ}{dt}(t, \omega) + h(t, \omega) \\ X(0) &= \xi_0(\omega), \quad X(1) = \xi_1(\omega), \end{aligned} \quad (4)$$

where ξ_0 and ξ_1 are arbitrary random variables.

where Z_t , $t \in [0, 1]$ is an arbitrary stochastic process defined on the $(\Omega, \mathcal{F}, P)$, with square integrable sample paths.

Then as we do for the boundary value problem of ordinary differential equations, by using the Green's function $K(t, s, \omega)$ corresponding to the above situation, we may get in a very formal way the following integral equation of Fredholm type;

$$X(t, \omega) = f(t, \omega) + \int_0^1 L(t, s, \omega) X(s) ds + \int_0^1 K(t, s, \omega) X(s) d_\varphi Z(s), \quad (5)$$

where $L(t, s, \omega) = h(s)K(t, s, \omega)$ and $\int d_\varphi Z(s)$ represents the stochastic integral of noncausal type with respect to an orthonormal basis $\{\varphi_n\}$ in $L^2(0, 1)$.

This is a very typical subject in the noncausal stochastic calculus, since in such situation we can no more suppose that the solution X_t is still causal (it i.e. adapted) to the natural filtration generated by the underlying fundamental process Z_t , hence the stochastic integral term $\int X_t dZ_t$ loses its meaning in the framework of the Itô's theory.

In the article of 1986 ([16]) the author studied this subject in connection with the boundary value problems of stochastic differential equations and he showed the existence and the uniqueness of solutions (see Theorem 1 in [16]), under some reasonable assumptions on the choice of the fundamental pair $(Z, \{\varphi_n\})$ and on the sample regularity of the kernels K and L .

As a natural extension of such SIE (stochastic integral equation) to the case where the Z_t , X_t are the random fields, namely the stochastic processes with multi-dimensional parameters $t \in J = [0, 1]^d$, we can think of the SIE for the random fields. Imagine for example the case where the driving force process Z_t is the Brownian sheet.

$$X(t, \omega) = f(t, \omega) + \int_J L(t, s, \omega) X(s) ds + \int_J K(t, s, \omega) X(s) d_\varphi Z(s), \quad (6)$$

For the review of this subject we would refer to the coming article [9], which will appear in another monograph of lecture notes of the same conference.

So far the examples of noncausal problems are given where the noncausality has entered into the situation through the noncausal quantities, such as the noncausal initial values, the noncausal coefficients in SDE, or the driving stochastic process that does not have the martingale or semi-martingale property. But the noncausal problem can arise even in the ordinary situation where all the random quantities are supposed to be causal, that is *adapted* to the natural filtration $\{\mathcal{F}_t\}$. Here is another example which is typical in this sense.

Example 4 – The SPDE called BPE

Given the real Brownian motion W_t and smooth coefficients

$$a(t, x), b(t, x), A(t, x), B(t, x), C(t, x), u_0(x), \quad (t, x) \in [0, T] \times R^1,$$

we consider the Cauchy problem of a stochastic partial differential equation as follows;

$$\frac{\partial}{\partial t} u + \left\{ a(t, x) + b(t, x) \frac{dW_t}{dt} \right\} \frac{\partial}{\partial x} u = A(t, x) u(t, x, \omega) \frac{dW_t}{dt} + Bu + C(t, x) \quad (7)$$

$$u(0, x, \omega) = u_0(x)$$

By the solution of this problem, we understand the random function $u(t, x, \omega)$ measurable in (t, x, ω) with respect to the field $B_{[0, T]} \times B_{R^1} \times \mathcal{F}$, especially causal in (t, ω) for each fixed $x \in R^1$, and satisfies the following weak solution equality with probability one for any test function $\varphi(t, x)$ (cf. [24]);

$$\begin{aligned} & \int \int_G [\{\varphi_t + (a\varphi)_x + A\varphi\} u(t, x, \omega) + C\varphi] dt dx \\ & + \int_{R^1} dx \int_0^T (b\varphi)_x u(t, x, \omega) d_* W_t + \int_{R^1} u_0(x) \varphi(0, x) dx = 0, \end{aligned} \quad (8)$$

where $G = [0, T] \times \mathbf{R}^1$, $\varphi_t = \frac{\partial}{\partial t}\varphi$, $\varphi_x = \frac{\partial}{\partial x}\varphi$ and the term $\int d_* W_t$ stands for the symmetric stochastic integral (see the Paragraph 3 below) as it is remarked in the introduction.

The equation of this type, called the "Brownian particle equation", was introduced by the author ([24]) in the study of wave propagation in random media as a mathematical model for such stochastic propagation phenomenon carried by the Brownian or diffusive particles, namely the propagation along the stochastic trajectory X_t governed by the SDE,

$$dX_t = a(t, X_t)dt + b(t, X_t)d_* W_t.$$

Now the noncausal situation arises in this problem when we try to construct the solution $u(t, x, \omega)$ by means of the *method of stochastic characteristics* which was first studied by the author; By the formal application of the well known method of characteristics in the theory of the partial differential equation of the first order, or of hyperbolic type, we will get in this case the following system of (symmetric) stochastic integral equations.

$$\begin{aligned} X^{(t,x)}(s) &= x + \int_s^t a(r, X^{(t,x)}(r))dr + \int_s^t b(r, X^{(t,x)}(r))d_* W_r, \quad (0 \leq s \leq t \leq T) \\ u(t, x, \omega) &= u_0(X^{(t,x)}(0)) + \int_0^t \{Bu(s, X^{(t,x)}(s), \omega) + C(s, X^{(t,x)}(s))\}ds \\ &\quad + \int_0^t Au(s, X^{(t,x)}(s), \omega)d_* W_s. \end{aligned} \tag{9}$$

Here the first equation gives the characteristic curve $X^{(t,x)}(s)$ ($s \leq t$) passing through the fixed point (t, x) , along which the phenomenon propagates, and the second equation is derived by integrating the PDE along that stochastic calculus. Now remember that the solution $u(t, x, \omega)$ is adopted to the σ -field $\mathcal{F}_t$, while the characteristic curve $X^{(t,x)}(s)$, $0 \leq s \leq t \leq T$ is measurable with respect to the σ -field $\mathcal{F}_t^s := \sigma\{W(t) - W(r); s \leq r \leq t\}$. The composed function $u(s, X^{(t,x)}(s), \omega)$ appeared in the second equation above is no more causal with respect to the Brownian motion and hence the term of stochastic integral loses its meaning.

Such are the examples of noncausal problems, for the treatment of which we need another theory of stochastic calculus that is free from the restriction of the *Causality*. The calculus introduced by the author in 1979 is one of such theories and has many advantageous properties compared to other calculus from the viewpoint of the tool for the modelling and analysis of noncausal random problems. We are going to give in what follows a resume of that noncausal theory of stochastic calculus, brief but sufficient for us to see that these problems can be treated in very natural way.

3 Review of the noncausal stochastic calculus

For the rigorous study of those noncausal problems listed in the previous paragraph, we need to introduce a noncausal stochastic calculus that is free from the restriction of causality and one of such calculus was introduced by the author in 1979 ([23], [18],[22],[20] and [15], etc.). We are going to give in this paragraph a rapid review of some fundamental results in the theory of noncausal stochastic calculus, mainly following the recent article [13]. For its special importance of the Brownian motion in the stochastic theory and also for the concreteness of the discussion, we will show mainly the case of the noncausal calculus with respect to the Brownian motion W_t , but as we will see (cf. Remark 2) easily the formalism can work even for the use of more general stochastic processes instead of the W_t .

3.1 Causal functions and the B-differentiability

Following the [25], we will say that an **H**-class random function $g(t, \omega)$ is differentiable with respect to the Brownian motion W_t (or B-differentiable) provided that there exists an **M**-class random function say $\frac{\partial}{\partial W_t}g(t, \omega)$ such that, for small enough $h > 0$,

$$\sup_{t,s,|t-s|<h} E|g(t, \omega) - g(s, \omega) - \int_s^t \frac{\partial}{\partial W_r}g(r, \omega)dW_r|^2 = o(h)$$

where the integral $\int dW$ stands for the Itô's stochastic integral. The function $\frac{\partial}{\partial W_t}g$ is called the B-derivative of the g . It is not difficult to see that if the function $g(t, \omega)$ is B-differentiable then its B-derivative is uniquely determined (see [25]). The B-differentiability of the random function with respect to the multi-dimensional Brownian motion is defined in a similar way.

Remark 1 Let $g(t, \omega)$ be a functional of the multi-dimensional Brownian motion, $\mathbf{W}_t = (W_t^1, W_t^2, \dots, W_t^n)$ where the W^i , ($1 \leq i \leq n$) are independent copies of the 1-dim. Brownian motion W_t . Then the B-derivative of such function, say $\nabla_w g$, can be defined in the following way: the $\nabla_w g = (\frac{\partial}{\partial W_t^1}g, \frac{\partial}{\partial W_t^2}g, \dots, \frac{\partial}{\partial W_t^n}g)^t$ is a causal random vector such that,

$$\sup_{t,s,|t-s|<h} E|g(t, \omega) - g(s, \omega) - \sum_{k=1}^n \int_s^t \frac{\partial}{\partial W_r^k}g(r, \omega)dW_r^k|^2 = o(h)$$

Here we notice that the Itô integral is defined for the causal random functions $f(t, \omega) \in \mathbf{M}$ and roughly speaking the symmetric integrals (i.e. $\mathcal{I}_{1/2}$ of Ogawa [25] and Stratonovich-Fisk integral) are defined for the causal and B-differentiable functions. That is, the symmetric integral $\mathcal{I}_{1/2}(f)$ of a B-differentiable function was introduced

as the limit (in probability) $\lim_{|\Delta| \rightarrow 0} \mathcal{I}_\Delta(f)$ of the sequence $\{\mathcal{I}_\Delta(f)\}$ of Riemannian sums,

$$\mathcal{I}_\Delta(f) = \sum_{t_i \in \Delta} f\left(\frac{t_i + t_{i+1}}{2}\right) \{W(t_{i+1}) - W(t_i)\}$$

where, $\Delta = \{0 \leq t_1 < \dots < t_n \leq 1\}$ is a partition of the interval $[0, 1]$ and $|\Delta| = \max_i(t_{i+1} - t_i)$.

The following result was established by the author in 1970,

Theorem 3.1 ([25]) *The limit (in probability) $\mathcal{I}_{1/2}(f) = \lim_{|\Delta| \rightarrow 0} \mathcal{I}_\Delta(f)$ exists and is represented in the following form:*

$$\mathcal{I}_{1/2}(f) = \frac{1}{2} \int_0^1 \frac{\partial}{\partial W_t} f \, dt + \int_0^1 f(t, \omega) \, dW_t$$

3.2 Noncausal stochastic integral

Given a random function $f(t, \omega) \in \mathbf{H}$ and an arbitrary complete orthonormal system $\{\varphi_n\}$ in $L^2([0, 1])$, we consider the formal random series

$$\sum_n \int_0^1 f(t, \omega) \varphi_n(t) dt \int_0^1 \varphi_n(t) dW_t.$$

The stochastic integral of noncausal type introduced by the author in 1979 ([23]), is given in the following way,

Definition 1 : A random function $f(t, \omega) \in \mathbf{H}$ is said to be integrable with respect to the basis $\{\varphi_n\}$ (or φ -integrable) when the random series above converges in probability and the sum, denoted by $\int_0^1 f(t, \omega) d_\varphi W_t$, is called the stochastic integral of noncausal type with respect to the basis $\{\varphi_n\}$.

Remark 2 The validity of the above definition is not limited to the case of Brownian motion or to other square integrable semi-martingales. Indeed it can apply even to the case of general square integrable processes say Z_t that do not possess the property of semi-martingale, as long as the quantities

$$\int_0^1 \varphi_n(t) dZ_t$$

are well defined. The simplest example is when we employ the system of Haar functions $\{H_{0,0}(t), H_{n,i}(t), 0 \leq i \leq 2^{n-1} - 1, n \in \mathbf{N}\}$ as orthonormal basis. Let us

remembre that the $H_{n,i}(t)$ are as follows;

$$\begin{aligned} H_{0,0}(t) &= 1, \\ H_{n,i}(t) &= 2^{(n-1)/2} \{ \mathbf{1}_{[2^{-n}2i, 2^{-n}(2i+1))}(t) - \mathbf{1}_{[2^{-n}(2i+1), 2^{-n}2(i+1))}(t) \} \\ 0 \leq i &\leq 2^{n-1} - 1, \quad n \geq 1. \end{aligned} \quad (10)$$

In this case the quantity above is defined in the natural way as follows;

$$\begin{aligned} \int_0^1 H_{0,0}(t) dZ_t &= Z(1) - Z(0), \\ \int_0^1 H_{n,i}(t) dZ_t &= 2^{n/2} [\{ Z(\frac{2i+1}{2^n}) - Z(\frac{2i}{2^n}) \} - \{ Z(\frac{2i+2}{2^n}) - Z(\frac{2i+1}{2^n}) \}] \end{aligned} \quad (11)$$

The noncausal integral with respect to the fractional Brownian motion can be introduced by this way (cf. [8]).

In general case, the way of convergence of the random series being conditional, the integrability and the sum should depend on the basis, even on the order of the same complete system of orthonormal functions. On the relation between the noncausal integrals with respect to different bases, very few is known except the following,

Theorem 3.2 (1984,[21]) *If the random function $f(t, \omega) \in \mathbf{H}$ is integrable in the L^1 - sense (ie. convergent in $L^1(\Omega, P)$ sense) with respect to the system of trigonometric functions,*

$$\{1, \sqrt{2} \cos 2n, \sqrt{2} \sin 2n\pi x; n \in \mathbf{N}\}.$$

Then the f is integrable with respect to the system of Haar functions and the value of two integrals coincide.

If the function is integrable with respect to any basis $\{\varphi_n\}$ and the sum does not depend on the choice of the basis, we will say that the function is *universally integrable* (or shortly *u-integrable*).

3.3 Equivalent expressions and variants

Here are some equivalent expressions and a possible variation of the above definition, which are worth to be remarked so that we can have a better understanding of the nature of our noncausal integral.

(a) As a limit of the sequence of random Stieltjes integrals:

Given the pair $(W_t, \{\varphi_n\})$ we introduce the sequence of approximation processes $W_n^\varphi(t)$ in the following way.

$$W_n^\varphi(t) = \sum_{k=1}^n \int_0^t \varphi_k(s) ds \int_0^1 \varphi_k(s) dW_s. \quad (12)$$

It is immediate to see that this gives a pathwise smooth approximation of the Brownian motion $W(t, \omega)$. Moreover, by virtue of the famous theorem due to K. Itô and M. Nishio [2], we know that for any choice of the basis $\{\varphi_n\}$ the sequence $\{W_n^\varphi(t)\}$ converges uniformly in $t \in [0, 1]$ as $n \rightarrow \infty$ with probability one. Now we notice that our noncausal integral can be expressed as the limit (in probability) of the sequence of random Stieltjes integrals;

Proposition 3.1 *It holds that,*

$$\int_0^1 f d_\varphi W_t := \lim \int_0^1 f dW_n^\varphi(t) \quad (\text{in probability}).$$

(b) Riemannian definition:

Let us take the Haar functions $\{H_{n,i}(t)\}$ for basis $\{\varphi_n\}$. This is a case of special interest because we have the following,

Lemma 3.1 (1984 [21]) *Let us define the approximation process $W_n^H(t)$ for this case by the following formula,*

$$W_n^H(t) = \sum_{k \leq n} \sum_{i=0}^{2^k-1} \int_0^1 H_{k,i}(s) dW_s \int_0^t H_{k,i}(s) ds.$$

Then each $W_n^H(t)$ is the Cauchy polygonal approximation of the process W_t taken over the set of dyadic points $\{k/2^n; 0 \leq k \leq 2^n\}$, that is,

$$W_n^H(t) = W\left(\frac{k}{2^n}\right) + 2^n \left\{ W\left(\frac{k+1}{2^n}\right) - W\left(\frac{k}{2^n}\right) \right\} \left(t - \frac{k}{2^n}\right), \quad \text{for } t \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right). \quad (13)$$

To check this, we introduce the indicator function, $\chi_{n,i}(t) = 2^{n/2} \mathbf{1}_{[2^{-n}i, 2^{-n}(i+1))}(t)$. It is immediate to see that

$$(\chi_{n,i}, H_{m,k}) = 0 \quad \text{for } \forall (m, k) \text{ with } m \geq n+1,$$

here the symbol $(\cdot, \cdot)$ denotes the inner product in $L^2(0, 1)$.

Therefore each $\chi_{n,i}$ should be represented as linear combination of the members $\{H_{m,k}, m \leq n\}$, say;

$$\begin{aligned} \chi_{n,i}(t) &= C(n, j; 0, 0) H_{0,0} + \sum_{1 \leq m \leq n} \sum_{k=0}^{2^{(m-1)}-1} C(n, i; m, k) H_{m,k}(t). \\ C(n, i; m, k) &= (\chi_{n,i}, H_{m,k}). \end{aligned}$$

It is also easy to see that we have the following relation;

$$\sum_{i=0}^{2^n-1} C(n, i; m, k) C(n, i; l, j) = \delta_{m,l} \delta_{k,j}.$$

Based on this relation we can get the following equality,

$$W_n^H(t) = \sum_{k=0}^{2^n-1} \Delta_k^n W \chi_{n,k}(t), \quad \Delta_k^n W = W\left(\frac{k+1}{2^n}\right) - W\left(\frac{k}{2^n}\right),$$

and this is what we want to see.

Now applying this result to the expression given in the Proposition 3.1 of (a), we see that in this case the defining formula of the noncausal integral is given as the Riemannian sum,

$$\int_0^1 f d_H W_t = \lim_{n \rightarrow \infty} \sum_{i=0}^{2^n-1} 2^n \int_{2^{-n}i}^{2^{-n}(i+1)} f(s) ds \cdot \{W(2^{-n}(i+1)) - W(2^{-n}i)\}. \quad (14)$$

This type of definition can be found in recent publications of some authors. But as we have seen in here ([21]), this is merely a special case of our integral.

(c) Let $D_n(t, s)$ be the kernel given by, $D_n(t, s) = \sum_{k=1}^n \varphi_k(t) \varphi_k(s)$, $(t, s \in [0, 1])$.

Then we have the following representation for the noncausal integral,

$$\int_0^1 f d_\varphi W(t) = \lim_{n \rightarrow \infty} \int_0^1 dt \int_0^1 f(t, \omega) D_n(t, s) dW_s \quad (\text{limit in probability}).$$

For the case of trigonometric functions, the kernel $D_n(t, s)$ is the Dirichlet kernel appearing in the theory of Fourier series.

(d) A generalization of the above view: Replace the kernels $\{D_n(t, s)\}$ in the above interpretation by any δ -sequence say $\{K_n(t, s)\}$, then we will get a generalized formula for the noncausal integral:

$$\int_0^1 f d_K W := \lim_{n \rightarrow \infty} \int_0^1 dt \int_0^1 f(t, \omega) K_n(t, s) dW_s$$

3.4 Condition for the integrability – in the framework of the Homogeneous Chaos theory

Let $\mathbf{H}_0$ be the totality of all random functions $f(t, \omega) \in \mathbf{H}$ such that, $E \int_0^1 |f(t, \omega)|^2 dt < \infty$. By Wiener-Itô's theory of Homogeneous Chaos, we know that such function $f \in \mathbf{H}_0$ can be decomposed into the sum of multiple Wiener integrals, that is:

There exists a set of kernels, say $\{k_n^f(t; t_1, \dots, t_n)\}_{n=0}^\infty$, such that $k_n^f \in L^2([0, 1]^{n+1})$

with $\sum_n n! \|k_n^f\|_{n+1}^2 < \infty$, symmetric in n -parameters $(t_1, \cdot, t_n) \in [0, 1]^n$ and that,

We will denote by $\mathbf{H}_1$ the totality of all $\mathbf{H}_0$ - functions $f(t, \omega)$ such that,

$\sum_{n=1}^{\infty} n! \|k_n^f\|_{n+1}^2 < \infty$. Given a function $f \in \mathbf{H}_1$ we introduce its stochastic derivative $\mathcal{D}f$ by the following formula,

$$\mathcal{D}f(t, s) = \sum_{n=1}^{\infty} n I_{n-1}(k_n^f(t; s, \cdot)).$$

Since $E \int_0^1 \int_0^1 (\mathcal{D}f(t, s))^2 dt ds = \sum n! \|k_n^f\|_{n+1}^2$, we notice that the stochastic derivative $\mathcal{D}f(t, s)$ is well defined for the $f \in \mathbf{H}_1$. Now we can state the condition for the φ -integrability of the $\mathbf{H}_1$ -class functions in the following theorem that was obtained by the author in 1984.

Theorem 3.3 ([22]) *Let $f \in \mathbf{H}_1$ and let $\{\varphi_n\}$ be an arbitrary orthonormal basis. Then the necessary and sufficient condition for the random function f to be φ -integrable is that the $\lim_{n \rightarrow \infty} \int_0^1 \int_0^1 \mathcal{D}f(t, s) D_n(t, s) dt ds$ exists in probability.*

3.5 Relation between symmetric and noncausal integrals

We call a random function $f(t, \omega)$ semi martingale when it admits the decomposition, $f(t, \omega) = a(t, \omega) + \int_0^t \hat{f} dW_s$ where $\hat{f} \in \mathbf{M}$ and $a(t)$ is such that almost every sample path is of bounded variation in t over $[0, 1]$. Notice that if $\sup_{t, s | t-s| < h} E|a(t) - a(s)|^2 = o(h)$ then f is B-differentiable.

The followings are the basic results concerning the relation between the symmetric integrals with the noncausal integral.

Theorem 3.4 ([18]) *Every causal B- differentiable function is integrable in non-causal sense with respect to the system of Haar functions and the sum coincides with that of the symmetric integrals:*

$$\int_0^1 f d_H W = \int_0^1 f dW + \frac{1}{2} \int_0^1 \hat{f} dt$$

We say that an orthonormal basis $\{\varphi_n\}$ is *regular* provided that it satisfies the next condition:

$$\sup_n \|u_n\|_2 < \infty, \text{ where } u_n(t) = \sum_{k \leq n} \varphi_k(t) \int_0^t \varphi_k(s) ds \quad (15)$$

Remark 3 Notice that this condition (15) is equivalent to the fact that,

$$w - \lim_{n \rightarrow \infty} u_n = \frac{1}{2} \quad (\text{in } L^2)$$

namely to the fact that, for any $f(t) \in L^2(0,1)$ it holds the following,

$$\lim_{n \rightarrow \infty} \int_0^1 u_n(t) f(t) dt = \frac{1}{2} \int_0^1 f(t) dt$$

Theorem 3.5 ([18]) *Every semi martingale (causal or not) becomes φ -integrable, iff the basis $\{\varphi_n\}$ is regular. In this case the noncausal integral coincides with the symmetric integrals.*

Related to this result is a natural and interesting question asking whether there can or can not be a basis $\{\varphi_n\}$ which is not regular. This question is affirmatively answered by P.Mayer and M.Mancino [4]. We can go on further. The next result shows that a smoothness in W_t of the integrand ensures the integrability with respect to any orthonormal basis.

Theorem 3.6 ([18]) *Every semi martingale that is twice B-differentiable, namely the B-derivative $\hat{f}$ is again a semi martingale, is u-integrable.*

In their articles [7], [5], of the authors M.Zakai and D.Nualart, the noncausal integral was referred as the *intrinsic* Ogawa integral. We like to remark at this stage that, precisely saying what they referred there was our noncausal integral for the case of u-integrable functions,

Now let us finish this paragraph by giving another result on the way of convergence of noncausal integrals for semi-martingales, for its usefulness in applications.

Proposition 3.2 (1985,[18]) *If the B-derivative $\hat{f}$ of the semi-martingale f , $df = \hat{f}dW_t + g(t, \omega)dt$, satisfies the following condition,*

$$P\left\{\int_0^1 \hat{f}^4(t)dt < \infty\right\} = 1.$$

Then the noncausal integral $\int_0^t f d_ W_t$ with respect to the regular basis converges uniformly in $t \in [0, 1]$.*

4 Applications to the noncausal SDEs

Now we like to show basic results concerning the Cauchy problem of the noncausal SDEs and the validity of the noncausal Itô formula, which will give us at the same time the appropriate answers to those noncausal problems listed in the Paragraph 2.

4.1 Noncausal Cauchy problem

First notice that the SDE in (3) becomes meaningful in the framework of the noncausal stochastic calculus, that is:

$$\begin{aligned} dX_t &= a(t, X_t, \eta(\omega))dt + b(t, X_t, \eta(\omega))d_\varphi W_t, \quad t \in (0, T], \\ X_0(\omega) &= \xi(\omega) \end{aligned} \quad (16)$$

here the $\{\varphi_n\}$ is a regular basis in $L^2(0, 1)$, which we will fix throughout the discussion.

We notice at this stage that when the parameters $\xi(\omega), \eta(\omega)$ are not random and the solution X_t can be supposed to be causal, then by virtue of the Theorem 3.5 the SDE in (16) is reduced to the usual SDE with symmetric integration,

$$\begin{aligned} dX_t &= a(t, X_t, \eta)dt + b(t, X_t, \eta)dW_t, \quad t \in (0, T], \\ X_0(\omega) &= \xi. \end{aligned} \quad (17)$$

The Cauchy problem for the noncausal SDE was first studied by the author [19] for such simple case where the parameter η is not random or does not appear in $a(t, x)$, $b(t, x)$ and only the initial data $\xi(\omega)$ arises as a noncausal factor.

$$\begin{aligned} dX_t &= a(t, X_t)dt + b(t, X_t)d_\varphi W_t, \quad t \in (0, T], \\ X_0(\omega) &= \xi(\omega). \end{aligned} \quad (18)$$

For this case, the results on the existence and a kind of uniqueness properties of the solution are proved under a milder assumption on the regularity of the coefficients $a(\cdot), b(\cdot)$ as follows:

Assumption 1 *The coefficients $a(t, x)$, $b(t, x)$ are sufficiently regular in such sense that,*

1. $a(t, x), \frac{\partial^2}{\partial x^2}b(t, x)$ are of C^1 -class,
2. $a(t, x), b(t, x)$ are sufficiently regular in the sense that the causal Cauchy problem 17 admits the unique strong solution $X(t, \omega; \xi)$ and that the $X(t, \omega; \xi)$ is continuous in (t, ξ) with probability one.

Notice that under such conditions the composite

$$\tilde{X}(t, \omega) = X(t, \omega; \xi(\omega))$$

of the strong solution $X(t, \xi, \omega)$ of the (17) and the random variable $\xi(\omega)$ is well defined, which we expect to be a solution of the noncausal Cauchy problem (18). In fact we have the following,

Theorem 4.1 (1985 [19]) *The composite $\tilde{X}(t, \omega)$ is a solution of the noncausal Cauchy problem (18).*

We have also found that this solution $\tilde{X}(t, \omega)$ verifies the Itô formula of noncausal type, that is:

Proposition 4.1 (1985, [17]) *For any function $F(x) \in C^4$ it holds the equality,*

$$dF(\tilde{X}_t) = F'(\tilde{X}_t)\{a(t, \tilde{X}_t)dt + b(t, \tilde{X}_t)d_\varphi W_t\} \quad 0 \leq t \leq 1$$

As an application of this we can show the following result that concerns the uniqueness of the solution for the noncausal problem (18),

Corollary 4.1 ([17]) *When the $b(x) \neq 0$, the composed function $\tilde{X}(t, \omega) = X(t, \omega; \xi(\omega))$ is the unique solution among all random functions verifying the Itô formula 4.1 of noncausal type.*

The proof of this together with that of the previously presented Proposition 4.1 will be given in the next paragraph for a more general case.

4.2 Discussions for the more general cases

We are going to give in this paragraph the results on the Cauchy problem for the more general case (16).

Assumption 2 *We suppose that the coefficients $a(t, x; \eta), b(t, x; \eta)$ are sufficiently regular in such sense that, for an arbitrary couple of parameters (ξ, η) the causal Cauchy problem 17 admits the unique strong solution $X(t, \omega; \xi, \eta)$ and that the $X(t, \omega; \xi, \eta)$ is continuous in (t, ξ, η) with probability one.*

Remark 4 *The assumption is satisfied when, for example, the $a(t, x; \eta), b(t, x; \eta)$ are of the C^4 -class in x , of C^1 -class in η and all derivatives are bounded on $[0, 1] \times R^1$.*

We also notice that under the Assumption 2 the composite

$$\tilde{X}(t, \omega) = X(t, \omega; \xi(\omega), \eta(\omega))$$

of the strong solution $X(t, \omega; \xi, \eta)$ of the (17) and the random variables $\xi(\omega), \eta(\omega)$ is well defined, and as in the previous case we expect this composite $\tilde{X}(t, \omega)$ to be a solution of the noncausal Cauchy problem (16). In fact we have the following,

Theorem 4.2 *The $\tilde{X}$ gives a noncausal solution of the noncausal Cauchy problem 16.*

For the verification of this, we need some preparations.

Proposition 4.2 *Let $f(t, \omega; \xi, \eta)$ ($\xi, \eta \in [-A, A]$) be a semi-martingale such that for each fixed (ξ, η) ,*

$$df(t, \omega; \xi, \eta) = g(t, \omega; \xi, \eta)dt + h(t, \omega; \xi, \eta)dW_t \quad (19)$$

where $g(\cdot), h(\cdot)$ are causal random functions satisfying the following condition,

$$P\left[\int_{-A}^A d\xi \int_{-A}^A d\eta \int_0^1 \{g^2(t, \omega; \xi, \eta) + h^2(t, \omega; \xi, \eta)\} dt < \infty\right] = 1$$

(i) *Then for any regular basis $\{\varphi_n\}$ in $L^2(0, 1)$, it holds the following equality,*

$$\lim_{n \rightarrow \infty} \int_{-A}^A d\xi \int_{-A}^A d\eta \left[\int_0^1 f(t, \omega; \xi, \eta) \{d_\varphi W(t) - dW_n^\varphi(t)\} \right]^2 = 0 \quad (\text{in probability}) \quad (20)$$

(ii) *Moreover if the coefficient $h(t, \omega; \xi, \eta)$ in the decomposition (19) again becomes a semi-martingale satisfying the same condition as the $f(\cdot)$, then the equality (20) still holds true for any basis $\{\varphi_n\}$.*

(Proof) Put $f = f_1 + f_2$ where,

$$f_1(t, \omega; \xi, \eta) = f(0, \omega; \xi, \eta) + \int_0^t g(s, \omega; \xi, \eta) ds$$

and

$$f_2(t, \omega; \xi, \eta) = \int_0^t h(s, \omega; \xi, \eta) dW_s.$$

Then we have for f_1 , the equality:

$$\begin{aligned} & \int_0^1 f_1(s, \omega; \xi, \eta) \{d_\varphi W(s) - dW_n^\varphi(s)\} \\ &= f_1(1, \omega; \xi, \eta) \{W(1) - W_n^\varphi(1)\} - \int_0^1 \{W(s) - W_n^\varphi(s)\} g(s, \omega; \xi, \eta) ds \end{aligned}$$

Hence with the help of the Theorem of Nishio-Itô we confirm that,

$$\lim_{n \rightarrow \infty} \int_{-A}^A d\xi \int_{-A}^A d\eta \left[\int_0^1 f_1(t, \omega; \xi, \eta) \{d_\varphi W(t) - dW_n^\varphi(t)\} \right]^2 = 0 \quad (\text{in probability})$$

For the term f_2 we have the decomposition,

$$\int_0^1 f_2(t, \omega; \xi, \eta) \{d_\varphi W(t) - dW_n^\varphi(t)\} = \sum_{i=1}^4 I_{i,n}(\xi, \eta)$$

where,

$$\begin{aligned} I_{1,n} &= \sum_{k=n+1}^{\infty} f_2(1, \omega; \xi, \eta) \tilde{\varphi}_k(1) Z_k, \quad (\tilde{\varphi}_k(t) = \int_0^t \varphi_k(s) ds) \\ I_{2,n} &= \sum_{k=n+1}^{\infty} \int_0^1 \tilde{\varphi}_k(t) \varphi_k(t) h(t, \omega; \xi, \eta) dt \\ I_{3,n} &= \sum_{k=n+1}^{\infty} \int_0^1 \varphi_k(t) dW(t) \int_0^t \tilde{\varphi}_k(s) h(s, \omega; \xi, \eta) dW(s) \\ I_{4,n} &= \sum_{k=n+1}^{\infty} \int_0^1 \tilde{\varphi}_k(t) h(t, \omega; \xi, \eta) dW(t) \int_0^t \varphi_k(s) dW(s) \end{aligned}$$

and, $Z_n = \int_0^1 \varphi_n(t) dW(t)$.

We are to show that: $\lim_{n \rightarrow \infty} \int_{-A}^A d\xi \int_{-A}^A I_{i,n}^2(\xi, \eta) = 0$ (in probability), $(1 \leq i \leq 4)$.

Since for the quantities $I_{i,n}$ ($i = 1, 3, 4$) this could be easily done by a usual routine, it would suffice to show the result only for the term $I_{2,n}$.

By taking the Remark3 into account, we see that for each fixed (ξ, η) we have,

$$\lim_{n \rightarrow \infty} I_{2,n}(\xi, \eta) = \lim_{n \rightarrow \infty} \int_0^1 h(t, \omega; \xi, \eta) \left\{ \frac{1}{2} - u_n(t) \right\} dt = 0$$

On the other hand we have,

$$I_{2,n}^2 \leq \left(\frac{1}{2} + 2U^2 \right) \int_0^1 h^2(t, \omega; \xi, \eta) dt \quad \text{where, } U = \sup_n \|u_n\|_{L^2} < \infty.$$

Hence we confirm the result, $\lim_{n \rightarrow \infty} \int_{-A}^A \int_{-A}^A I_{2,n}^2(\xi, \eta) d\xi d\eta = 0$ $\square$

Now given the unique solution $X(t, \omega; \xi, \eta)$ of the causal problem 17, we introduce the sequence of random functions in the following way,

$$X_n^\varphi(t, \omega; \xi, \eta) = \xi + \int_0^t a(s, X(s, \omega; \xi, \eta); \eta) ds + \int_0^t b(s, X(s, \omega; \xi, \eta); \eta) dW_n^\varphi(s) \quad (21)$$

where W_n^φ is the approximate process of the Brownian motion introduced in the previous paragraph.

We easily see by the Theorem 3.5 that for each fixed $t, (\xi, \eta)$, we have $\lim_{n \rightarrow \infty} X_n^\varphi(t, \omega; \xi, \eta) = X(t, \omega; \xi, \eta)$, (in probability). Moreover we can see that this convergence is uniform in (ξ, η) on every finite set $C_A = [-A, A] \times [-A, A]$.

Proposition 4.3 *For an arbitrarily large $A > 0$ it holds the following relation at each fixed $t \in [0, 1]$,*

$$\lim_{n \rightarrow \infty} \sup_{(\xi, \eta) \in C_A} |X_n^\varphi(t, \omega; \xi, \eta) - X(t, \omega; \xi, \eta)| = 0 \quad (\text{in probability})$$

(Proof) Put

$$\Delta_n(t, \omega; \xi, \eta) = X_n^\varphi(t, \omega; \xi, \eta) - X(t, \omega; \xi, \eta)$$

From equations (17), (21) we obtain the following:

$$\Delta_n(t, \omega; \xi, \eta) = \int_0^t b(X(s; \xi, \eta); \eta) \{dW_n^\varphi(s) - d_\varphi W(s)\} \quad (22)$$

On the other hand we have the following expression,

$$\begin{aligned} \Delta_n(t, \omega; \xi, \eta) &= \int_{-A}^\xi d\xi_1 \int_{-A}^\eta \frac{\partial^2}{\partial \xi \partial \eta} \Delta_n(t, \omega; \xi_1, \eta_1) d\eta_1 \\ &+ \int_{-A}^\xi \frac{\partial}{\partial \xi} \Delta_n(t, \omega; \xi_1, -A) d\xi_1 + \int_{-A}^\eta \frac{\partial}{\partial \eta} \Delta_n(t, \omega; -A, \eta_1) d\eta_1 + \Delta_n(t, \omega; -A, -A) \end{aligned}$$

which implies that,

$$\sup_{(\xi, \eta) \in C_A} |\Delta_n(t, \omega; \xi, \eta)| \leq J_1(n) + J_2(n) + J_3(n)$$

where

$$\begin{aligned} J_1(n) &= 4A^2 \int_{-A}^A d\xi_1 \int_{-A}^A d\eta_1 \left| \frac{\partial^2}{\partial \xi \partial \eta} \Delta_n(t, \omega; \xi_1, \eta_1) \right|^2 + |\Delta_n(t, \omega; -A, -A)| \\ J_2(n) &= 2A \int_{-A}^A \left| \frac{\partial}{\partial \xi} \Delta_n(t, \omega; \xi_1, -A) \right|^2 d\xi_1, \quad J_3(n) = 2A \int_{-A}^A \left| \frac{\partial}{\partial \eta} \Delta_n(t, \omega; -A, \eta_1) \right|^2 d\eta_1 \end{aligned}$$

We are to show that for each fixed t these $J_1(n), J_2(n), J_3(n)$ tend to zero in probability as $n \rightarrow \infty$. Since at this stage the parameters ξ, η remain as deterministic constants, we notice that the $X(t, \omega; \xi, \eta)$ is causal and derivable in ξ, η . In fact under the assumption (4) on the regularity of the coefficients $a(\cdot), b(\cdot)$ it is easy to verify that the derivatives,

$$X_1(t) = \frac{\partial}{\partial \xi} X(t, \omega; \xi, \eta), \quad X_2(t) = \frac{\partial}{\partial \eta} X(t, \omega; \xi, \eta), \quad X_3(t) = \frac{\partial^2}{\partial \xi \partial \eta} X(t, \omega; \xi, \eta),$$

are given as the solutions of the following symmetric type SDEs, which can be solved explicitly:

$$\begin{cases} dX_1(t) = a_x(t, X, \eta)X_1(t)dt + b_x(t, X, \eta)X_1(t)dW_t, \\ X_1(0) = 1 \\ \\ dX_2(t) = \{a_\eta(t, X, \eta)dt + b_\eta(t, X, \eta)dW_t\} \\ \quad + \{a_x(t, X, \eta)X_2(t)dt + b_x(t, X, \eta)X_2(t)dW_t\}, \\ X_2(0) = 0, \end{cases}$$

$$\begin{cases} dX_3(t) = \{a_{xx}(t, X, \eta)X_2(t) + a_{x,\eta}(t, X, \eta)\}X_1(t)dt \\ \quad + \{b_{xx}(t, X, \eta)X_2(t) + b_{x,\eta}(t, X, \eta)\}X_1(t)dW_t \\ \quad + a_x(t, X, \eta)X_3(t)dt + b_x(t, X, \eta)X_3(t)dW_t, \\ X_3(0) = 0 \end{cases}$$

This combined with the expression (22) would imply that the quantity $\Delta_n(t, \omega; \xi, \eta)$ is derivable in ξ, η and that the order of the derivation in ξ, η and the integration is exchangeable. For example,

$$\frac{\partial^2}{\partial \xi \partial \eta} \Delta_n = \int_0^t \frac{\partial^2}{\partial \xi \partial \eta} b(s, X(s; \xi, \eta)) \{dW_n^\varphi(s) - dW_\varphi(s)\}$$

Hence by virtue of the Proposition 4.2 we only need to show that the following quantities,

$$\frac{\partial^2}{\partial \xi \partial \eta} b(X(t; \xi, \eta); \eta), \quad \frac{\partial}{\partial \xi} b(X(t; \xi, -A); -A), \quad \frac{\partial}{\partial \eta} b(X(t; -A, \eta); \eta)$$

are semi-martingales satisfying the condition in that Proposition. Since this can be verified by a simple routine work, we see that we are done. $\square$

Now we are going to give the proof for our Theorem,

(Proof) Fix a positive A in an arbitrary way and put,

$$\begin{aligned} \xi_A(\omega) &= \xi(\omega)1_{C_A}(\xi(\omega)) - A1_{(-\infty, -A]}(\xi(\omega)) + A1_{[A, \infty)}(\xi(\omega)) \\ \eta_A(\omega) &= \eta(\omega)1_{C_A}(\eta(\omega)) - A1_{(-\infty, -A]}(\eta(\omega)) + A1_{[A, \infty)}(\eta(\omega)) \end{aligned}$$

For an arbitrary positive ϵ we have,

$$\begin{aligned} &P\{|X_n^\varphi(t, \omega; \xi(\omega), \eta(\omega)) - X(t, \omega; \xi(\omega), \eta(\omega))| > \epsilon\} \\ &\leq P\{|X_n^\varphi(t, \omega; \xi_A(\omega), \eta_A(\omega)) - X(t, \omega; \xi_A(\omega), \eta_A(\omega))| > \epsilon\} \\ &\quad + P(|\xi(\omega)| > A) + P(|\eta(\omega)| > A) \end{aligned}$$

Since $|\xi_A(\omega)|, |\eta_A(\omega)| \leq A$, we see that

$$\lim_{n \rightarrow \infty} P\{|X_n^\varphi(t, \omega; \xi_A(\omega), \eta_A(\omega)) - X(t, \omega; \xi_A(\omega), \eta_A(\omega))| > \epsilon\} = 0$$

by virtue of the Proposition 4.3. The A being arbitrary this implies that,

$$\lim_{n \rightarrow \infty} P\{|X_n^\varphi(t, \omega; \xi(\omega), \eta(\omega)) - X(t, \omega; \xi(\omega), \eta(\omega))| > \epsilon\} = 0 \quad \square$$

5 Question of uniqueness – Noncausal Itô formula

The noncausal solution of the problem (16), $\tilde{X}(t, \omega) = X(t, \omega; \xi(\omega), \eta(\omega))$ constructed in the Theorem 4.2, has a remarkable property as stated in the next,

Theorem 5.1 (Noncausal Itô formula) *For any random variable $\zeta(\omega)$ and any function $F(x, y)$, which is differentiable in (x, y) and of C^4 -class in x with bounded derivatives, it holds the following equality:*

$$dF(\tilde{X}_t, \zeta(\omega)) = (\partial_x F)(\tilde{X}_t, \zeta(\omega))\{a(\tilde{X}_t; \eta(\omega))dt + b(\tilde{X}_t; \eta(\omega))d_\varphi W_t\} \quad (23)$$

(Proof) Let $X(t, \omega; \xi, \eta)$ be the unique solution of the causal SDE (17) with deterministic parameters (ξ, η) . Then by the usual Itô formula for causal functions, we have for each fixed deterministic parameters (ξ, η, ζ) , the following relation:

$$F(X(t; \xi, \eta), \zeta) = F(\xi, \zeta) + \int_0^t (\partial_x F)(X(s, \omega; \xi, \eta), \zeta)\{a(X_s; \eta)ds + b(X_s; \eta)dW_s\} \quad (24)$$

Here the stochastic integral $\int dW_t$ stands for the causal symmetric integral. Given this we introduce the approximation sequence as follows,

$$F^n(t, \omega; \xi, \eta, \zeta) = F(\xi, \zeta) + \int_0^t (\partial_x F)(X(s, \omega; \xi, \eta), \zeta)\{a(X_s; \eta)ds + b(X_s; \eta)dW_n^\varphi(s)\} \quad (25)$$

Following the same argument as in the proof of Proposition 4.2, we would easily verify that for each fixed $t \in [0, 1]$ the sequence $F^n(t, \omega; \xi, \eta, \zeta)$ converges to $F(X(t, \omega; \xi, \eta), \zeta)$ in probability as $n \rightarrow \infty$, uniformly in $(\xi, \eta, \zeta) \in C'_A$ on any finite set $C'_A = [-A, A]^3$. Hence we confirm, again following the same argument as in the proof of the Theorem 4.2, that for each fixed t the sequence $F^n(t, \omega; \xi(\omega), \eta(\omega), \zeta(\omega))$ converges in probability to the $F(X(t, \omega; \xi(\omega), \eta(\omega), \zeta(\omega)) = F(\tilde{X}(t, \omega), \zeta(\omega))$. Now from the equation (25) we see that the following limit,

$$\lim_{n \rightarrow \infty} \int_0^t (\partial_x F)(X(s, \omega; \xi(\omega), \eta(\omega)), \zeta(\omega))b(X(s, \omega; \xi(\omega), \eta(\omega)); \eta(\omega))dW_n^\varphi(s)$$

should converge in probability to the limit,

$$\int_0^t (\partial_x F)(\tilde{X}(s, \omega), \zeta(\omega))b(\tilde{X}(s, \omega); \eta(\omega))dW_\varphi(s)$$

by definition of the $\tilde{X}(t, \omega)$ and by definition of the noncausal integral with respect to the basis $\{\varphi_n\}$.

Thus from this fact we get the desired equality (23), by letting $n \rightarrow \infty$ on both sides

of the equality (25). $\square$

As we have mentioned in the previous paragraph, this fact that the solution $\tilde{X}(t, \omega) = X(t, \omega; \xi(\omega), \eta(\omega))$ of the noncausal problem (16) satisfies the Itô formula of noncausal type (23) would give us a partial answer to the question of uniqueness of the solution of our noncausal problem. In fact we have the following result that is valid for the case of 1-dimensional SDE.

Corollary 5.1 *If the $b(t, x; \eta)$ does not depend on the t and $b(x; \eta) > 0$ (or < 0) for all (t, x, η) , then the solution $\tilde{X}(t, \omega)$ is unique among the all random functions that verify the noncausal Itô formula (23). We will call such solution the regular solution of the Cauchy problem.*

(Proof) Without loss of generality we suppose that $b(\cdot) > 0$. Put $Y(t) = F(\tilde{X}(t))$ where $F(x)$ is as follows,

$$F(x) = \int_0^x \frac{dy}{b(y, \eta)}.$$

Then we have, $\tilde{X}(t, \omega) = F^{-1}(Y(t, \omega))$. By applying the noncausal Itô formula to the function $Y(t)$ we get,

$$Y(t) = F(\xi(\omega)) + \int_0^t \left(\frac{a}{b}\right)(F^{-1}(Y(s)); \eta(\omega)) ds + W(t)$$

Since this is merely a family of ordinary integral equations parametrized by the ω , we see the uniqueness of its solution $Y(t)$ for each fixed ω and hence the uniqueness of the $\tilde{X}(t, \omega)$. This completes the proof. $\square$

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