

Forking in Generic Structures

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Abstract

Baldwin の問題とは, **superstable** であるが ω -stable でない **generic** 構造が存在するか, という問題である ([1]). この問題の部分的結果として, **generic** 構造が **saturated** のときは, 理論が **superstable** ならば必ず ω -stable となることがわかった (定理 10).

1 Preliminaries

Many papers [2,3,4,8] have laid out the basics of generic structures. So we do not explain all of those details here.

Let L be a countable relational language and $\mathbf{K}^*$ a class of the finite L -structures. Then $\delta : \mathbf{K}^* \rightarrow \mathbb{R}^+$ is said to be a *preimension*, if (i) for all $AB \in \mathbf{K}^*$, $\delta(A/B) \leq \delta(A/A \cap B)$; (ii) if $A \cong B \in \mathbf{K}^*$, then $\delta(A) = \delta(B)$; (iii) $\delta(\emptyset) = 0$; (iv) If $A \subset A', B \subset B', C \subset C'$ and A', B', C' are pairwise disjoint, then $\delta(B/A) - \delta(B/AC) \leq \delta(B'/A') - \delta(B'/A'C')$, where $\delta(A/B)$ denotes $\delta(AB) - \delta(B)$.

Let $A \subset B \in \mathbf{K}^*$. Suppose that A is finite. Then A is said to be *closed* in B (in symbol, $A \leq B$), if $\delta(X/A) \geq 0$ for any finite $X \subset B - A$. In general, A is said to be closed in B , if $A \cap X \leq B \cap X$. We define the *closure* of A in B by $\text{cl}_B(A) = \bigcap \{C : A \subset C \leq B, |C| < \omega\}$. We define a *dimension* of A in B by $d_B(A) = \delta(\text{cl}_B(A))$.

Let $\mathbf{K}$ be a subclass of $\mathbf{K}^*$ that is closed under substructures, and M a saturated $\mathbf{K}$ -generic structure.

$\mathbf{K}$ is said to have *finite closures*, if there are no chains $A_0 \subset A_1 \subset \dots$ of elements of $\mathbf{K}$ with $\delta_\alpha(A_{i+1}) < \delta_\alpha(A_i)$ for each $i < \omega$. If $\mathbf{K}$ has finite closures, then we can see that there exists a unique $\mathbf{K}$ -generic structure M , and moreover that any finite set of M has finite closures. On the other hand, it can be seen that if a $\mathbf{K}$ -generic structure M is saturated then $\mathbf{K}$ has finite closures. We summarize our situation.

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Assumption $\mathbf{K} = (K, \leq)$ is derived from a predimension δ such that $\mathbf{K}$ is closed under substructures. M is a saturated $\mathbf{K}$ -generic structure.

2 Smallness of algebraic types

Definition Let AB be finite L -structure. Then

- (i) A pair (B, A) is said to be **$\mathbf{K}$ -normal**, if $A \leq AB \in \mathbf{K}$ and $A \cap B = \emptyset$.
- (ii) A $\mathbf{K}$ -normal pair (B, A) is said to be *minimal*, if $\delta(C/A) > \delta(B/A)$ for any non-empty proper subset C of B .
- (iii) A $\mathbf{K}$ -normal pair (B, A) is said to be *weakly small*, if whenever $A \subset C, B \subset D$ and (D, C) is $\mathbf{K}$ -normal, then $\delta(D/C) \geq \delta(B/C)$.
- (iv) A $\mathbf{K}$ -normal pair (B, A) is said to be *pseudo-small*, if whenever $A \subset C$ and (B, C) is $\mathbf{K}$ -normal, then $\delta(B/C) \geq \delta(B/A)$.
- (v) A $\mathbf{K}$ -normal pair (B, A) is said to be *small*, if whenever $A \subset C, B \subset D$ and (D, C) is $\mathbf{K}$ -normal, then $\delta(D/C) \geq \delta(B/A)$.

Note 1 A $\mathbf{K}$ -normal pair (B, A) is small if and only if it is weakly small and pseudo-small.

Lemma 2 Let (B, A) be $\mathbf{K}$ -minimal with $A \leq AB \leq M$. If $\text{tp}(B/A)$ is algebraic, then (B, A) is weakly $\mathbf{K}$ -small.

Proof Suppose by way of contradiction that (B, A) is not weakly $\mathbf{K}$ -small. Then there are $C \supset A$ and $D \supset B$ such that (D, C) is $\mathbf{K}$ -normal and $\delta(D/C) < \delta(B/C)$.

Claim 1: There is a set $\{B_i\}_{i < \omega}$ of copies of B with the following conditions:

- (i) $B_i \cong_{CB_0 \dots B_{i-1}} B$ for each $i < \omega$;
- (ii) $CB_0 \dots B_i, CB_0 \dots B_{i-1}D \leq CB_0 \dots B_iD \in \mathbf{K}$ for each $i < \omega$;
- (iii) $D, B_0, B_1, B_2, \dots$ are pairwise disjoint.

Proof of Claim 1: We construct $\{B_i\}_{i < \omega}$ inductively. Suppose that $\{B_i\}_{i \leq n}$ has been defined. By (ii), $CB_0 \dots B_n \leq CB_0 \dots B_nD \in \mathbf{K}$, and so we have $CB_0 \dots B_n \leq CB_0 \dots B_nB \in \mathbf{K}$. By the amalgamation property, we can take a copy B_{n+1} of B over $CB_0 \dots B_n$ such that

$$(*) \quad CB_0 \dots B_nD, CB_0 \dots B_nB_{n+1} \leq CB_0 \dots B_nB_{n+1}D \in \mathbf{K}.$$

Hence B_{n+1} satisfies (i) and (ii). On the other hand, $B_0, B_1, \dots, B_{n+1}$ are pairwise disjoint, since $B_{n+1} \cong_{CB_0 \dots B_n} B$ and $B \subset D$. So, to see that (iii) holds it is enough to show that $B' = B_{n+1} \cap D = \emptyset$. If $B' = B_{n+1}$ would hold, then we have $B_{n+1} \subset D$, and so $CB_{n+1} \not\leq CD$, since $\delta(D/C) < \delta(B/C) = \delta(B_{n+1}/C)$. This contradicts (*), and hence we have $B' \neq B_{n+1}$. By (*) again, we have $CB_0 \dots B_nD \leq CB_0 \dots B_nB_{n+1}D$, and so $AB' \leq AB_{n+1}$. Since (B, A) is a minimal pair, we have $B' = \emptyset$. (End of Proof of Claim 1)

Claim 2: $AB, AB_j \leq AB_0 \dots B_iB \in \mathbf{K}$ for $j \leq i < \omega$

Proof: We prove by induction on i . By (ii) of claim 1, $AB_0 \dots B_iB \leq AB_0 \dots B_{i+1}B$.

By induction hypothesis, we have $AB, AB_j \leq AB_0 \dots B_i B$ for $j \leq i$. Hence $AB, AB_j \leq AB_0 \dots B_{i+1} B$ for $j \leq i$. So, it is enough to show that $AB_{i+1} \leq AB_0 \dots B_{i+1} B$. By induction hypothesis again, we have $AB \leq AB_0 \dots B_i B$. From (i) of claim 1, it follows that $AB_{i+1} \leq AB_0 \dots B_{i+1}$. By (ii) of claim 1, $AB_0 \dots B_{i+1} \leq AB_0 \dots B_{i+1} B$. Hence we have $AB_{i+1} \leq AB_0 \dots B_{i+1} B$. (End of Proof of Claim 2)

We show that $\text{tp}(B/A)$ is non-algebraic. By claim 2, we can assume that $AB, AB_j \leq ABB_0 \dots B_i \leq M$ for each i, j with $j \leq i < \omega$. So we have $\text{tp}(B_j/A) = \text{tp}(B/A)$ for each $j \leq i$. By (iii) of claim 1, B_j 's are pairwise disjoint. Hence $\text{tp}(B/A)$ is not algebraic.

Definition We say that $\mathbf{K}$ is *closed under δ -substructures*, if for any disjoint A, B, C with $ABC \in \mathbf{K}$, there is a copy B^* of B over A with $\delta(B^*/CA) = \delta(B^*/A)$.

Lemma 3 Assume that $\mathbf{K}$ is closed under δ -substructures. Let (B, A) be $\mathbf{K}$ -minimal with $A \leq AB \leq M$. If $\text{tp}(B/A)$ is algebraic, then (B, A) is pseudo- $\mathbf{K}$ -small.

Proof Suppose by way of contradiction that (B, A) is not pseudo- $\mathbf{K}$ -small. Then there is $C \supset A$ such that (B, C) is $\mathbf{K}$ -normal and $\delta(B/C) < \delta(B/A)$.

Claim: There is a set $\{B_i\}_{i < \omega}$ of copies of B over A with the following conditions:

- (i) $C \leq CB_j \leq CB_0 B_1 \dots B_i \in \mathbf{K}$ for each $j \leq i < \omega$
- (ii) $B_0, B_1, B_2, \dots$ are pairwise disjoint.
- (iii) $B_i \cap C = \emptyset$ for each $i < \omega$;
- (iv) $\delta(B_i/C) = \delta(B_i/A)$ for each $i < \omega$.

Proof: Suppose that $\{B_i\}_{i \leq n}$ has been defined. By our assumption, we have $C \leq CB \in \mathbf{K}$, and by (i) we have $C \leq CB_0 B_1 \dots B_n \in \mathbf{K}$. So we can take a copy B^* of B over C such that $CB_0 \dots B_n, CB^* \leq CB_0 \dots B_n B^* \in \mathbf{K}$. By (iv), $\delta(B_i/C) = \delta(B_i/A)$ for each $i \leq n$. On the other hand, we have $\delta(B^*/C) < \delta(B^*/A)$. So we have $B_i \neq B^*$ for all $i \leq n$. Since (B, A) is $\mathbf{K}$ -minimal, B and B_i 's are pairwise disjoint. Since $\mathbf{K}$ is closed under δ -substructures, B_{n+1} with $B_{n+1} \cong_{AB_0 B_1 \dots B_n} B^*, CB_0 B_1 \dots B_n B_{n+1} \in \mathbf{K}$ and $\delta(B_{n+1}/C) = \delta(B_{n+1}/A)$. Then we can see that (i)–(iv) hold. (End of Proof of Claim)

By claim, we have $AB_j \leq AB_0 \dots B_i (\in \mathbf{K})$ for $j \leq i < \omega$. So we can assume that $AB_j \leq AB_0 \dots B_i \leq M$ for each i, j with $j \leq i < \omega$. Thus we have $\text{tp}(B_j/A) = \text{tp}(B/A)$ for each $j \leq i$. By (ii) of claim, B_j 's are pairwise distinct. Hence $\text{tp}(B/A)$ is not algebraic.

Lemma 4 If (B, A) and (C, BA) are $\mathbf{K}$ -small, then so is (BC, A) .

Proof Take any $\mathbf{K}$ -normal pair (E, D) such that $BC \subset E$ and $A \subset D$. Then note that $(E - B, BD)$ is $\mathbf{K}$ -normal. (Proof: Take any $X \subset E - B$. Note

that (XB, D) is $\mathbf{K}$ -normal since (E, D) is so. Since (B, A) is $\mathbf{K}$ -small, we have $\delta(X/BD) = \delta(XB/D) - \delta(B/D) \geq \delta(XB/D) - \delta(B/A) \geq 0$. Hence $(E - B, BD)$ is $\mathbf{K}$ -normal.) Since (C, BA) is $\mathbf{K}$ -small, we have $\delta(E/BD) \geq \delta(C/BA)$. On the other hand, since (B, A) is $\mathbf{K}$ -small and (B, D) is $\mathbf{K}$ -normal, we have $\delta(B/D) = \delta(B/A)$. It follows that $\delta(BC/A) = \delta(C/AB) + \delta(B/A) \leq \delta(E/BD) + \delta(B/D) = \delta(E/D)$. Hence (BC, A) is $\mathbf{K}$ -small.

Theorem 5 Assume that $\mathbf{K}$ is closed under δ -substructures. Let (B, A) be $\mathbf{K}$ -normal with $AB \leq M$. If $\text{tp}(B/A)$ is algebraic, then (B, A) is $\mathbf{K}$ -small.

Proof Let $\text{tp}(B/A)$ be algebraic. Take a sequence $A = B_0 \leq B_0B_1 \leq \dots \leq B_0B_1\dots B_n = AB$ with (B_{i+1}, B_i) $\mathbf{K}$ -minimal for each i . Since each $\text{tp}(B_{i+1}/B_0\dots B_i)$ is algebraic, it is $\mathbf{K}$ -small by lemma 2 and 3. So, by lemma 4, $(B, A) = (B_0B_1\dots B_n, B_0)$ is $\mathbf{K}$ -small.

3 Forking and dimension

In this section, we assume that $\mathbf{K}$ is closed under δ -substructures.

Lemma 6 Let $A \subset B$. If B is closed, then so is $B \cup \text{acl}(A)$.

Proof We can assume that A, B are finite. It is enough to show that BA^* is closed for any finite A^* with $A \subset A^* \leq \text{acl}(A)$. Let $A' = A^* \cap B$. Since $\text{tp}(A^*/A')$ is algebraic and $A' \leq A^*$ is closed, $(A^* - A', A')$ is small. Then we can see that BA^* is closed as follows. If not, then there is finite $X \subset N - BA^*$ with $\delta(X/BA^*) < 0$. Then we have $0 \leq \delta(XA^*/B) = \delta(X/BA^*) + \delta(A^*/B) < \delta(A^*/B) \leq \delta(A^*/A')$. This contradict that $(A^* - A', A')$ is small.

Fact([7], [8]) Let B, C be closed and $A = B \cap C$ algebraically closed. Then the following are equivalent.

- (i) $d(B/A) = d(B/C)$;
- (ii) B and C are free over A , and BC is closed;
- (iii) $\text{tp}(B/C)$ does not fork over A .

Lemma 7 Assume that $\mathbf{K}$ is closed under δ -substructures. Let B, C be closed and $A = B \cap C$. If B, C are free over A and BC is closed, then $\text{tp}(B/C)$ does not fork over A .

Proof By lemma 6, $B\text{acl}(A), C\text{acl}(A)$ is closed. Since BC is closed, by lemma 6 again, $BC\text{acl}(A)$ is closed. So, by fact, to show that $\text{tp}(B/C)$ does not fork over A , it is enough to prove that $\text{acl}(A)B, \text{acl}(A)C$ are free over $\text{acl}(A)$. Take finite closed $B_0 \subset B, C_0 \subset C$ such that B_0C_0 is closed. Let $A_0 = B_0 \cap C_0$. Take

finite closed $A^* \subset \text{acl}(A_0)$. Let $D = A^* - B_0C_0$ and $A' = A^* \cap B_0C_0$.

Claim 1: $\delta(B_0C_0/A^*) = \delta(B_0C_0/A')$.

Proof: Since $\text{tp}(D/A')$ is algebraic and (D, A') is normal, (D, A') is small. So we have $\delta(D/B_0C_0) = \delta(D/A')$. Then $\delta(B_0C_0/A^*) - \delta(B_0C_0/A') = \delta(D/B_0C_0) - \delta(D/A') = 0$. (End of Proof of Claim 1)

Claim 2: $\delta(B_0/C_0A^*) = \delta(B_0/A^*)$.

Proof: Since B, C are free over A , we can see that $\delta(C_0/A'B_0) = \delta(C_0/A')$. Then $\delta(B_0/C_0A^*) = \delta(B_0C_0/A^*) - \delta(C_0/A^*) \geq \delta(B_0C_0/A') - \delta(C_0/A') = \delta(B_0C_0/A') - \delta(C_0/A'B_0) = \delta(B_0/A') \geq \delta(B_0/A^*)$. Hence $\delta(B_0/C_0A^*) = \delta(B_0/A^*)$.

By claim 2, $\text{Bacl}(A), \text{Cacl}(A)$ is free over $\text{acl}(A)$.

Lemma 8 Assume that $\mathbf{K}$ is closed under δ -substructures. Let A, B, C be such that B, C are closed and $A = B \cap C$. Suppose that $\text{tp}(B/C)$ does not fork over A . Then

- (i) B, C are free over A (and moreover $\text{acl}(B), \text{acl}(C)$ are free over $\text{acl}(A)$);
- (ii) $B \cup C \cup \text{acl}(A)$ is closed.

Proof Let $A^* = \text{acl}(A), B^* = \text{acl}(B)$ and $C^* = \text{acl}(C)$. Clearly $\text{tp}(B^*/A^*)$ does not fork over A^* , and so $B^* \cap C^* = A^*$.

(i) By fact, B^* and C^* are free over A^* . So we obtain that B and C are free over A^* . Let $B' = B \cap A^*, C' = C \cap A^*$. Note that $\delta(B/B'C) = \delta(B/B'C')$. First we show that B' and C are free over A . If not, then there are finite closed $A_0 \subset A, B'_0 \subset B', C_0 \subset C$ such that B'_0, C_0 are not free over $A_0 = B'_0 \cap C_0$. We can assume that $\text{tp}(B'_0/A_0)$ is algebraic. By theorem 5, $(B'_0 - A_0, A_0)$ is small. So we have $\delta(B'_0/A_0) = \delta(B'_0/C_0)$. This contradicts that B'_0 and C_0 are not free over A_0 . Thus B' and C are free over A . Similarly we see that B and C' are free over A . Then $\delta(B/C) = \delta(BB'/C) = \delta(B/B'C) + \delta(B'/C) = \delta(B/B'C') + \delta(B'/C') = \delta(B/C') = \delta(B/A)$. Hence B and C are free over A .

(ii) By fact, we obtain that B^*C^* is closed. If BCA^* is not closed, then there are finite $X \subset B^*C^* - BCA^*, B_0 \subset B, C_0 \subset C, A_0^* \subset A^*$ such that $\delta(X/B_0C_0A_0^*) < 0$. By lemma, BA^*, CA^* are closed, and hence we can assume that $B_0A_0^*, C_0A_0^*$ are closed. Let $X_B = X \cap B^*$ and $X_C = X \cap C^*$. Then $\delta(X_B/B_0C_0X_CA_0^*) = \delta(X_B/B_0A_0^*)$ and $\delta(X_C/B_0C_0A_0^*) = \delta(X_C/C_0A_0^*)$ since B^*, C^* are free over A^* . Therefore we have $\delta(X/B_0C_0A_0^*) = \delta(X_B/B_0C_0X_CA_0^*) + \delta(X_C/B_0C_0A_0^*) = \delta(X_B/B_0A_0^*) + \delta(X_C/C_0A_0^*) \geq 0 + 0 = 0$. A contradiction.

Lemma 9 Assume that $\text{Th}(M)$ is superstable. Then for any countable model N and $p \in S(N)$ there is finite $A \subset N$ such that $p|A$ is stationary.

Proof Take a realization $\bar{b}$ of p . By superstability, there is finite $X \subset N$ such that p does not fork over X . Let $B = \text{cl}(X\bar{b})$ and $A = B \cap N$. Clearly $\text{tp}(\bar{b}/N)$ does not fork over A . We show that $\text{tp}(\bar{b}/A)$ is stationary. Take any $\bar{b}'$ such

that $\text{tp}(\bar{b}'/A) = \text{tp}(\bar{b}/A)$ and $\text{tp}(\bar{b}'/N)$ does not fork over A . Let $B' = \text{cl}(\bar{b}'A)$. Then we have $B \cong_A B'$. Note that $B' \cap N = A$. By lemma 8, B, N and B', N are free over A respectively, and so we have $B \cong_N B'$. By lemma 8 again, $BN, B'N$ are closed since $\text{acl}(A) \subset N$. It follows that $\text{tp}(BN) = \text{tp}(B'N)$ and hence $\text{tp}(b/N) = \text{tp}(b'/N)$.

By lemma 9, we have the following theorem.

Theorem 10 Let L be a countable relational language. Let $\mathbf{K} = (K, \leq)$ be a class of finite L -structures that is derived from a predimension δ and that is closed under substructures. Let M be a saturated $\mathbf{K}$ -generic structure. If $\text{Th}(M)$ is superstable, then it is ω -stable.

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