

Singularities of nullsphere Gauss map for spacelike surface in nullcone 3-space

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1 Introduction

The nullcone in Minkowski 4-space is one kind of Minkowski pseudo-sphere, which is similar with the sphere in Euclidean 4-space. In [6], Izumiya has studied the details of spacelike hypersurface in the nullcone by Legendrian dualities. Our aim in this article is to study spacelike surfaces in nullcone 3-space by the method similar to that in [5].

We shall assume throughout the whole article that all maps and manifolds are C^∞ unless the contrary is explicitly stated.

Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) | x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ be a 4-dimensional vector space. For any two vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, the *pseudo-scalar product* of $\mathbf{x}$ and $\mathbf{y}$ is defined by $\langle \mathbf{x}, \mathbf{y} \rangle = -x_1 y_1 + \sum_{i=2}^4 x_i y_i$. $(\mathbb{R}^4, \langle \cdot, \cdot \rangle)$ is called a *Minkowski 4-space* and written by $\mathbb{R}_1^4$. A vector $\mathbf{x}$ in $\mathbb{R}_1^4 \setminus \{0\}$ is called *spacelike*, *lightlike* or *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle$ is positive, zero or negative respectively. The norm of a vector $\mathbf{x} \in \mathbb{R}_1^4$ is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}_1^4$, we say $\mathbf{x}$ *pseudo-perpendicular* to $\mathbf{y}$ if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. For a vector $\mathbf{v} \in \mathbb{R}_1^4$ and a real number c , a hyperplane with pseudo normal $\mathbf{v}$ is defined by $HP(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}_1^4 | \langle \mathbf{x}, \mathbf{v} \rangle = c\}$. $HP(\mathbf{v}, c)$ is called a *timelike hyperplane*, a *spacelike hyperplane* or a *lightlike hyperplane* if $\mathbf{v}$ is timelike, spacelike or lightlike respectively. Now, *hyperbolic 3-space* is defined by $H_1^3 = \{\mathbf{x} \in \mathbb{R}_1^4 | \langle \mathbf{x}, \mathbf{x} \rangle = -1\}$, *de Sitter 3-space* is defined by $S_1^3 = \{\mathbf{x} \in \mathbb{R}_1^4 | \langle \mathbf{x}, \mathbf{x} \rangle = 1\}$ and the *nullcone 3-space* is defined by $NC^3 = \{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}_1^4 | x_1 \neq 0, \langle \mathbf{x}, \mathbf{x} \rangle = 0\}$. The *3-dimension nullcone with vertex λ* in $\mathbb{R}_1^4$ is defined by $NC_\lambda^3 = \{\mathbf{x} \in \mathbb{R}_1^4 | \langle \mathbf{x} - \lambda, \mathbf{x} - \lambda \rangle = 0\}$. If $\mathbf{x} = (x_1, x_2, x_3, x_4)$ is a lightlike vector, then $x_1 \neq 0$. Therefore we have $\tilde{\mathbf{x}} = \left(1, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_4}{x_1}\right) \in S_+^2 = \{\mathbf{x} \in \mathbb{R}_1^4 | \langle \mathbf{x}, \mathbf{x} \rangle = 0, x_1 = 1\}$. S_+^2 is called the *nullcone unit 2-sphere*.

For any $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}_1^4$, we define a vector $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3$ by

$$\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ x_1^1 & x_1^2 & x_1^3 & x_1^4 \\ x_2^1 & x_2^2 & x_2^3 & x_2^4 \\ x_3^1 & x_3^2 & x_3^3 & x_3^4 \end{vmatrix},$$

where e_1, e_2, e_3, e_4 is the canonical basis of $\mathbb{R}_1^4$ and $\mathbf{x}_i = (x_i^1, x_i^2, x_i^3, x_i^4)$. It can easily check that $\langle \mathbf{x}, \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \rangle = \det(\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, so that $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3$ is pseudo orthogonal to any $\mathbf{x}_i (i = 1, 2, 3)$.

We fix an orientation and timelike orientation of $\mathbb{R}_1^4$ (i.e., a 4-volume form dV , and future time-like vector field, have been chosen). Let $X : U \rightarrow NC^3$ be an embedding, where U is an open subset of $\mathbb{R}^2$. Denote $M = X(U)$ and identify M with U by the embedding X . We say X a *spacelike surface* if X_{u_1} and X_{u_2} are spacelike vectors. Therefore, the tangent space $T_p M$ of M is a spacelike subspace (i.e., consists of spacelike vectors) for any point $p \in M$. In this case, the pseudo-normal space $N_p M$ is a timelike plane (i.e.,

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Lorentz plane). Denote by $N(M)$ the pseudo-normal bundle over M . Since this is a trivial bundle, we can arbitrarily choose a future directed unit timelike normal section $\mathbf{n}^T(u) \in N_p M \cap H_1^3$, where $p = X(u)$. Therefore we can define a spacelike unit normal section $\mathbf{n}^S(u)$ by

$$\mathbf{n}^S(u) = \frac{\mathbf{n}^T(u) \wedge X_{u_1}(u) \wedge X_{u_2}(u)}{\|\mathbf{n}^T(u) \wedge X_{u_1}(u) \wedge X_{u_2}(u)\|} \in S_1^3,$$

and we have $\langle \mathbf{n}^T, \mathbf{n}^S \rangle = 0$. Although we could also choose $-\mathbf{n}^S(u)$ as a spacelike unit normal section with the above properties, we fix the direction $\mathbf{n}^S(u)$ throughout this article. $(\mathbf{n}^T, \mathbf{n}^S)$ is called a *future directed normal frame* along $M = X(U)$. Clearly, the vector $\mathbf{n}^T \pm \mathbf{n}^S(u)$ is lightlike. Since $\{X_{u_1}, X_{u_2}\}$ is a basis of $T_p M$, the system $\{X_{u_1}, X_{u_2}, \mathbf{n}^T, \mathbf{n}^S\}$ provides a basis for $T_p \mathbb{R}_1^4$.

$X \in N_p M$, $N_p M$ is a Lorentzian plane and $X(U)$ is a regular surface, so $\tilde{X}(u) = \widetilde{\mathbf{n}^T + \mathbf{n}^S}(u)$ for any $u \in U$ or $\tilde{X}(u) = \widetilde{\mathbf{n}^T - \mathbf{n}^S}(u)$ for any $u \in U$.

Here, we only consider the case of $\tilde{X}(u) = \widetilde{\mathbf{n}^T - \mathbf{n}^S}(u)$ for $u \in U$. The case of $\tilde{X}(u) = \widetilde{\mathbf{n}^T + \mathbf{n}^S}(u)$ can be discussed similarly. Define two maps of $M = X(U)$ as

$$NG_M^\pm : U \rightarrow S_+^2, \quad NG_M^\pm(u) = \widetilde{\mathbf{n}^T \pm \mathbf{n}^S}(u),$$

each one of these maps is called *nullsphere Gauss map*. Under the identification of M with U through X , we have the linear mapping $d_p(\widetilde{\mathbf{n}^T \pm \mathbf{n}^S}) : T_p M \rightarrow T_p \mathbb{R}_1^4 = T_p M \oplus N_p M$. Consider the orthogonal projections $\pi^t : T_p M \oplus N_p M \rightarrow T_p M$ and $\pi^n : T_p M \oplus N_p M \rightarrow N_p M$. Define $d_p(\widetilde{\mathbf{n}^T \pm \mathbf{n}^S})^t = \pi^t \circ d_p(\widetilde{\mathbf{n}^T \pm \mathbf{n}^S})$ and $d_p(\widetilde{\mathbf{n}^T \pm \mathbf{n}^S})^n = \pi^n \circ d_p(\widetilde{\mathbf{n}^T \pm \mathbf{n}^S})$. The linear transformations $S_p^\pm(\mathbf{n}^T, \mathbf{n}^S) = -d_p(\widetilde{\mathbf{n}^T \pm \mathbf{n}^S})^t$ and $d_p(\widetilde{\mathbf{n}^T \pm \mathbf{n}^S})^n$ are respectively called the $(\mathbf{n}^T, \mathbf{n}^S)$ -*shape operator* and the *normal connection with respect to $(\mathbf{n}^T, \mathbf{n}^S)$* of $M = X(U)$ at $p = X(u)$.

The eigenvalues of $S_p^\pm(\mathbf{n}^T, \mathbf{n}^S)$ denoted by $\{\kappa_i^\pm(\mathbf{n}^T, \mathbf{n}^S)(p)\} (i = 1, 2)$ are called the $(\mathbf{n}^T, \mathbf{n}^S)$ -*nullsphere principal curvature* with respect to $(\mathbf{n}^T, \mathbf{n}^S)$ at p . Then the *nullsphere Gauss-Kronecker curvature* with respect to $(\mathbf{n}^T, \mathbf{n}^S)$ at $p = X(u)$ is defined as

$$K_n^\pm(\mathbf{n}^T, \mathbf{n}^S)(p) = \det S_p^\pm(\mathbf{n}^T, \mathbf{n}^S).$$

We say that a point $p = X(u)$ is a $(\mathbf{n}^T, \mathbf{n}^S)$ -*umbilic point* if all the principal curvatures coincide at p and thus $S_p^\pm(\mathbf{n}^T, \mathbf{n}^S) = \kappa^\pm(\mathbf{n}^T, \mathbf{n}^S)I|_{T_p M}$ for some function $\kappa^\pm$. We say that $M = X(U)$ is totally $(\mathbf{n}^T, \mathbf{n}^S)$ -umbilic if all points on M are $(\mathbf{n}^T, \mathbf{n}^S)$ -umbilic.

We deduce now the nullcone Weingarten formula. Since X_{u_1} and X_{u_2} are spacelike vectors, we have a Riemannian metric (*the first fundamental form*) on M defined by $ds^2 = \sum_{i=1}^2 g_{ij} du_i du_j$, where $g_{ij}(u) = \langle X_{u_i}, X_{u_j} \rangle$ for any $u \in U$. We also have a *nullcone second fundamental invariant* with respect to the normal vector field $(\mathbf{n}^T, \mathbf{n}^S)$ defined by $h_{ij}^\pm(\mathbf{n}^T, \mathbf{n}^S)(u) = \langle -(\widetilde{\mathbf{n}^T \pm \mathbf{n}^S})_{u_i}(u), X_{u_j}(u) \rangle$ for any $u \in U$.

Proposition 1.1. *Under the above notations, we have the following nullcone Weingarten formula with respect to $(\mathbf{n}^T, \mathbf{n}^S)$:*

$$(a) \widetilde{(\mathbf{n}^T \pm \mathbf{n}^S)}_{u_i} = \frac{\mp(\mathbf{n}_1^T \pm \mathbf{n}_1^S)(\mathbf{n}_{u_i}^S, \mathbf{n}^T) - (\mathbf{n}_1^T \pm \mathbf{n}_1^S)_{u_i}}{(\mathbf{n}_1^T \pm \mathbf{n}_1^S)^2} (\mathbf{n}^T \pm \mathbf{n}^S) - \sum_{j=1}^2 h_i^{j\pm}(\mathbf{n}^T, \mathbf{n}^S) X_{u_j};$$

$$(b) \pi^t \circ (\widetilde{\mathbf{n}^T \pm \mathbf{n}^S})_{u_i} = - \sum_{j=1}^2 h_i^{j\pm}(\mathbf{n}^T, \mathbf{n}^S) X_{u_j}.$$

Here, $h_i^{j\pm}(\mathbf{n}^T, \mathbf{n}^S) = h_{ik}^\pm(\mathbf{n}^T, \mathbf{n}^S) g^{kj}$, $g^{kj} = (g_{kj})^{-1}$ and $\mathbf{n}^i = (\mathbf{n}_1^i, \mathbf{n}_2^i, \mathbf{n}_3^i, \mathbf{n}_4^i) (i = T, S)$.

As a corollary of the above proposition, we have an explicit expression of the nullsphere Gauss-Kronecker curvature by Riemannian metric and the nullcone second fundamental invariant.

Corollary 1.2. *Under the same notations as in the above proposition, the nullsphere Gauss-Kronecker curvature is given by*

$$K_n^\pm(\mathbf{n}^T, \mathbf{n}^S)(u) = \frac{\det(h_{ij}^\pm(\mathbf{n}^T, \mathbf{n}^S)(u))}{\det(g_{\alpha\beta})}.$$

If $K_n^\pm(\mathbf{n}^T, \mathbf{n}^S)(u_0) = 0$, the point $p_0 = X(u_0)$ is called a $(\mathbf{n}^T, \mathbf{n}^S)$ -nullcone parabolic point of $X : U \rightarrow NC^3$. And we say that a point p_0 is a $(\mathbf{n}^T, \mathbf{n}^S)$ -nullcone flat point if it is a $(\mathbf{n}^T, \mathbf{n}^S)$ -nullcone umbilical point and $K_n^\pm(\mathbf{n}^T, \mathbf{n}^S)(u_0) = 0$.

Theorem 1.3. $K_n^-(\mathbf{n}^T, \mathbf{n}^S)(u) \neq 0$.

2 Nullsphere height function

The nullsphere height function family on $M = X(U)$ is defined by

$$H : U \times S_+^2 \rightarrow \mathbb{R}, \quad H(u, v) = \langle X(u), v \rangle.$$

The Hessian matrix of the nullsphere height function $h_{v_0} = H(u, v_0)$ at u_0 is denoted by $\text{Hess}(h_{v_0})(u_0)$.

Proposition 2.1. Let H be a nullsphere height function on M . Then

- (1) $\partial h_{v_0} / \partial u_i(u_0) = 0 (i = 1, 2)$ if and only if $v_0 = \widetilde{\mathbf{n}^T \pm \mathbf{n}^S}(u_0)$.
- (2) $\partial h_{v_0} / \partial u_i(u_0) = \det \text{Hess}(h_{v_0})(u_0) = 0 (i = 1, 2)$ if and only if $v_0 = \widetilde{\mathbf{n}^T \pm \mathbf{n}^S}(u_0)$ and $K_n^\pm(\mathbf{n}^T, \mathbf{n}^S)(u_0) = 0$.
- (3) p_0 is a nullcone flat point if and only if $\text{rank} \text{Hess}(h_{v_0})(u_0) = 0$.

Corollary 2.2. For a point $p_0 = X(u_0) \in M$, the following conditions are equivalent:

- (1) The point $p_0 \in M$ is a $(\mathbf{n}^T, \mathbf{n}^S)$ -nullcone parabolic point.
- (2) The point $p_0 \in M$ is a singular point of the nullsphere Gauss map $NG_M^\pm$.
- (3) $K_n^\pm(\mathbf{n}^T, \mathbf{n}^S)(u_0) = 0$.
- (4) $\det \text{Hess}(h_{v_0})(u_0) = 0$ for $v_0 = \widetilde{\mathbf{n}^T \pm \mathbf{n}^S}(u_0)$.

Corollary 2.3. NG_M^- is a regular nullsphere Gauss map.

Consider now the particular case of a surface $M \subset NC^3$. Given a vector $\mathbf{v} \in S_+^2$ (resp. S_1^3, H_1^3) and a number c , denoted by $S(\mathbf{v}, c)$ the null hyperhorosphere (resp. null equidistant hyperplane, null hypersphere) determined by the intersection of the hyperplane $HP(\mathbf{v}, c)$ with NC^3 .

Proposition 2.4. Let M be a spacelike surface in NC^3 . If NG_M^- is constant, then M degenerate to a straight line.

We now define a family of functions

$$\tilde{H} : U \times NC^3 \rightarrow \mathbb{R}, \quad \tilde{H}(u, v) = \langle X(u), \tilde{\mathbf{v}} \rangle - v_1,$$

where $\mathbf{v} = (v_1, v_2, v_3, v_4)$. $\tilde{H}$ is called the extended nullsphere height function of $M = X(U)$. The Hessian matrix of the extended nullsphere height function $\tilde{h}_{v_0} = \tilde{H}(u, v_0)$ at u_0 is denoted by $\text{Hess}(\tilde{h}_{v_0})(u_0)$.

Proposition 2.5. Let M be a spacelike surface in NC^3 . $\tilde{H}$ is the extended nullsphere height function of M . For $v_0 \in NC^3$, we have the following:

- (1) $\tilde{h}_{v_0}(p_0) = \frac{\partial \tilde{h}_{v_0}}{\partial u_i}(p_0) = 0$ if and only if $\tilde{v}_0 = \widetilde{\mathbf{n}^T \pm \mathbf{n}^S}(u_0)$ and $v_1 = \langle X(u_0), \widetilde{\mathbf{n}^T \pm \mathbf{n}^S}(u_0) \rangle$.
- (2) $\tilde{h}_{v_0}(p_0) = \frac{\partial \tilde{h}_{v_0}}{\partial u_i}(p_0) = \det \text{Hess} \tilde{h}_{v_0}(p_0) = 0$ if and only if $\tilde{v}_0 = \widetilde{\mathbf{n}^T \pm \mathbf{n}^S}(u_0)$, $v_1 = \langle X(u_0), \widetilde{\mathbf{n}^T \pm \mathbf{n}^S}(u_0) \rangle$ and $K_n^\pm(\mathbf{n}^T, \mathbf{n}^S)(p_0) = 0$.

The assertions of proposition 2.5 means that the discriminant set of the extended nullsphere height function $\tilde{H}$ is given by $\mathcal{D}_{\tilde{H}} = \{\mathbf{v} \mid \mathbf{v} = \langle X(u), \widetilde{\mathbf{n}^T \pm \mathbf{n}^S}(u) \rangle (\widetilde{\mathbf{n}^T \pm \mathbf{n}^S}(u))\}$. Therefore we now define a pair of singular surfaces in NC^3 by $NP_M^\pm(u) = \langle X(u), \widetilde{\mathbf{n}^T \pm \mathbf{n}^S}(u) \rangle (\widetilde{\mathbf{n}^T \pm \mathbf{n}^S}(u))$, each one of $NP_M^\pm$ is called the nullcone pedal surface of $X(U) = M$. A singularity of the nullcone pedal surface exactly corresponds to a singularity of the nullsphere Gauss map.

Corollary 2.6. NP_M^- is a zero map.

This work is only a preparation for further studying, in the following, we will discuss some geometrical properties of spacelike curve from singularity theory viewpoint.

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