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Indivisibility of special values of zeta functions associated to real quadratic fields

Dedicated to Professor Tatsuo Kimura on the occasion of his 60-th birthday

By

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Abstract

We discuss some aspects of indivisibility of the special values of Dedekind zeta functions at negative odd integers associated to real quadratic fields. These values are closely related to the orders of certain cohomology groups and algebraic K-groups.

We show that, for an even number n and a prime p under some conditions, a quantitative result for the distribution of real quadratic fields whose special values of the L-functions at $1-n$ are indivisible by p.

§ 1. Introduction

We shall consider indivisibility properties of special values of Dedekind zeta functions associated to real quadratic fields at negative integers.

We use the following notation in the sequel. For any set S, we denote by $\#S$ the cardinality of S. We fix an algebraic closure $\overline{\mathbb{Q}}$ of the field of rational numbers $\mathbb{Q}$ in the field of complex numbers $\mathbb{C}$. We assume that any number field (an algebraic extension of $\mathbb{Q}$) is a subfield of $\overline{\mathbb{Q}}$. For any number field k, we denote by $\zeta_k(s)\,(s \in \mathbb{C})$ the Dedekind zeta function of k. We denote by $L(s,\chi)\,(s \in \mathbb{C})$ the Dirichlet L-function associated to a Dirichlet character χ.

For an imaginary quadratic field $\mathbb{Q}(\sqrt{D})$, $D < 0$, with the Kronecker symbol $\chi_D(\cdot) = (D/\cdot)$, the special value $L(0,\chi_D)$ of the Dirichlet L-function at 0 is essentially

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the class number $h(D)$ of $\mathbb{Q}(\sqrt{D})$. There are many investigations concerning indivisibility of the class numbers of quadratic fields. Among these studies, the Cohen-Lenstra heuristics [CL84] predicts that, for a fixed odd prime p,

$$\lim_{X\to\infty} \frac{\#\{-X < D < 0 \mid p \nmid h(D)\}}{\#\{-X < D < 0\}} = \prod_{n\geq 1}(1-p^{-n}), \tag{1.1}$$

where D runs through fundamental discriminants. There is also real quadratic counterpart:

$$\lim_{X\to\infty} \frac{\#\{0 < D < X \mid p \nmid h(D)\}}{\#\{0 < D < X\}} = \prod_{n\geq 2}(1-p^{-n}). \tag{1.2}$$

We are interested in the indivisibility of the special values of Dedekind zeta functions at negative integers because these values are also closely related to the orders of certain cohomology groups and the orders of algebraic K-groups.

Byeon's results [Bye03] (*cf.* Bruinier [Bru99, corollary 2]) can be stated that, combining Birch-Tate conjecture (now a consequence of Iwasawa main conjecture proved by Mazur-Wiles [MW84]), for any odd prime p, there are infinitely many real quadratic fields $\mathbb{Q}(\sqrt{D})$ with the ring of integers O_D such that the order of the second algebraic K-group $\mathrm{K}_2(O_D)$ is not divisible by p:

$$\#\{\mathbb{Q}(\sqrt{D}) \mid D > 0,\ p \nmid \#\mathrm{K}_2(O_D)\} = \infty.$$

We show that, for an even number n and a prime p under some conditions, a quantitative result for the distribution of real quadratic fields whose special values of the L-functions at $1-n$ are indivisible by p (Theorem 3.1). This is an analogue of the Cohen-Lenstra heuristics on the distribution of ideal class groups of quadratic fields.

We deduce similar statements for indivisibility of orders of certain cohomology groups and K-groups (corollary 3.5).

There is an imaginary quadratic counterpart, which will be treated in separate article.

Remark. We treated in [Kim06] the other question: for an odd prime p and a natural number n, are there infinitely many real or imaginary quadratic fields whose special values of Dedekind zeta functions or Dirichlet L-functions at $1-n$ are divisible by p?

In some cases, this question has been known to be true. Carlitz [Car59, Theorem 4] showd that if p is a rational prime such that $p^e | n$ but $p \nmid f$, f being the conductor of a Dirichlet character χ, then p^e divides the numerator of the n-th generalized Bernoulli number associated to χ.

For real quadratic fields, Coates-Lichtenbaum [CL73, Example 6, §7] showed that, for a prime $p \equiv 3 \pmod 4$, the value $\zeta_{\mathbb{Q}(\sqrt{pf})}(-p^r(p-1)/2)$ is divisible by p^{r+1}, provided that $p \nmid f$, and f is not quadratic residue modulo p (*cf.* Carlitz (*loc. cit.* Theorem 6)).

§ 2. Preliminaries

We need some more notations. For an extension of fields K/k, $[K : k]$ means the degree of K/k. If the extension K/k is Galois extension, we denote its Galois group by $\mathrm{Gal}(K/k)$. As usual, $\mathbb{Z}$ means the ring of rational integers, $\mathbb{Z}_{\geq 0}$ is the set of nonnegative rational integers, $\mathbb{N} = \mathbb{Z}_{>0}$ is the set of natural numbers. For a rational prime p, $\mathrm{ord}_p(a)$ is the additive valuation of a rational integer a, and $|a|_p = p^{-\mathrm{ord}_p(a)}$ is the multiplicative valuation.

For any number field k, $D(k)$ denote the discriminant, $\mathrm{Cl}(k)$ denote the class group, $h(k) = \#\mathrm{Cl}(k)$ denote the class number, O_k denote the ring of integers (when $k = \mathbb{Q}(\sqrt{D})$ is a quadratic field, we abbreviate $O_D = O_{\mathbb{Q}(\sqrt{D})}$).

For any natural number m, $\zeta_m \in \overline{\mathbb{Q}}$ is a primitive m-th root of unity. Let $w_r(k)$ (where $r \in \mathbb{N}$) be

$$w_r(k) := \max\{m \in \mathbb{Z}_{\geq 0} | \exp(\mathrm{Gal}(k(\zeta_m)/k)) \mid r\},$$

where $\exp(G)$ is the exponent of a finite group G. We abbreviate $w_r(D) = w_r(\mathbb{Q}(\sqrt{D}))$ for a quadratic field $\mathbb{Q}(\sqrt{D})$.

For any even natural number r and for any positive fundamental discriminant D, we denote

$$\xi_r(D) := w_r(D)\zeta_{\mathbb{Q}(\sqrt{D})}(1-r).$$

It is known by Coates-Sinnott [CS77] that

$$\xi_r(D) \in \mathbb{Z}. \tag{2.1}$$

Let $H(r, N)$, $(r, N \in \mathbb{Z}_{\geq 0})$ denote the generalized class number function defined by Cohen [Coh75]. This is given by the following formula:

$$H(r,N) = \begin{cases} 0 & \text{if } N \not\equiv 0, 1 \pmod 4, \\ \zeta(1-r) = -\frac{B_r}{r} & \text{if } N = 0, \\ L(1-r, \chi_D) \sum_{d|f} \mu(d)\chi_D(d) d^{r-1} \sigma_{2r-1}\left(\frac{f}{d}\right) & \text{otherwise,} \end{cases}$$

where $(-1)^r N = Df^2$, D is a fundamental discriminant, χ_D is the Kronecker character associated to the quadratic field $\mathbb{Q}(\sqrt{D})$, $\mu(\cdot)$ is the Möbius function, $\sigma_k(\cdot)$ is the sum of the k-th power of divisors. Note that $H(r, N)$ is a rational number: $H(r, N) \in \mathbb{Q}$.

We write $M_k(\Gamma, \chi)$ for the space of modular forms of weight k and character χ on a congruence subgroup Γ of the special linear group $\mathrm{SL}_2(\mathbb{Z})$.

Let $f(z) = \sum_{n=0}^{\infty} a(n)q^n$, $q = e^{2\pi iz}$ be a q-expansion at infinity of $f(z) \in M_k(\Gamma, \chi)$. We write

$$f(z) \in M_k(\Gamma, \chi) \cap \mathbb{Q}[[q]] \quad (\textit{resp. } f(z) \in M_k(\Gamma, \chi) \cap \mathbb{Z}[[q]])$$

if all of the coefficients $a(n)$ are rational numbers (*resp.* rational integers).

For a prime p and modular forms $f(z) = \sum_{n=0}^{\infty} a(n)q^n$, $g(z) = \sum_{n=0}^{\infty} b(n)q^n \in M_k(\Gamma, \chi) \cap \mathbb{Z}[[q]]$, we denote $f(z) \equiv g(z) \pmod{p}$ if and only if $a(n) \equiv b(n) \pmod{p}$ for all n.

Let $g = \sum_{n=0}^{\infty} a(n)q^n \in \mathbb{Z}[[q]]$ be a formal power series with coefficients in $\mathbb{Z}$ for a variable q. Let $\mathrm{ord}_p(g)$ be an order of g at p:

$$\mathrm{ord}_p(g) := \min\{n \geq 0 | \, a(n) \not\equiv 0 \pmod{p}\}.$$

We need Sturm's theorem ([Stu87]) on congruence for modular forms. This is proven for the integral weight case, but it holds for half integral weight case taking appropriate powers.

Theorem 2.1 (Sturm's theorem). *If a modular form* $g(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]]$ *satisfies*

$$ord_p(g) \geq \kappa(k, N) := \frac{k}{12}[\Gamma_0(1) : \Gamma_0(N)],$$

then

$$g(z) \equiv 0 \pmod{p}, \quad \textit{i.e. for all } n, a(n) \equiv 0 \pmod{p}.$$

The generating function of $H(r, N)$ is a modular form $\mathcal{H}_r(z)$ of half integral weight for the principal congruence subgroup $\Gamma_0(4)$ of level 4. This is of weight $r + (1/2)$ and trivial character, with rational coefficients (*cf.* Cohen (*loc. cit.*, Theorem 3.1)):

$$\mathcal{H}_r(z) := \sum_{N \geq 0} H(r, N)q^N \in M_{r+\frac{1}{2}}(\Gamma_0(4), 1) \cap \mathbb{Q}[[q]], \quad q = e^{2\pi iz}.$$

Lemma 2.2. *Notations are the same as above. There exists a rational integer* $M(r) \in \mathbb{Z}$ *such that all of the coefficients of* $M(r)\mathcal{H}_r(z)$ *are rational integers. Further, if a prime* p *satisfies* $(p-1)/2 \nmid r$, *then* $p \nmid M(r)$.

Proof. It can be shown that there exists an integer $M \in \mathbb{Z}$ such that

$$M\mathcal{H}_r(z) \in M_{r+\frac{1}{2}}(\Gamma_0(4), 1) \cap \mathbb{Z}[[q]]$$

by "bounded denominators" principle (*cf.* Serre-Stark [SS77, lemma 8]). The assertion of the lemma is more precise.

This follows from the lemma by Harris and Segal [HS75, §4, p. 28]. To recall, let F be a number field, $W \subset \overline{\mathbb{Q}}^{\times}$ be the group of all root of unity. Letting the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ act on W by $\sigma * \zeta = \sigma^r(\zeta)$, $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, $\zeta \in W$, we see $w_r(F) = \#W^{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}$, the order of the fixed subgroup. For a prime l, let $w_r^{(l)}(F)$ be the l-primary factor of $w_r(F)$, l^m be the order of the group of l-power-th root of unity in $F(\zeta_l)$ and $s = [F(\zeta_l) : F]$. In this setting, they proved that, if $r \equiv 0 \pmod{s}$, *e. g.* $r = sl^{\lambda}t$, $(l,t) = 1$, then $w_r^{(l)} = l^{m+\lambda}$. If $r \not\equiv 0 \pmod{s}$, then $w_r^{(l)} = 1$.

In our situation, $F = \mathbb{Q}(\sqrt{D})$ is a real quadratic field. $s = [\mathbb{Q}(\sqrt{D}, \zeta_l) : \mathbb{Q}(\sqrt{D})]$ is either $(l-1)/2$ or $l-1$. Thus the prime l which affects $w_r(D)$ satisfies $(l-1)/2 \mid r$ or $(l-1) \mid r$. Combining this with (2.1) shows that, we can take $M(r) = \prod_l l^{a(l)}$, where the product run through a prime which $(l-1)/2 \mid r$ or $(l-1) \mid r$, and accordingly, $a(l) = \mathrm{ord}_l(2r/(l-1)) + 1$ or $a(l) = \mathrm{ord}_l(r/(l-1)) + 1$. □

§ 3. Main Results

In this section, we extend Theorem 2 of K. Ono [Ono99] to the case of special values of Dirichlet L-functions at negative integers.

For the Dirichlet character χ_D associated to a quadratic field $\mathbb{Q}(\sqrt{D})$, let $B(r, \chi_D)$ denote the r-th generalized Bernoulli number associated to χ_D.

Theorem 3.1. *Let r be an even natural number, $p > 3$ a prime which satisfies $(p-1)/2 \nmid r$. Suppose there is a fundamental discriminant D_0 coprime to p such that $|B(r, \chi_{D_0})|_p = 1$.*

Then there exist an arithmetic progression $\{r_p + nt_p | n \in \mathbb{Z}\}$ with $(r_p, t_p) = 1$ and a constant $\kappa(p)$ depending on p which satisfy all of the conditions below:

For any prime $l \equiv r_p \pmod{t_p}$, there exists an integer d_l such that $1 \leq d_l \leq \kappa(p)l$ which satisfy the following two conditions:

1. *$D_l = d_l lp$ is a fundamental discriminant,*
2. *$p \nmid \xi_r(D)$.*

Proof. Retaining notations as above, we have

$$M(r)\mathcal{H}_r(z) \in M_{r+\frac{1}{2}}(\Gamma_0(4), 1) \cap \mathbb{Z}[[q]].$$

Let us define some modular forms from $\mathcal{H}_r(z)$.

$$\begin{aligned}\mathcal{F}_r(z) &:= M(r)\mathcal{H}_r(z) - M(r)(V_p|U_p|\mathcal{H}_r(z)) \\ &= M(r) \sum_{(n,p)=1} H(r,n)q^n \in M_{r+\frac{1}{2}}(\Gamma_0(4p^2), \chi_0).\end{aligned}$$

Take a prime Q so that $Q \neq p$, $(D_0/Q) = -1$, where $(D_0/\cdot)$ is the Kronecker symbol. Then, we define another modular form

$$\begin{aligned}\mathcal{G}_p(z) &:= \mathcal{F}_r(z) \otimes \left(\frac{\cdot}{Q}\right) \\ &= M(r) \sum_{(n,p)=1} \left(\frac{n}{Q}\right) H(r,n) q^n.\end{aligned}$$

Finally, we put

$$\begin{aligned}G_r(z) &:= \frac{1}{2}\left(\mathcal{G}_p(z) \otimes \left(\frac{\cdot}{Q}\right) - \mathcal{G}_p(z)\right) \\ &= M(r) \sum_{(n,p)=1,\ \left(\frac{n}{Q}\right)=-1} H(r,n) q^n.\end{aligned}$$

By the assumption and lemma 2.2, the coefficients of q^{D_0} of $G_r(z)$ is not congruent to 0 modulo p. Thus we see

$$G_r(z) \not\equiv 0 \pmod{p}.$$

The coefficients $H(r,0), H(r,n^2), H(r,pn)$ of $G_r(z)$ are annihilated.

Let l be any rational prime. We define two linear maps U_l, V_l from $M_k(\Gamma_0(N),\chi)$ to $M_k(\Gamma_0(lN),\chi(4p/\cdot))$. For any $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma,\chi)$, we define

$$(U_l|f)(z) = \sum_{n=0}^{\infty} a(ln) q^n,$$

$$(V_l|f)(z) = \sum_{n=0}^{\infty} a(n) q^{ln}.$$

We define $(U_l|G_r)(z), (V_l|G_r)(z)$ by the following;

$$\begin{aligned}(U_l|G_r)(z) &:= \sum_{n=1}^{\infty} u_{r,l}(n) q^n \\ &= M(r) \sum_{\left(\frac{ln}{Q}\right)=-1,\ \left(\frac{ln}{p}\right)=1} H(r,ln) q^n \qquad (3.1)\end{aligned}$$

$$\begin{aligned}(V_l|G_r)(z) &:= \sum_{n=1}^{\infty} v_{r,l}(n) q^{ln} \\ &= M(r) \sum_{\left(\frac{n}{Q}\right)=-1,\ \left(\frac{n}{p}\right)=1} H(r,n) q^{ln} \qquad (3.2)\end{aligned}$$

Then we see

$$(U_l|G_r)(z), (V_l|G_r)(z) \in M_{r+\frac{1}{2}}(\Gamma_0(4p^2Q^4l), \left(\frac{4l}{\cdot}\right)).$$

We need Sturm's theorem 2.1 to proceed. It is well known that $[\Gamma_0(1):\Gamma_0(N)] = N\prod_{p|N}(1+p^{-1})$. In our situation, $N = 4p^2Q^4l$, thus we have

$$\begin{aligned}[\Gamma_0(1):\Gamma_0(4p^2Q^4l)] &= 4p^2Q^4l\frac{3}{2}\left(1+\frac{1}{p}\right)\left(1+\frac{1}{Q}\right)\left(1+\frac{1}{l}\right)\\ &= 6pQ^3(p+1)(Q+1)(l+1).\end{aligned}$$

The Sturm bound $\kappa(r+(1/2), 4p^2Q^4l)$ is thus given by

$$\kappa(r+\frac{1}{2}, 4p^2Q^4l) = \frac{r+\frac{1}{2}}{12}6pQ^3(p+1)(Q+1)(l+1) = \frac{2r+1}{4}p(p+1)Q^3(Q+1)(l+1).$$

For brevity we put

$$\kappa(r,p) = \frac{2r+1}{4}p(p+1)Q^3(Q+1).$$

By Sturm's theorem 2.1, we see that, for sufficiently large prime $l \gg 0$, if

$$g \in M_{r+(1/2)}(\Gamma_0(4p^2Q^4l), (4l/\cdot)) \cap \mathbb{Z}[[q]]$$

and $\mathrm{ord}_p(g) > \kappa(p,r)l$, then $g \equiv 0 \pmod{p}$.

Let $l \neq p$ be a prime with $(l/Q) = 1$. If $(n/Q) \neq -1$ or $(n,p) \neq 1$, by (3.1) and (3.2),

$$u_{r,l}(nl) = v_{r,l}(nl) = 0.$$

For n with $(n/Q) = -1$ and $(n,p) = 1$, we see that

$$u_{r,l}(nl) = M(r)H(r, nl^2), \tag{3.3}$$

$$v_{r,l}(nl) = M(r)H(r,n). \tag{3.4}$$

Then, if $(l/Q) = 1$, $l \gg 0$, for all $n \leq \kappa(p,r)$,

$$u_{r,l}(nl) = M(r)(1 - \chi_{D_n}(l)l^{r-1} + l^{2r-1})H(r,n), \tag{3.5}$$

where D_n is a discriminant of $\mathbb{Q}(\sqrt{(-1)^r n})$.

Let us define a set $S_{p,r}$ by

$$S_{p,r} := \left\{ D_n \,|\, n \leq \kappa(p,r),\ \left(\frac{n}{Q}\right) = 1,\ (n,p) = 1 \right\}.$$

If $n < \kappa(p,r)$, the $D_n m^2$-th coefficients of $G_r(z)$ is 0, and it does not affect the arguments after here.

There exists an arithmetic progression $\{r_p + nt_p \,|\, n \in \mathbb{Z}\}$, $(r_p, t_p) = 1$, $p \mid t_p$ for which the three conditions hold:

1. $\chi_{D_n}(l) = 1$, for all prime $l \equiv r_p \pmod{t_p}$ and for all $D_n \in S_{p,r}$.
2. $(l/Q) = 1$ for all $l \equiv r_p \pmod{t_p}$.
3. $r_p^r \not\equiv 1 \pmod{p}$.

By (3.3), (3.4), (3.5), for all prime $l \equiv r_p \pmod{t_p}$, we see that

$$u_{r,l}(nl) \equiv (1 - r_p^{r-1} + r_p^{2r-1}) v_{r,l}(nl) \pmod{p}.$$

holds for all $n \leq \kappa(p,r)$.

If $l \nmid n$, then $v_{r,l}(n) = 0$. Thus we see that the following implication holds: There is no n with $n \leq \kappa(p,r)l$, $l \nmid n$ and

$$u_{r,l}(n) = M(r)H(r,nl) \not\equiv 0 \pmod{p},$$

then

$$(U_l|G_r)(z) \equiv (1 - r_p^{r-1} + r_p^{2r-1})(V_l|G_r)(z) \pmod{p}. \tag{3.6}$$

(We shall show that (3.6) does not hold, there is thus an $n \leq \kappa(p,r)l$, $l \nmid n$ and $u_{r,l}(n) \not\equiv 0 \pmod{p}$.)

By a fundamental properties of $H(r,N)$ and by the definitions of $u_{r,l}(N), v_{r,l}(N)$, if $l \equiv r_p \pmod{t_p}$,

$$u_{r,l}(D_0 l^3) \equiv M(r)L(1-r, \chi_{D_0}) \times (*),$$

where the factor (*) is given below: Since $u_{r,l}(D_0 l^3)$ is $D_0 l^4$-th coefficient of $M(r)\mathcal{H}_r(z)$, and $(-1)^r D_0 l^4 = (-1)^r D_0 (l^2)^2$,

$$(*) = \sum_{d|l^2} \mu(d)\chi_{D_0}(d) d^{r-1} \sigma_{2r-1}\left(\frac{l^2}{d}\right).$$

The summation is calculated as follows:

$$\begin{aligned}
&\sum_{d|l^2} \mu(d)\chi_{D_0}(d) d^{r-1}\sigma_{2r-1}\left(\frac{l^2}{d}\right) \\
&= \sum_{d=1,l,l^2} \mu(d)\chi_{D_0}(d) d^{r-1}\sigma_{2r-1}\left(\frac{l^2}{d}\right) \\
&= \sum_{d=1,l} \mu(d)\chi_{D_0}(d) d^{r-1}\sigma_{2r-1}\left(\frac{l^2}{d}\right) \quad \text{(Note that } \mu(l^2) = 0\text{)} \\
&= \sigma_{2r-1}(l^2) - \chi_{D_0}(l) l^{r-1}\sigma_{2r-1}(l) \\
&= 1 + l^{2r-1} + l^{2(2r-1)} - l^{r-1}(1 + l^{2r-1}) \quad \text{(Note that } \chi_{D_0}(l) = 1 \text{ for all } l \equiv r_p \pmod{t_p}\text{)} \\
&= 1 + l^{2r-1} + l^{2(2r-1)} - l^{r-1} - l^{3r-2}.
\end{aligned}$$

On the other hand, it is plain to see

$$v_{r,l}(D_0 l^3) = M(r)H(r, D_0 l^3) = M(r)L(1-r, \chi_{D_0})(1 + l^{2r-1} - \chi_{D_0}(l) l^{r-1})$$
$$= M(r)L(1-r, \chi_{D_0})(1 + l^{2r-1} - l^{r-1}).$$

We see that it holds that

$$u_{r,l}(D_0 l^3) \not\equiv v_{r,l}(D_0 l^3) \pmod{p}$$

if and only if

$$1 + l^{2r-1} + l^{2(2r-1)} - l^{r-1} - l^{3r-2} \not\equiv 1 + l^{2r-1} - l^{r-1} \pmod{p}$$

holds. The last congruence is equivalent to

$$l^{2(2r-1)} - l^{3r-2} = l^{4r-2} - l^{3r-2} = l^{3r-2}(l^r - 1) \not\equiv 0 \pmod{p}.$$

Since $l \equiv r_p \pmod{t_p}$, $p \mid t_p$, $l \equiv r_p \pmod{p}$. We assumed $r_p^r \not\equiv 1 \pmod{p}$, $l^r \not\equiv 1 \pmod{p}$ indeed holds.

We have

$$(U_l | G_r)(z) \not\equiv (V_l | G_r)(z) \pmod{p}.$$

There is an n with $1 \leq n \leq \kappa(p,r)l = ((2r+1)/4)p(p+1)Q^4(Q+1)l$ such that $(n, l) = 1$ and

$$u_{p,l}(n) = M(r)H(r, nl) \not\equiv 0 \pmod{p}.$$

(Note also that $v_{r,l}(n) = 0$ since $(n, l) = 1$).

We can plainly see that there is a positive fundamental discriminant $D_l = d_l lp$ such that $p \nmid M(r)H(r, D_l) = M(r)L(1-r, \chi_{D_l})$. As we see in the proof of lemma 2.2, if $(p-1)/2 \nmid r$ then $p \nmid w_r(D)$, thus $p \nmid \xi_r(D_l)$ holds. □

Corollary 3.2. *Under the same assumptions of the theorem 3.1, namely, let $r \in \mathbb{N}$ be an even natural number, $p > 3$ a prime which satisfies $(p-1)/2 \nmid r$. Suppose there is a fundamental discriminant D_0 prime to p such that $|B(r, \chi_{D_0})|_p = 1$. Then the following inequality holds:*

$$\#\{0 < D < X \mid p \nmid \xi_r(D)\} \gg \frac{\sqrt{X}}{\log X} \quad (X \gg 0).$$

Proof. What we should prove is that $D_l = d_l lp$ is different for each prime $l \equiv r_p \pmod{t_p}$ (notations are the same as in the proof of theorem 3.1). This is the same as the proof of theorem 1 of Ono (*loc. cit.*), so we omit this. □

Corollary 3.3. *For any prime $p \geq 7$, the following inequality holds:*

1.
$$\#\{0 < D < X | p \nmid \xi_2(D)\} \gg \frac{\sqrt{X}}{\log X} \quad (X \gg 0).$$

2.
$$\#\{0 < D < X | p \nmid \xi_4(D)\} \gg \frac{\sqrt{X}}{\log X} \quad (X \gg 0).$$

Proof. (1). Since any prime $p \geq 7$ satisfies $(p-1)/2 \nmid 2$, it is enough to show that there is a fundamental discriminant D_0 coprime to p such that $|B(2, \chi_{D_0})|_p = 1$. We can take $D_0 = 8$ so that $B(2, \chi_8) = 2$.

(2). Since for any prime $p \geq 7$ satisfies $(p-1)/2 \nmid 4$, taking $D_0 = 5$, we see $B(4, \chi_5) = -8$. □

A version of Lichtenbaum conjecture (proved by Wiles [Wil90]) asserts that the special values of Dedekind zeta functions at negative integers are intimately connected with arithmetic of number fields. To state the relation between them, we need some notations.

For a number field K and a prime p, we denote the ring of p-integers in K by $O_K[1/p]$. Let $\mu_{p^m}(n)$ be the n-th Tate twist of the étale sheaf of p^m-th roots of unity on the étale site $(O_K[1/p])_{\text{ét}}$. Let $\mathrm{H}^2_{\text{ét}}(O_K[1/p], \mathbb{Z}_p(n)) = \varprojlim_m \mathrm{H}^2_{\text{ét}}(O_K[1/p], \mu_{p^m}(n))$ be the second étale cohomology group (the projective limit are taken with respect to the map induced by the p-th power map $\mu_{p^{m+1}}(n) \to \mu_{p^m}(n)$). For any rational numbers $a, b \in \mathbb{Q}$ and for a prime p, we denote $a \sim_p b$ when if a/b is a p-adic unit.

Then the relation alluded above is:

$$w_n(K)\zeta_K(1-n) \sim_p \#\mathrm{H}^2_{\text{ét}}(O_K[1/p], \mathbb{Z}_p(n))$$

for any totally real number field K, an even natural number n and for any odd prime p. We thus have, for an even natural number n and a positive fundamental discriminant D,

$$\xi_n(D) \sim_p \#\mathrm{H}^2_{\text{ét}}(O_D[1/p], \mathbb{Z}_p(n)).$$

It is plain to see the following corollary:

Corollary 3.4. *Under the same assumptions of theorem 3.1, the following inequality holds:*

$$\#\{0 < D < X | \mathrm{H}^2_{\text{ét}}(O_D[1/p], \mathbb{Z}_p(n)) = \{0\}\} \gg \frac{\sqrt{X}}{\log X} \quad (X \gg 0).$$

By corollary 3.3, for any prime $p \geq 7$ and for $n = 2, 4$, without any other assumptions, we have

$$\#\{0 < D < X \mid \mathrm{H}^2_{\acute{e}t}(O_D[1/p], \mathbb{Z}_p(n)) = \{0\}\} \gg \frac{\sqrt{X}}{\log X} \quad (X \gg 0).$$

Quillen's conjecture predicts that the following isomorphism between the p-part of the $2m-2$-th algebraic K-group $\mathrm{K}_{2m-2}(O_D)$ of O_D and $\mathrm{H}^2_{\acute{e}t}(O_K[1/p], \mathbb{Z}_p(m))$ holds for all even integers $m \geq 2$ (*cf.* Soulé [Sou79, I.1]):

$$p\text{-torsion part of } \mathrm{K}_{2m-2}(O_K) \cong \mathrm{H}^2_{\acute{e}t}(O_K[1/p], \mathbb{Z}_p(m)). \tag{3.7}$$

We see the corollary which extends Byeon's theorem mentioned in the introduction:

Corollary 3.5. *Under the same assumptions on p and n as in the theorem 3.1, and assuming the isomorphism* (3.7),

$$\#\{0 < D < X \mid p \nmid \#\mathrm{K}_{2n-2}(O_D)\} \gg \frac{\sqrt{X}}{\log X} \quad (X \gg 0)$$

holds.

By corollary 3.3, for any prime $p \geq 7$ and for $m = 2, 6$, only assuming the isomorphism (3.7) *we have*

$$\#\{0 < D < X \mid p \nmid \#\mathrm{K}_m(O_D)\} \gg \frac{\sqrt{X}}{\log X} \quad (X \gg 0).$$

Remark. It may be interesting to consider, for a given odd prime p and natural number n, how often p does not divide $\xi_n(D)$ while discriminant D varies. As we saw in this paper, if n is even, this question is equivalent (up to Quillen's conjecture (3.7)) to the growth of

$$\#\{0 < D < X \mid p \nmid \#K_{2n-2}(O_D)\} \qquad (X \to \infty).$$

Further we can ask whether these quantities have *density* in the set of fundamental discriminants similar to the Cohen-Lenstra heuristics (1.1), (1.2) referred in the introduction.

Let us define a ratio $d(r, p, X)$ by

$$d(r, p, X) := \frac{\#\{0 < D < X \mid p \nmid \xi_r(D)\}}{\#\{0 < D < X\}},$$

for even positive integer r, a prime p and positive number X. Numerical computation shows that

$$d(4, 11, 25210001) = 0.899236, \quad d(4, 13, 25210001) = 0.917296,$$
$$d(4, 17, 25210000) = 0.937675, \quad d(4, 19, 25210001) = 0.94455,$$
$$d(4, 23, 25210001) = 0.954569.$$

These values seems rather near to the $(1-(1/p))$ times right hand side of (1.2), namely the right hand side of (1.1), but the author do not have any reason.

Remark. One can consider indivisibility of (algebraic part of) special values of zeta or L-functions not only for quadratic fields but for quadratic twists of, for example, a Galois representation associated to an elliptic curve or a modular form. For indivisibility of central critical values of modular L-functions, see Chida [Chi04] and papers cited therein.

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References

[Bru99] Jan Hendrik Bruinier, *Nonvanishing modulo l of Fourier coefficients of half-integral weight modular forms*, Duke Math. J. **98** (1999), no. 3, 595–611. MR 2000d:11061

[Bye03] Dongho Byeon, *Indivisibility of special values of Dedekind zeta functions of real quadratic fields*, Acta Arith. **109** (2003), no. 3, 231–235. MR 1 980 259

[Car59] L. Carlitz, *Arithmetic properties of generalized Bernoulli numbers*, J. Reine Angew. Math. **202** (1959), 174–182. MR 22 #20

[Chi04] M. Chida, *Indivisibility of Fourier coefficients of half integral weight modular forms*, preprint (2004).

[CL73] J. Coates and S. Lichtenbaum, *On l-adic zeta functions*, Ann. of Math. (2) **98** (1973), 498–550. MR MR0330107 (48 #8445)

[CL84] H. Cohen and H. W. Lenstra, Jr., *Heuristics on class groups of number fields*, Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983), Springer, Berlin, 1984, pp. 33–62. MR 85j:11144

[Coh75] Henri Cohen, *Sums involving the values at negative integers of L-functions of quadratic characters*, Math. Ann. **217** (1975), no. 3, 271–285. MR 52 #3080

[CS77] J. Coates and W. Sinnott, *Integrality properties of the values of partial zeta functions*, Proc. London Math. Soc. (3) **34** (1977), no. 2, 365–384. MR MR0439815 (55 #12697)

[HS75] B. Harris and G. Segal, *K_i groups of rings of algebraic integers*, Ann. of Math. (2) **101** (1975), 20–33. MR MR0387379 (52 #8222)

[Kim06] I. Kimura, *Divisibility of orders of K_2 groups associated to quadratic fields*, Demonstratio Math. (2006), no. 39, 277–284.

[MW84] B. Mazur and A. Wiles, *Class fields of abelian extensions of* $\mathbf{Q}$, Invent. Math. **76** (1984), no. 2, 179–330. MR 85m:11069

[Ono99] Ken Ono, *Indivisibility of class numbers of real quadratic fields*, Compositio Math. **119** (1999), no. 1, 1–11. MR 1 711 515

[Sou79] C. Soulé, *K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale*, Invent. Math. **55** (1979), no. 3, 251–295. MR MR553999 (81i:12016)

[SS77] J.-P. Serre and H. M. Stark, *Modular forms of weight* 1/2, Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), Springer, Berlin, 1977, pp. 27–67. Lecture Notes in Math., Vol. 627. MR MR0472707 (57 #12400)

[Stu87] Jacob Sturm, *On the congruence of modular forms*, Number theory (New York, 1984–1985), LNM 1240, Springer, Berlin, 1987, pp. 275–280. MR 88h:11031

[Wil90] A. Wiles, *The Iwasawa conjecture for totally real fields*, Ann. of Math. (2) **131** (1990), no. 3, 493–540. MR 91i:11163