

On some conditions of starlikeness

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Abstract

Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be analytic in the unit disk U . Yuan Chun Fang proved that

$$\left| \frac{f''(z)}{f'(z)} \right| < m \implies \operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad (z \in U),$$

where $m(= 2.83 \cdots)$ is the best possible. In this paper, we generalize this theorem.

1. Introduction

Let A_p denote the class of functions of the form

$$(1) \quad f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots \quad (p \in \mathbb{N}),$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$, and M_p denote the class of functions

$$(2) \quad f(z) = z^{-p} + a_{-p+1}z^{-p+1} + a_{-p+2}z^{-p+2} + \cdots \quad (p \in \mathbb{N}),$$

which are analytic in punctured disk $U - \{0\}$. Then we denote two classes of starlike functions as follows:

$$(3) \quad A_p^* = \{f \in A_p : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0, z \in U\},$$

$$(4) \quad M_p^* = \{f \in M_p : \operatorname{Re} \frac{zf'(z)}{f(z)} < 0, z \in U\}.$$

For the class A_1 , Singh and Singh [4] showed the following theorem.

Theorem A Let $f(z) \in A_1$, then

$$(5) \quad \left| \frac{f''(z)}{f'(z)} \right| < \frac{3}{2} \quad (z \in U) \implies f(z) \in A_1^*.$$

He used Jack's Lemma. Mocanu [3,p.338] showed

Theorem B Let $g(z) = \frac{e^{\lambda z} - 1}{\lambda}$.

$$(6) \quad g(z) \in A_1^* \iff |\lambda| \leq m = 2.8329 \dots$$

Where m is the least positive solution of the following equation

$$(7) \quad \cos \sqrt{x^2 - 1} + \sqrt{x^2 - 1} \sin \sqrt{x^2 - 1} - \frac{1}{e} = 0.$$

Miller and Mocanu [2] proved, using their theory of first order differential subordination, that

Theorem C Let $f(z) \in A_1$. Then

$$(8) \quad \left| \frac{f''(z)}{f'(z)} \right| < 2 \implies f(z) \in A_1^*.$$

In the same article [2], they posed the interesting question of finding the maximum value of k for which

$$(9) \quad \left| \frac{f''(z)}{f'(z)} \right| < k \implies f(z) \in A_1^*.$$

From the above two theorems, $2 \leq k \leq m$. Some Mathematicians improved the lower bound of k . And recently, Yuan Chun Fang [4] showed

Theorem D Let $f(z) \in A_1$. Then

$$(10) \quad \left| \frac{f''(z)}{f'(z)} \right| < m \quad (z \in U) \implies f(z) \in A_1^*.$$

The result is sharp, with the extremal function

$$(11) \quad G(z) = \frac{e^{mz} - 1}{m}.$$

The purpose of this paper is to obtain similar theorems for A_p and M_p .

Theorem 1 If $f(z) \in A_p$ satisfies

$$(12) \quad \left| \frac{f''(z)}{f'(z)} - \frac{p-1}{p} \frac{f'(z)}{f(z)} \right| \leq m \quad (z \in U),$$

then f belongs to A_p^* . The result is sharp, with the extremal function

$$(13) \quad G_1(z) = \left(\frac{e^{mz} - 1}{m} \right)^p.$$

Theorem 2 If $f(z) \in M_p$ satisfies

$$(14) \quad \left| \frac{f''(z)}{f'(z)} - \frac{p+1}{p} \frac{f'(z)}{f(z)} \right| \leq m \quad (z \in U),$$

then f belongs to M_p^* . The result is sharp, with the extremal function

$$(15) \quad G_2(z) = \left(\frac{m}{e^{mz} - 1} \right)^p.$$

2. Proof of Theorem 1

We use the following lemma due to Miller and Mocanu [2, Theorem 2].

Lemma let h be convex in U and θ and ϕ be analytic in a domain D . Let p be analytic in U , with $p(0) = h(0) = \theta(p(0))$ and $p(U) \subset D$. If the differential equation

$$(16) \quad \theta(q(z)) + zq'(z)\phi(q(z)) = h(z)$$

has a univalent solution in U that satisfies $q(0) = h(0)$ and

$$(17) \quad \theta(q(z)) \prec h(z),$$

then the relation

$$(18) \quad \theta(p(z)) + zp'(z)\phi(p(z)) \prec h(z)$$

implies $p(z) \prec q(z)$. The function q is the best dominant of (18).

Suppose $f(z) \in A_p^*$ satisfies (12), then we have

$$(19) \quad \frac{zf''(z)}{f'(z)} - \frac{p-1}{p} \frac{zf'(z)}{f(z)} \prec mz.$$

Let put

$$p(z) = \frac{1}{p} \frac{zf'(z)}{f(z)}, \quad q(z) = \frac{1}{p} \frac{zG_1'(z)}{G_1(z)}$$

$$h(z) = 1 + mz, \quad \theta(z) = z, \quad \text{and} \quad \phi(z) = \frac{1}{z}.$$

Then we have

$$(20) \quad q(z) = \frac{zG_1'(z)}{G_1(z)} = m \frac{ze^{mz}}{e^{mz} - 1}.$$

$$(21) \quad q(z) + \frac{zq'(z)}{q(z)} = 1 + mz,$$

and

$$(22) \quad p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} - \frac{p-1}{p} \frac{zf'(z)}{f(z)}.$$

From (19) and (22) we obtain (18), and from (21) we obtain (16). It yields $p(z) \prec q(z)$, and so $\frac{zf'(z)}{f(z)} \prec \frac{zG_1'(z)}{G_1(z)}$. Therefore, from Theorem B we obtain that

$$\frac{zf'(z)}{f(z)} > 0 \quad (z \in U).$$

Concerning the extremal function, we have

$$\frac{zG_1'''(z)}{G_1'(z)} - \frac{p-1}{p} \frac{G_1 f'(z)}{G_1(z)} = m.$$

This implies that $G_1(z)$ is extremal.

Proof of Theorem 2 is similar, so we omit.

References

- [1] Yuan Chun Fang, Univalence of analytic solutions to some first order differential equations, *Acta Mathematica Sinica (Chinese, Chinese summary)*. **35**(4), 483-491.
- [2] S.S. Miller and P.T. Mocanu, On some class of first-order differential subordinations, *Michigan Math.J.* **32**(1985), 185-194.
- [3] P.T. Mocanu, Asupra razei de stelaritate a funcțiilor univalente, *Stud. Cer. Mat.(Cluj)*, **11**(1960), 337-341.
- [4] R.Singh and S. Singh, Starlikeness and convexity of certain integrals, *Ann. Univ. Mariae Curie- Skłodowska.*, **35**(1981), 145-147. *Colloq. Math.* **47**(1982), 309-314.