

Some Results on the CR property of non-E-overlapping and depth-preserving TRS's

Michio Oyamaguchi
Faculty of Engineering
Mie University

(email: mo@info.mie-u.ac.jp)

Hiroshi Gomi
Faculty of Engineering
Mie University and
Oki TechnoSystems Laboratory, Inc.
(email: gomi@info.mie-u.ac.jp)

Abstract

A term rewriting system (TRS) is said to be depth-preserving if for any rewrite rule and any variable appearing in the both sides, the maximal depth of the variable occurrences in left-hand-side is greater than or equal to that of the variable occurrences in the right-hand-side, and to be strongly depth-preserving if it is depth-preserving and for any rewrite rule and any variable appearing in the left-hand-side, all the depths of the variable occurrences in the left-hand-side are the same. This paper shows that there exists non-E-overlapping and depth-preserving TRS's which do not satisfy the Church-Rosser property, but all the non-E-overlapping and strongly depth-preserving TRS's satisfy the Church-Rosser property.

1 Introduction

A term-rewriting system (TRS) is a set of directed equations (called rewrite rules). A TRS is Church-Rosser (CR) if any two interconvertible terms reduce to some common term by applications of the rewrite rules. Church-Rosser is an important property in various applications of TRS's and has received much attention so far [1-5,8-15]. Although the CR property is undecidable for general TRS's, many sufficient conditions for ensuring this property have been obtained [1,3,5,8-15]. For example, for noetherian (i.e. terminating) TRS's, the CR property is decidable and reduces to joinability of the critical pairs [5], and for nonterminating and linear TRS's, some sufficient conditions (e.g., nonoverlapping) have been given [3, 11].

On the other hand, for nonlinear and nonterminating TRS's, only a few results on the CR property have been obtained. Our previous paper [9,10,13] may be pioneer ones which have first given nontrivial conditions for the CR property. In [10], it was shown that if TRS's are non-E-overlapping (stronger than nonoverlapping) and right-ground, then they are CR. Here, a TRS is right-ground if no variables occur in the right-hand-side of a rewrite rule. This result is compared with an example given by G.Huet [3], i.e., a nonoverlapping, right-ground and non-CR TRS with the three rules: $f(x, x) \rightarrow a, f(x, g(x)) \rightarrow b, c \rightarrow g(c)$. Here, f, g, a, b, c are function symbols and x is a variable. The above result was extended in [9,13,14,15] and it was shown that if TRS's are non-E-overlapping and simple-right-linear, then they are CR. Here, a TRS is simple-right-linear if for any rewrite rule, the right-hand-side is linear (i.e., any variable occurs at most once in the term) and no variables occurring more than once in the left-hand-side occur in the right-hand-side. Moreover, it was shown that even if simple-right-linear TRS's are E-overlapping, some additional conditions ensure that they are CR [9,13,15].

However, these results were restricted to those on the CR property of subclasses of right-linear TRS's. On the other hand, if we omit the right-linearity condition, then it has been shown that

only the non-E-overlapping condition is insufficient for ensuring the CR property of TRS's. For example, the following non-E-overlapping TRS R_1 is not CR: $R_1 = \{f(x, x) \rightarrow a, g(x) \rightarrow f(x, g(x)), c \rightarrow g(c)\}$ given by Barendregt and Klop. Here, f, g, a, c are function symbols and x is a variable.

In this paper, we consider the CR property of nonlinear, nonterminating and depth-preserving TRS's. Here, a TRS is depth-preserving if for each rule $\alpha \rightarrow \beta$ and any variable x appearing in both α and β , the maximal depth of the x occurrences in α is greater than or equal to that of the x occurrences in β ([6]). For example, TRS $R_2 = \{f(x, g(x)) \rightarrow h(k(x), x)\}$, where x is a variable, is depth-preserving, since the maximal depths of the x occurrences of the left-hand-side and of the right-hand-side are 2 and 2, respectively.

We first show that only the non-E-overlapping and depth-preserving properties are insufficient for ensuring the CR property. That is, the following TRS R_3 is not CR: $R_3 = \{f(x, x) \rightarrow a, c \rightarrow h(c, g(c)), h(x, g(x)) \rightarrow f(x, h(x, g(c)))\}$ where x is a variable. Note that R_3 is non-E-overlapping and depth-preserving, but R_3 is not CR, since $c \rightarrow h(c, g(c)) \rightarrow^* a$ and $c \rightarrow^* h(a, g(a))$, but a and $h(a, g(a))$ are not joinable. Note that R_3 is also non-duplicating, since for each rule the number of x occurrences of the left-hand side $\geq$ that of the right-hand side. Thus, non-E-overlapping, non-duplicating and depth-preserving conditions do not necessarily ensure CR.

Next, we introduce the notion of strongly depth-preserving property (stronger than the depth-preserving one). A TRS R is strongly depth-preserving if R is depth-preserving and for each $\alpha \rightarrow \beta$ and for any variable x appearing in α , all the depths of the x occurrences in α are the same. For example, TRS $R_4 = \{h(g(x), g(x)) \rightarrow f(x, h(x, g(c)))\}$ is strongly depth-preserving, since R_4 is depth-preserving and all the depths of x occurrences of the left-hand side are 2.

In this paper, we prove that non-E-overlapping and strongly depth-preserving TRS's are CR. For example, the following three TRS's R'_1 , R'_3 and R_5 are ensured to be CR:

$$\begin{aligned} R'_1 &= \{f(x, x) \rightarrow a, c \rightarrow g(c), g(x) \rightarrow f(x, x)\} \\ R'_3 &= \{f(x, x) \rightarrow a, c \rightarrow h(c, g(c)), h(g(x), g(x)) \rightarrow f(x, h(x, g(c)))\} \\ R_5 &= \{f(x, x) \rightarrow h(x, x, x)\} \end{aligned}$$

This paper is organized as follows. Section 2 is devoted to definitions. In Section 3, we explain how to prove the above main theorem. In Section 4, we make concluding remarks about the strongly depth-preserving property.

2 Definitions

The following definitions and notations are similar to those in [3, 10]. Let X be a set of variables, F be a finite set of operation symbols and T be the set of terms constructed from X and F .

Definitions of $\langle O(M), M/u, M[u \leftarrow N], V(M), O_x(M) \rangle$

For a term M , we use $O(M)$ to denote the set of occurrences (positions) of M , and M/u to denote the subterm of M at occurrence u , and $M[u \leftarrow N]$ to denote the term obtained from M by replacing the subterm M/u by term N , $V(M)$ to denote the set of variables in M , $O_x(M)$ to denote the set of occurrences of variable $x \in V(M)$.

Definitions of $\langle \bar{O}(M) \rangle$

$\bar{O}(M)$ is the set of non-variable occurrences, i.e.,
 $\bar{O}(M) = O(M) - \cup_{x \in V(M)} O_x(M)$

Definition of $\langle h(M) \text{ — height of } M \rangle$

For a term M , $h(M) = \text{Max}\{|u| \mid u \in O(M)\}$. $h(M)$ is called "height of M ".

Example.

$$h(f(g(x))) = 2, h(a) = 0, h(g(x)) = 1.$$

Definition of $\langle \text{TRS} \rangle$

A term-rewriting system (TRS) is a set of directed equations (called rewrite rules).

Definition of $\langle \text{depth-preserving TRS } R \rangle$

TRS R is depth-preserving

if $\forall \alpha \rightarrow \beta \in R \forall x \in V(\alpha) \quad \text{Max}\{|v| \mid v \in O_x(\beta)\} \leq \text{Max}\{|u| \mid u \in O_x(\alpha)\}$

Note

TRS R is depth-preserving if and only if R is locally increasing, i.e., $\exists l \geq 0$ such that $\forall \alpha \rightarrow \beta \in R \forall \sigma : X \rightarrow T$, if $h(\sigma(\alpha)) < h(\sigma(\beta))$ then $h(\sigma(\alpha)) \leq l$

Definition of $\langle \text{strongly depth-preserving TRS } R \rangle$

TRS R is strongly depth-preserving

if R is depth-preserving and satisfies that $\forall \alpha \rightarrow \beta \in R \forall x \in V(\alpha) \forall u, v \in O_x(\alpha) |u| = |v|$ hold.

Definition of $\langle \text{parallel-one-step } \leftrightarrow \rangle$

$$\begin{aligned} M \leftrightarrow N \quad \text{iff} \quad & \exists U \subseteq O(M) \text{ s.t.} \\ & \forall u, v \in U \quad u \neq v \Rightarrow u|v \text{ (disjoint)} \\ & \forall u \in U \quad M/u \leftrightarrow N/u \\ & N = M[u \leftarrow N/u, u \in U] \end{aligned}$$

where $M/u \leftrightarrow N/u$ is one step reduction between $\{M/u, N/u\} = \{\sigma(\alpha), \sigma(\beta)\}$ for some $\alpha \rightarrow \beta \in R$ and $\sigma : X \rightarrow T$.

In this case, let $R(M \leftrightarrow N) = U$.

(Note. $U = \phi$ is allowed.)

Example.

Let $R = \{a \rightarrow c\}$, then $f(c, g(a)) \leftrightarrow f(a, g(c))$.

We assume that $\gamma : M_0 \leftrightarrow M_1 \leftrightarrow \dots \leftrightarrow M_n$ in the following definitions.

Definition of $\langle R(\gamma), MR(\gamma), u\text{-invariant} \rangle$

$$R(\gamma) = \{u_i \mid u_i \in R(M_i \leftrightarrow M_{i+1}) (0 \leq i \leq n)\}$$

$MR(\gamma)$ is the set of minimal occurrences in $R(\gamma)$.

For $u \in O(M_0)$, if there exists no $v \in R(\gamma)$ such that $v \leq u$, then γ is said to be u -invariant.

Definition of $\langle \text{composition, cut of reduction sequence} \rangle$

Let $\delta : N_0 \leftrightarrow N_1 \leftrightarrow \dots \leftrightarrow N_k$. If $M_n = N_0$, then the composition of γ and δ , i.e., $M_0 \leftrightarrow M_1 \leftrightarrow \dots \leftrightarrow M_n (= N_0) \leftrightarrow N_1 \leftrightarrow \dots \leftrightarrow N_k$ is denoted by $(\gamma; \delta)$.

Let γ be u -invariant, then the cut sequence of γ at u is

$$\gamma/u = (M_0/u \leftrightarrow M_1/u \leftrightarrow \dots \leftrightarrow M_n/u).$$

Definition of $\langle H(\gamma) \rangle$ — the height of reduction sequence γ

$$H(\gamma) = \text{Max}\{h(M_i) \mid 0 \leq i \leq n\}$$

Example.

Let $\gamma : f(c) \leftrightarrow f(g(c)) \leftrightarrow a$, then $H(\gamma) = h(f(g(c))) = 2$.

Definition of $\langle |\gamma|_p \rangle$ — the number of parallel reduction steps of γ

$$|\gamma|_p = n$$

Note.

If $\delta : M \leftrightarrow M$, then $|\delta|_p = 1$.

Example.

Let $\gamma : f(c) \leftrightarrow f(g(c)) \leftrightarrow a$, then $|\gamma|_p = 2$.

Definition of $\langle \text{net}(\gamma) \rangle$

$\text{net}(\gamma)$ is the sequence obtained from γ by removing all $M_i \leftrightarrow M_{i+1}$ satisfying $M_i = M_{i+1}$, $0 \leq i < n$.

Example.

Let $\gamma : f(c) \leftrightarrow f(g(c)) \leftrightarrow a \leftrightarrow a$, then $\text{net}(\gamma) : f(c) \leftrightarrow f(g(c)) \leftrightarrow a$.

Definition of $\langle |\gamma|_{np} \rangle$

$$|\gamma|_{np} = |\text{net}(\gamma)|_p$$

Definitions of $\langle \text{left}(\gamma, h), \text{right}(\gamma, h), \text{width}(\gamma, h), \text{ldis}(\gamma, h), \text{rdis}(\gamma, h) \rangle$

$\text{left}(\gamma, h)$	$=$	$\text{Min}\{i \mid h(M_i) = h\}$	if $\exists i \ (0 \leq i \leq n)$ s.t. $h(M_i) = h$ and $\forall j (0 \leq j < i) \ h(M_j) < h$
	$=$	$\perp$	otherwise
$\text{right}(\gamma, h)$	$=$	$\text{Max}\{i \mid h(M_i) = h\}$	if $\exists i \ (0 \leq i \leq n)$ s.t. $h(M_i) = h$ and $\forall j \ (i < j \leq n) \ h(M_j) < h$
	$=$	$\perp$	otherwise
$\text{left}(\gamma, h) \downarrow$	$\stackrel{\text{def}}{=}$	$\text{left}(\gamma, h) \neq \perp$	
$\text{right}(\gamma, h) \downarrow$	$\stackrel{\text{def}}{=}$	$\text{right}(\gamma, h) \neq \perp$	
$\text{left}(\gamma, h) \uparrow$	$\stackrel{\text{def}}{=}$	$\text{left}(\gamma, h) = \perp$	
$\text{right}(\gamma, h) \uparrow$	$\stackrel{\text{def}}{=}$	$\text{right}(\gamma, h) = \perp$	
$\text{width}(\gamma, h)$	$=$	$\text{right}(\gamma, h) - \text{left}(\gamma, h)$	if $\text{left}(\gamma, h) \downarrow \wedge \text{right}(\gamma, h) \downarrow$
	$=$	$\text{right}(\gamma, h) - \text{left}(\gamma, h')$	if $\text{left}(\gamma, h) \uparrow \wedge \text{right}(\gamma, h) \downarrow$ $h' = \text{Min}\{h' \mid h' > h \wedge \text{left}(\gamma, h') \downarrow\}$
	$=$	$\text{right}(\gamma, h') - \text{left}(\gamma, h)$	if $\text{left}(\gamma, h) \downarrow \wedge \text{right}(\gamma, h) \uparrow$ $h' = \text{Min}\{h' \mid h' > h \wedge \text{right}(\gamma, h') \downarrow\}$
	$=$	$\perp$	otherwise

$$\begin{aligned}
l\text{dis}(\gamma, h) &= n - \text{left}(\gamma, h) && \text{if } \text{left}(\gamma, h) \downarrow \\
&= \perp && \text{otherwise} \\
r\text{dis}(\gamma, h) &= \text{right}(\gamma, h) && \text{if } \text{right}(\gamma, h) \downarrow \\
&= \perp && \text{otherwise} \\
l\text{dis}(\gamma, h) \downarrow &\stackrel{\text{def}}{=} l\text{dis}(\gamma, h) \neq \perp \\
r\text{dis}(\gamma, h) \downarrow &\stackrel{\text{def}}{=} r\text{dis}(\gamma, h) \neq \perp \\
l\text{dis}(\gamma, h) \uparrow &\stackrel{\text{def}}{=} l\text{dis}(\gamma, h) = \perp \\
r\text{dis}(\gamma, h) \uparrow &\stackrel{\text{def}}{=} r\text{dis}(\gamma, h) = \perp
\end{aligned}$$

In fig.1, we illustrate *width*, *ldis* and *rdis* with examples.

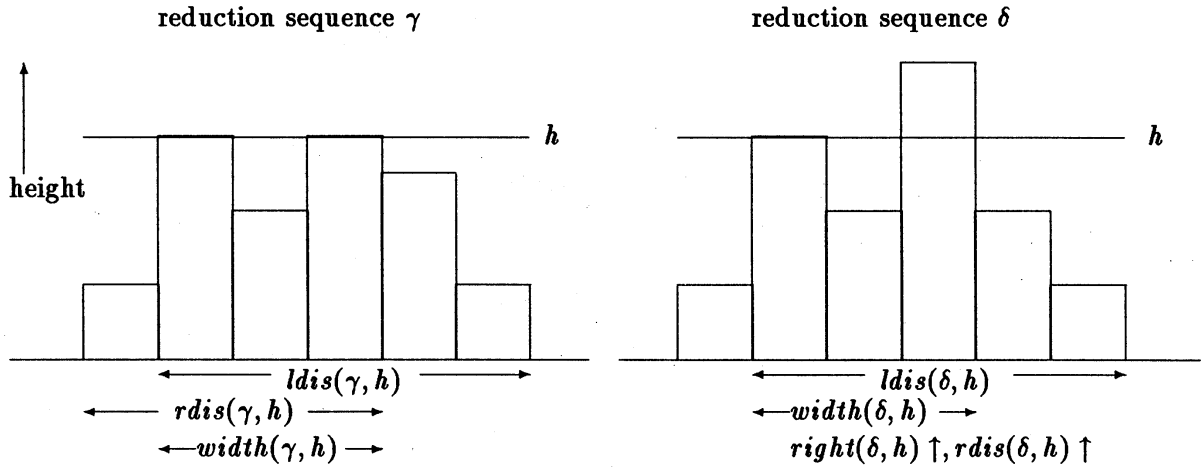


Fig.1 Definitions of *ldis*, *rdis*, *width*.

Example.

Let $\gamma : f(c) \leftrightarrow f(g(g(c))) \leftrightarrow f(g(c)) \leftrightarrow f(f(g(g(c)))) \leftrightarrow f(f(c)) \leftrightarrow g(c)$. Then
 $\text{left}(\gamma, 1) = 0$, $\text{left}(\gamma, 2) \uparrow$, $\text{ldis}(\gamma, 1) = 5$, $\text{ldis}(\gamma, 2) \uparrow$,
 $\text{right}(\gamma, 1) = 5$, $\text{right}(\gamma, 3) \uparrow$, $\text{right}(\gamma, 0) \uparrow$, $\text{rdis}(\gamma, 1) = 5$, $\text{rdis}(\gamma, 3) \uparrow$,
 $\text{width}(\gamma, 1) = \text{right}(\gamma, 1) - \text{left}(\gamma, 1) = 5$, $\text{width}(\gamma, 2) = 3$, $\text{width}(\gamma, 3) = 2$, $\text{width}(\gamma, 4) = 0$

Definition of $\langle K(\gamma), W(\gamma) \rangle$

$$\begin{aligned}
K(\gamma) &= \{(h, \text{ldis}(\gamma, h)) \mid \text{ldis}(\gamma, h) \downarrow\} \\
W(\gamma) &= \{(h, \text{width}(\gamma, h)) \mid \text{width}(\gamma, h) \downarrow\}
\end{aligned}$$

Notation

We denote by $\gamma[\delta'/\delta]$ the sequence obtained from reduction sequence γ by replacing the subsequence or cut sequence δ of γ by sequence δ' .

3 Assertions

In this section, we explain how to prove that non-E-overlapping and strongly depth-preserving TRS R is CR. For this purpose, we need the following five assertions $S(k)$, $S'(k)$, $P(k)$, $Q(k)$, $Q'(k)$ for $k \geq 0$.

Assertion $S(k)$

Let $\gamma : M_0 \leftarrow^+ M_1 \leftarrow^+ \dots \leftarrow^+ M_k$ where $|\gamma|_p = k$, $M_0 = \sigma(\beta)$, $M_1 = \sigma(\alpha)$, $M_{k-1} = \sigma'(\alpha)$, $M_k = \sigma'(\beta)$ for some rule $\alpha \rightarrow \beta \in R$ and mappings σ, σ' and $\bar{\gamma} : M_1 \leftarrow^+ {}^*M_{k-1}$ is ε -invariant.

Then $\exists \delta : \sigma(\beta) \leftarrow^+ {}^*\sigma'(\beta)$ such that

- (i) $|\delta|_p \leq k - 2$
- (ii) If β is a variable, then $H(\delta) < H(\gamma)$.
Otherwise, δ is ε -invariant and $H(\delta) \leq H(\gamma)$.
- (iii) $\forall h \geq 0$ if $ldis(\delta, h) \downarrow$, then
 $\exists h' \geq h$ such that $ldis(\gamma, h') \downarrow$ and $ldis(\delta, h) < ldis(\gamma, h')$.

Assertion $S'(k)$

Let $\gamma : M_0 \leftarrow^+ M_1 \leftarrow^+ \dots \leftarrow^+ M_k$
where $|\gamma|_p = k$, $M_0 = \sigma(\beta)$, $M_1 = \sigma(\alpha)$, $M_{k-1} = \sigma'(\alpha)$, $M_k = \sigma'(\beta)$ for some rule $\alpha \rightarrow \beta \in R$ and mappings σ, σ' and $\bar{\gamma} : M_1 (= \sigma(\alpha)) \leftarrow^+ {}^*M_{k-1} (= \sigma'(\alpha))$ is ε -invariant.

Then $\exists \delta : \sigma(\beta) \leftarrow^+ {}^*\sigma'(\beta)$ such that

- (i) $|\delta|_p = |\gamma|_p$, $|\delta|_{np} \leq |\gamma|_{np} - 2$
- (ii) If β is a variable, then $H(\delta) < H(\gamma)$.
Otherwise, δ is ε -invariant and $H(\delta) \leq H(\gamma)$.
- (iii) $\forall h \geq 0$ if $left(\delta, h) \downarrow$, then
 $\exists h' \geq h$ such that $left(\gamma, h') \downarrow$ and $left(\gamma, h') \leq left(\delta, h)$.
If $right(\delta, h) \downarrow$, then
 $\exists h' \geq h$ such that $right(\gamma, h') \downarrow$ and $right(\delta, h) \leq right(\gamma, h')$.

Assertion $P(k)$

Let $\gamma : \sigma(\beta) \leftarrow \sigma(\alpha) \leftarrow^+ {}^*M$ for some rule $\alpha \rightarrow \beta \in R$ and mapping σ where $H(\gamma) = k$ and $\bar{\gamma} : \sigma(\alpha) \leftarrow^+ {}^*M$ is ε -invariant.

Then, if β is not a variable, then

$\exists \delta : \sigma(\beta) \leftarrow^+ {}^*N \leftarrow^+ {}^*M$ for some N such that

$H(\delta) \leq k$, $M \rightarrow^* N$ and $\delta' : \sigma(\beta) \leftarrow^+ {}^*N$ is ε -invariant.

If β is a variable, then $\exists \delta : \sigma(\beta) \leftarrow^+ {}^*N \leftarrow^+ {}^*M$ for some N such that

$H(\delta) \leq k$, $M \rightarrow^* N$ and $H(\delta') < k$ for $\delta' : \sigma(\beta) \leftarrow^+ {}^*N$

Assertion $Q(k)$

Let $\gamma : M \leftarrow^+ {}^*N$ where $H(\gamma) \leq k$.

Then, $\exists \delta : M \leftarrow^+ {}^*L \leftarrow^+ {}^*N$ such that $H(\delta) \leq k$, $M \rightarrow^* L$ and $N \rightarrow^* L$.

Assertion $Q'(k)$

Let $\gamma_i : M \multimap^* M_i$, where $H(\gamma_i) \leq k$, $1 \leq i \leq n$.

Then, $\exists \delta : M \multimap^* N$ such that $H(\delta) \leq k$ and $\forall i$ ($1 \leq i \leq n$) $M_i \rightarrow^* N$.

The assertions $S(k)$ and $S'(k)$ are similar to the Elimination lemma in [7]. For any reduction sequence $\gamma : \sigma(\beta) \leftarrow \sigma(\alpha) \multimap^* \sigma'(\alpha) \rightarrow \sigma'(\beta)$ for some rule $\alpha \rightarrow \beta$ and mappings σ, σ' where $\bar{\gamma} : \sigma(\alpha) \multimap^* \sigma'(\alpha)$ is ε -invariant, $S(k)$ ensures that there exists $\delta : \sigma(\beta) \multimap^* \sigma'(\beta)$ such that $|\delta|_p \leq |\gamma|_p - 2$, $H(\delta) \leq H(\gamma)$ (where δ is ε -invariant or $H(\delta) < H(\gamma)$) and $K(\delta) \ll K(\gamma)$. Here, $\ll$ is the multiset ordering of a lexicographic ordering $<$. And $S'(k)$ ensures that there exists $\delta' : \sigma(\beta) \multimap^* \sigma'(\beta)$ such that $|\delta|_p = |\gamma|_p$, $|\delta|_{np} \leq |\gamma|_{np} - 2$, $H(\delta) \leq H(\gamma)$ (where δ is ε -invariant or $H(\delta) < H(\gamma)$) and $W(\delta) \leq W(\gamma)$. Here, $\leq$ is $\ll$ or $=$.

To prove these assertions, we use the following properties for *left*, *right*, *width*.

Property 1

Let $\gamma : M_0 \multimap M_1 \multimap \dots \multimap M_k$,
 $\delta : N_0 \multimap N_1 \multimap \dots \multimap N_k$.

1. Assume that for $h > 0$, $left(\delta, h) \downarrow$ and there exists j such that $j \leq left(\delta, h)$ and $h(M_j) \geq h$.
 Then, there exists $h' \geq h$ such that $left(\gamma, h') \downarrow$ and $left(\gamma, h') \leq left(\delta, h)$.
2. Assume that for $h > 0$, $right(\delta, h) \downarrow$ and there exists j such that $right(\delta, h) \leq j$ and $h(M_j) \geq h$.
 Then, there exists $h' \geq h$ such that $right(\gamma, h') \downarrow$ and $right(\gamma, h') \geq right(\delta, h)$.

Property 2

If $H(\gamma) > H(\delta)$, then $K(\gamma) \gg K(\delta)$ and $W(\gamma) \gg W(\delta)$.

Here, $\gg$ is the multiset ordering of a lexicographic ordering $>$.

These proofs are obvious by the definitions of left, right and width, etc.

We first prove $S(k)$ and $S'(k)$ by induction on $k \geq 0$, where k is the number of parallel reduction steps of γ . In the case of $k > 2$, we prove $S(k)$ and $S'(k)$ by induction on $weight(\gamma)$ which is defined as follows:

$$weight(\gamma) = \sum_{\gamma_i \in \Gamma} |\gamma_i|_{np}$$

where $\Gamma = \{\gamma_i \mid \gamma_i = \bar{\gamma}/u_i \text{ for some } u_i \in MR(\bar{\gamma}) \cap \bar{O}(\alpha)\}$,
 $\bar{\gamma} : \sigma(\alpha) \multimap^* \sigma'(\alpha)$.

1. Basis, i.e., the case of $weight(\gamma) = 0$
 The proof is straightforward.
2. Induction step, i.e., the case of $weight(\gamma) > 0$
 Let $\gamma_1 = \bar{\gamma}/u_1 : L_1 \multimap L_2 \dots \multimap L_{k-1}$ where $\gamma_1 \in \Gamma$ and $L_i = M_i/u_1$, $1 \leq i \leq k-1$.
 Then, there exist i, j such that $1 \leq i < j < k-1$ and
 $\delta_1 : L_i \multimap L_{i+1} \dots \multimap L_j \multimap L_{j+1}$
 where $L_i = \theta(\beta')$, $L_{i+1} = \theta(\alpha')$, $L_j = \theta'(\alpha')$, $L_{j+1} = \theta'(\beta')$ for some rule $\alpha' \rightarrow \beta'$ and mappings θ, θ' .

By the induction hypothesis $S(k')$, where $k' = |\delta_1|_p$, there exists $\eta_1 : L_i \leftrightarrow^* L_{j+1}$ satisfying the conditions (i), (ii) and (iii). Let $\eta'_1 = ((L_i \leftrightarrow L_i \cdots \leftrightarrow L_i); \eta_1)$ where $|\eta'_1|_p = |\delta_1|_p$.

Let $\gamma' = \gamma[\eta'_1/\delta_1]$. Then, obviously $\text{weight}(\gamma) > \text{weight}(\gamma')$ holds. Hence, by the induction hypothesis that $S(k)$ holds for γ' , it follows that $S(k)$ holds for γ .

The proof of $S'(k)$ is similar to that of $S(k)$.

We then prove that $Q(k) \Rightarrow Q'(k)$ for all $k \geq 0$. Using these results, we can prove $P(k) \wedge Q(k)$ by induction on $k \geq 0$.

Outline of the proof of $P(k) \wedge Q(k)$.

We first prove $P(k)$. Basis: $k = 0$. The proof is obvious.

Induction step: Let $\gamma : M_0 \leftrightarrow M_1 \leftrightarrow M_2 \cdots \leftrightarrow M_n$ where $H(\gamma) = k$, $M_0 = \sigma(\beta)$, $M_1 = \sigma(\alpha)$ and $M_n = M$. Let $\bar{\gamma} : M_1 \leftrightarrow M_2 \cdots \leftrightarrow M_n$. We prove $P(k)$ by induction on the following $\text{weight}(\gamma)$.

$$\text{weight}(\gamma) = \bigsqcup_{\gamma_i \in \Gamma} K(\text{net}(\gamma_i^R))$$

where $\Gamma = \{\gamma_i \mid \gamma_i = \bar{\gamma}/u_i \text{ for some } u_i \in MR(\bar{\gamma}) \cap \bar{O}(\alpha)\}$.

Here, γ_i^R is the reverse sequence of γ_i .

Note that if $\Gamma = \phi$, then $\text{weight}(\gamma) = \phi$.

1. Basis: the case of $\text{weight}(\gamma) = \phi$, i.e., all the reductions of γ occur in the variable parts of $\sigma(\alpha)$.

We can prove $P(k)$ by using the induction hypothesis $Q(k-1)$ and the strongly depth-preserving property.

2. Induction step: the case of $\text{weight}(\gamma) \gg \phi$ i.e., some reduction occurs in the non variable part.

By the definition of γ_1^R , then there exists an ε -reduction.

Let $\delta = \text{net}(\gamma_1^R) : (L_0 \leftrightarrow L_1 \cdots \leftrightarrow L_m)$ where $m \leq n$, $L_0 = M_n/u_1$, $L_m = M_1/u_1$.

There are two cases depending on whether there exists

$$\xi : L_i (= \sigma'(\beta')) \xleftarrow{\varepsilon} L_{i+1} (= \sigma'(\alpha')) \leftrightarrow^* L_j (= \sigma''(\alpha')) \xrightarrow{\varepsilon} L_{j+1} (= \sigma''(\beta'))$$

for some i, j ($1 \leq i < j < m$), where $L_{i+1} \leftrightarrow^* L_j$ is ε -invariant.

- (a) The case in which δ includes such ξ .

By $S(|\xi|_p)$, there exists $\xi' : L_i \leftrightarrow^* L_{j+1}$ satisfying the conditions (i), (ii), (iii).

Let $\delta' = \delta[\xi'/\xi]$ and $\gamma' = \gamma[\gamma'_1/\gamma_1]$ where $\text{net}(\gamma_1^R) = \delta'$ and $\text{net}(\gamma_1^R) = \delta$.

By $\text{weight}(\gamma) \gg \text{weight}(\gamma')$, the induction hypothesis for γ' ensures that $P(k)$ holds for γ .

- (b) The case in which δ does not include such ξ .

In this case, δ includes ε -reductions, but the direction of the ε -reductions is left-to-right by the non-E-overlapping property.

Using a finite number of the induction hypothesis $P(k')$, $k' < k$, we can prove that there exists $\eta : L_0 \leftrightarrow^* N \leftrightarrow^* L_i$ for some term N and i ($0 < i \leq m$) such that $H(\eta) \leq H(\delta)$, $L_0 \rightarrow^* N$ and either $i = m$ and $\eta' : N \leftrightarrow^* L_i$ is ε -invariant or $H(\eta') < H(\delta_i)$ holds where $\eta' : N \leftrightarrow^* L_i$ and $\delta_i : L_0 \leftrightarrow L_1 \cdots \leftrightarrow L_i$.

Let $\bar{\delta} = \delta[\eta'/\delta_i]$. Then, $\bar{\delta}$ is ε -invariant or $K(\delta) \gg K(\bar{\delta})$ holds. Let $\gamma' = \gamma[\gamma'_1/\gamma_1]$ where $\bar{\delta} = \text{net}(\gamma_1'^R)$ and $\delta = \text{net}(\gamma_1^R)$. Then, $\text{weight}(\gamma) \gg \text{weight}(\gamma')$ holds, so that the induction hypothesis $P(k)$ for γ' ensures that $P(k)$ holds for γ .

Next, we prove $Q(k)$ by induction on $(H(\gamma), W(\gamma), \varepsilon(\gamma))$, where $\varepsilon(\gamma)$ is the number of ε -reductions in γ and $W(\gamma) = \{(h, \text{width}(\gamma, h)) \mid \text{width}(\gamma, h) \downarrow\}$.

If $H(\gamma) \leq k - 1$ or γ has no ε -reductions, then the proof can be reduced to that of $Q(k - 1)$. So, let $H(\gamma) = k$ and γ has ε -reductions.

There are two cases depending on whether there exists a subsequence $\gamma_1 : N_1 \xleftarrow{\varepsilon} N_2 \xleftrightarrow{*} N_3 \xrightarrow{\varepsilon} N_4$ of γ for some $N_i, 1 \leq i \leq 4$, where $N_2 \xleftrightarrow{*} N_3$ is ε -invariant.

1. The case in which γ includes such γ_1 .

In this case, we apply $S(|\gamma_1|_p)$ or $S'(|\gamma_1|_p)$ and obtain $\delta_1 : N_1 \xleftrightarrow{*} N_4$ satisfying the conditions (i), (ii) and (iii).

Let $\gamma' = \gamma[\delta_1/\gamma_1]$. Then, either $W(\gamma) \gg W(\gamma')$ or $W(\gamma) = W(\gamma')$ and δ_1 is ε -invariant. In either case, the induction hypothesis for γ' ensures that $Q(k)$ holds for γ .

2. The case in which γ does not include such γ_1 .

We can prove this case by using $P(k)$ and $Q(k - 1)$. But, the details are omitted.

Since $Q(k), k > 0$, ensures that TRS R is CR, we have the following our main theorem.

Main Theorem

A TRS R is CR if R is non-E-overlapping and strongly depth-preserving.

Matsuura et al.[6] showed that if a TRS R is non- ω -overlapping and depth-preserving, then R is non-E-overlapping, so that we have the following corollary.

Corollary

A TRS R is CR if R is non- ω -overlapping and strongly depth-preserving.

Note

Whether R is non- ω -overlapping or not can be checked efficiently.

4 Concluding Remarks

In this paper, we have shown that there exists a non-E-overlapping and depth-preserving TRS which is not CR, but all the non-E-overlapping and strongly depth-preserving TRS's satisfy the CR property.

Finally, we make a comment on the strongly depth-preserving property. This property is defined by the depth-preserving property and the condition that for each rule $\alpha \rightarrow \beta$ and for any $x \in V(\alpha)$, all the depths of the x occurrences in α are the same. By replacing the restriction on α by that on β , we can define an analogous property. That is, this new property is defined by the depth-preserving property and the condition that for each rule $\alpha \rightarrow \beta$ and for any $x \in V(\beta)$, all the depths of the x occurrences in β are the same. However, this new property and non-E-overlapping do not necessarily ensure CR. For example, TRS $R_6 = \{f(g(x), x) \rightarrow a, c \rightarrow h(c, g(c)), h(x, g(x)) \rightarrow f(g(x), h(x, g(c)))\}$ is non-E-overlapping and satisfies this new condition, but R_6 is not CR.

It will be a next step following the work of this paper to study the CR property of E-overlapping and strongly depth-preserving TRS, that is, to find restriction conditions that E-critical pairs must satisfy for ensuring the CR property of strongly depth-preserving TRS's.

References

- [1] M.Dauchet, T.Heuillard, P.Lescanne and S.Tison, Decidability of the Confluence of Finite Ground Term Rewrite Systems and of Other Related Term Rewrite Systems, *Inform. Comput.* 88. pp.187-201(1990).
- [2] N.Dershowitz and J.-P.Jouannaud, Rewrite Systems, *Handbook of Theoretical Computer Science*, Vol.B,ed. J.van Leeuwen, pp.243-320, North-Holland, Amsterdam(1990).
- [3] G.Huet, Confluent reductions:abstract properties and applications to term rewrite systems, *J.ACM*,27, pp.797-821(1980).
- [4] G.Huet and C.Oppen, Equations and Rewrite Rules:A Survey, *Formal Language Theory: Perspectives and Open Problems*,ed. R.V.Book, pp.349-393, Academic Press, New York(1980).
- [5] D.Knuth and P.Bendix, Simple word problems in universal algebra, *Computational Problems in Abstract Algebra*,ed. J.Leech, pp.263-297, Pergamon Press, Elmsford(1970).
- [6] K.Matsuura, M.Oyamaguchi and Y.Ohta, On the E-overlapping property of nonlinear term rewriting systems, *LA symposium(Winter)*(1994).
- [7] M.Ogawa and S.Ono, On the uniquely converging property of nonlinear term rewriting systems, *Technical Report of IEICE, COMP 89-7*, pp.61-70(1989).
- [8] M.Oyamaguchi, The Church-Rosser property for ground term rewriting systems is decidable, *Theoret. Comput. Sci.*49, pp.43-79(1987).
- [9] M.Oyamaguchi, On the Church-Rosser property of nonlinear and non-terminating term rewriting systems, *LA Symposium(Summer)*(1992) and *Hatoyama Workshop on Rewriting*(1992).
- [10] M.Oyamaguchi and Y.Ohta, On the Confluent Property of Right-Ground Term Rewriting Systems, *Trans,IEICE Japan*, J76-D-I, pp.39-45(1993).
- [11] B.K.Rosen, Tree-manipulating systems and Church-Rosser theorems, *J,ACM*, 20, pp.160-187(1973).
- [12] Y.Toyama, On the Church-Rosser property for the direct sum of term rewriting systems, *J.ACM* 34, pp.128-143(1987).
- [13] Y.Toyama and M.Oyamaguchi, Church-Rosser Property and Unique Normal Form Property of Non-Duplicating Term Rewriting Systems, *LA Symposium(Winter)* (1993) and *4th International Workshop on CTRS*(1994).
- [14] Y.Ohta, M.Oyamaguchi and Y.Toyama, On the Church-Rosser property of simple-right-linear TRS's, *Trans. IEICE Japan*, J78-D-I, pp.263-268(1995).
- [15] M.Oyamaguchi and Y.Toyama, On the Church-Rosser property of E-overlapping and simple-right-linear TRS's, *Technical Report of IEICE, COMP 94-29*, pp.47-56(1994).