

A conjugacy class of regular operators

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1- Let $X = \mathbb{C}_{(t,x)}^{1+n}$ with coordinates $t \in \mathbb{C}$, $x = (x_1, \dots, x_n) \in \mathbb{C}^n$, $(t, x; \tau, \xi)$ the symplectic coordinate of T^*X and $\mathcal{E}_X$ the sheaf of microdifferential operators on T^*X . In Kashiwara and Oshima's study of regular systems (cf. [K-O]), the following definition occurs (with a slightly different vocabulary): a matrix of microdifferential operators $A(x, D_x)$ is *essentially of nonpositive order* if there exists $\nu > 0$ such that the coefficients of *any* power of A are microdifferential operators of order at most ν . It is shown in [K-O] that any regular system of microdifferential equations $\mathcal{M}$ with regular singularities along $V = \{t = \xi_1 = \dots = \xi_r = 0\}$, is a quotient of a system of the form $(tD_t - A(x, D_x))u = D_{x_1} = \dots = D_{x_r} = 0$, with A essentially of nonpositive order, which enables one to define the monodromy of $\mathcal{M}$. Actually, it is shown that A may be chosen such that it has the following slightly more precise form:

$$(1) \quad \begin{cases} [A, D_t] = [A, t] = 0, & \text{and } A \text{ consists of blocks } A_{ij} \text{ of } N_i \times N_j \\ \text{matrices of differential operators such that } A_{ij} = 0 \text{ for } i > j \\ \text{and } A_{ii} = A_{ii}(x) \text{ is a matrix of holomorphic functions.} \end{cases}$$

When investigating e.g. distribution solutions of regular systems from a microlocal point of view, that is, solutions with values in the sheaf $\mathcal{C}_{\mathbb{R}^n}^f$ of tempered microfunctions on $\mathbb{R}^n$, one is led to look for the simplest normal form over some extension of $\mathcal{E}_X$ to a ring of operators acting on $\mathcal{C}_{\mathbb{R}^n}^f$ (see remark 2) below).

In particular one has on $\mathcal{C}_{\mathbb{R}^n}^f$ an action of $\mathcal{E}_X^{\mathbb{R},f}$, the ring of tempered microlocal operators (cf. [An]) and the purpose of this note is to prove the

Theorem 1. *Let $A = A(x, D_x)$ be a matrix of differential operators such that (1) holds. Then*

- (i) $(D_t I_N)^{A(x, D_x)}$ is a well defined invertible matrix operator over $\mathcal{E}_{X, (0; dt)}^{\mathbb{R}, f}$, with inverse $(D_t I_N)^{-A(x, D_x)}$, and
- (ii) one has $(D_t I_N)^{A(x, D_x)}(tD_t I_N - A(x, D_x))(D_t I_N)^{-A(x, D_x)} = tD_t I_N$.

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2- Before going into the proof let us recall the following facts about $\mathcal{E}_X^{\mathbb{R},f}$. Let X be a n -dimensional complex manifold. The sheaf of rings $\mathcal{E}_X^{\mathbb{R},f}$ is the tempered version defined in [An] of the sheaf of rings $\mathcal{E}_X^{\mathbb{R}}$ on T^*X of holomorphic microlocal operators of [S-K-K], of which it is a subsheaf.

One has $\gamma^{-1}\gamma_*\mathcal{E}_X^{\mathbb{R},f} = \mathcal{E}_X$. Also $\mathcal{E}_X^{\mathbb{R},f}$ is faithfully flat over $\mathcal{E}_X$.

We are going to make use of a few topics of the theory of symbols of holomorphic microlocal operators as developed by Kataoka and Aoki (see [Ao] and the literature quoted there), which we adapt to the framework of $\mathcal{E}_X^{\mathbb{R},f}$.

Let $x_0^* = (x_0; \xi_0) \in \overset{\circ}{T}^*X$ and U a $\mathbb{R}_{>0}$ -conical open neighborhood of x_0^* . Denote by

$$S(U) \quad (\text{resp. } S^f(U), \text{ resp. } R(U)),$$

the space of holomorphic functions $p(x, \xi)$ on U such that for any compactly generated cone $U' \subset U$ one has :

$$\begin{cases} \text{for any } \varepsilon > 0 \quad p(x, \xi) = O(e^{\varepsilon|\xi|}) \text{ on } U' \\ \text{(resp. there exists } m > 0 \text{ such that } p(x, \xi) = O(|\xi|^m) \text{ on } U'), \\ \text{(resp. there exists } \delta > 0 \text{ such that } p(x, \xi) = O(e^{-\delta|\xi|}) \text{ on } U'). \end{cases}$$

The notations $S(U)$, $R(U)$ are borrowed from [Ao].

Proposition 2.

(i) (cf. [Ao]) *There is an isomorphism of vector spaces*

$$\varinjlim_{U \ni x_0^*} S(U)/R(U) \xrightarrow{\sim} \mathcal{E}_{X, x_0^*}^{\mathbb{R}}.$$

(ii) *The isomorphism of (i) induces an isomorphism*

$$\varinjlim_{U \ni x_0^*} S^f(U)/R(U) \xrightarrow{\sim} \mathcal{E}_{X, x_0^*}^{\mathbb{R},f}.$$

In a local coordinate system $(x_1, \dots, x_n)$, the above morphisms take x_i to x_i , ξ_i to D_{ξ_i} and $x_i \xi_i$ to $x_i D_{\xi_i}$. In fact (ii) is easily deduced from the calculation in [Ao].

A representative $p(x, \xi) \in S(U)$ of an operator $P \in \mathcal{E}_{X, x_0^*}^{\mathbb{R}}$ for a suitable neighborhood U of x_0^* is called a *symbol* of P , and the lower bound of the $m \in \mathbb{R}$ such that $p(x, \xi) = O(|\xi|^m)$ in a conical neighborhood of x_0^* is called the order of P at x_0^* , e.g. $m < \infty$ iff $P \in \mathcal{E}_{X, x_0^*}^{\mathbb{R},f}$. As noted by Aoki, (i) of Proposition 2 entails that if $p(x, \xi) \in S(U)$ satisfies $p(x, \xi) = o(|\xi|)$ in a conical neighborhood of x_0^* then $\exp(p(x, \xi))$ is a symbol of an operator of $\mathcal{E}_{X, x_0^*}^{\mathbb{R}}$. By (ii) we get also :

$$(2) \quad \begin{cases} \text{if } p(x, \xi) \in S(U) \text{ satisfies } p(x, \xi) = O(\log |\xi|) \\ \text{in a conical neighborhood of } x_0^*, \\ \text{then } \exp(p(x, \xi)) \text{ is a symbol of an operator of } \mathcal{E}_{X, x_0^*}^{\mathbb{R},f} \end{cases}$$

For example, if $X = \mathbb{C}_x^n$ with coordinates $x = (x_1, \dots, x_n)$, one has $\exp(x_1 \log D_{x_1}) \in \mathcal{E}_{X, dx_1}^{\mathbb{R}, f}$ whereas $\exp(x_1 \sqrt{D_{x_1}}) \in \mathcal{E}_{X, dx_1}^{\mathbb{R}} \setminus \mathcal{E}_{X, dx_1}^{\mathbb{R}, f}$.

3- Proof of theorem 1. We have to estimate the growth order in τ of the symbol of $\exp(A(x; \xi) \log \tau)$. This is done by a straightforward use of norms of matrices of microdifferential operators. (Of course, if $N = 1$, the theorem is already implied by (2).)

For a differential operator $P(x, D_x)$ of order $\leq l$ we denote as usually by $N_l(P, s) = N_l(P(x; \xi), s)$ the Boutet de Monvel and Krée formal norm of P defined as the series

$$N_l(P, s) = \sum_{\alpha, \beta, k} \frac{2}{(2n)^k} \frac{k!}{(|\alpha| + k)! (|\beta| + k)!} |\partial_x^\alpha \partial_\xi^\beta P_{l-k}(x; \xi)| s^{2k + |\alpha + \beta|},$$

where $P_{l-k}(x; \xi)$ denotes the symbol of the homogenous part of order $l - k$ of P , and s is an independant variable.

If $P = (P_{ij})$ is $N \times N$ matrix whose entries are (micro-) differential operators of order at most l , we denote, as usually by $N_l(P, s)$ the matrix whose entries are the $N_l(P_{ij}, s)$. Recall then, that if Q is another $N \times N$ matrix of (micro-) differential operators of order at most l' one has

$$N_{l+l'}(PQ, s) \ll N_l(P, s) N_{l'}(Q, s),$$

where the symbol $\ll$ means that each entry of the matrix on the right side is a majorant series of the corresponding entry of the matrix on the left side.

We will need the following two estimates. Let $P(x, D_x)$ be a matrix of *differential* operators of order at most l .

Fix $(x_0; \xi_0) \in T^*\mathbb{C}^n$. By using Cauchy inequalities, it is easy to see that one may find:

a conic neighborhood V of $(x_0; \xi_0)$, a constant $M > 0$ and a matrix function $c(s) = \sum a_j s^j$ holomorphic near $s = 0$, where a_j a constant matrix with nonnegative entries,

such that

$$(3) \quad N_l(P, s) \ll M(1 + |\xi|^l) c(s), \text{ uniformly in } (x; \xi) \in V.$$

If $a = (a_{ij})$ is a $N \times N$ matrix of complex numbers we use the notation

$$\|a\| = \sup_{i,j} |a_{ij}|.$$

Then if P is a matrix of differential operators of order at most l as before, and if $s = r$ with $0 < r < 1$, we get from the definition the obvious estimate

$$(4) \quad \|P(x; \xi)\| \leq \sum_{0 \leq k \leq l} \|P_{l-k}(x; \xi)\| \leq \frac{l! (2n)^l}{2 r^{2l}} N_l(P, r),$$

where $P_{l-k}(x; \xi)$ is the matrix of symbols of order $l - k$ of P and $P(x; \xi) = \sum_k P_{l-k}(x; \xi)$ is the total symbol matrix.

Proof of (i). The matrix $A = A(x, D_x)$ being as in (1), we may write $A = A_0 + B$ where $A_0 = A_0(x)$ is the matrix of holomorphic functions consisting of the diagonal blocks A_{ii} of A , and $B := A - A_0$ consists of the blocks B_{ij} with $B_{ij} = A_{ij}$ if $i < j$ and zero otherwise. Let ν be the least integer $1 \leq \nu \leq N - 1$ such that $B_{ij} = 0$ for all i, j such that $i + \nu \leq j$.

For multi-indices $\alpha = (\alpha_1, \alpha_2, \dots)$, $\beta = (\beta_1, \beta_2, \dots) \in \mathbb{N}^{(\mathbb{N})}$, we set

$$\varphi^{\alpha, \beta}(A_0, B) = A_0^{\alpha_1} B^{\beta_1} A_0^{\alpha_2} B^{\beta_2} \dots \in \mathbf{M}_N(\mathcal{E}_X).$$

Now because of the particular forms of A_0 and B we get that $\varphi^{\alpha, \beta}(A_0, B) = 0$ for $|\beta| > \nu$. Hence for any integer $m \geq 0$, we have

$$A^m = (A_0 + B)^m = \sum_{|\alpha|+|\beta|=m, |\beta| \leq \nu} \varphi^{\alpha, \beta}(A_0, B).$$

Let l be the maximum order of the entries of B , then the above formula implies that for any integer $m \geq 0$, the matrix A^m has entries of order $\leq \nu l$, and we have

$$N_{\nu l}(A^m, s) << \sum_{0 \leq k \leq \nu} \binom{m}{k} N_0(A_0, s)^{m-k} N_l(B, s)^k.$$

Fixing $(0; \xi_0) \in T^*\mathbb{C}^n$, and making use of (3) and of its notations, one may find $M > 0$ and a matrix function $c(s)$ holomorphic near $s = 0$ such that

$$N_l(B, s)^k << M(1 + |\xi|^l)^\nu c(s)$$

holds for $0 \leq k \leq \nu$ and $(x; \xi)$ in a conic neighborhood V of $(0; \xi_0)$.

Thus

$$(5) \quad N_{\nu l}(A^m, s) << M(1 + |\xi|^l)^\nu c(s)(1 + N_0(A_0, s))^m,$$

for any integer $m \geq 0$ and any $(0; \xi_0) \in V$.

Reducing the conic neighborhood V if necessary, we may choose $0 < r < 1$ and a constant $M_1 > 0$ such that $\|N_0(A_0, r)\| \leq M_1$ uniformly in $(0; \xi_0) \in V$. Then, since

$$\|A^m(x; \xi)\| \leq \frac{(\nu l)! (2n)^{\nu l}}{2 r^{2\nu l}} N_{\nu l}(A^m, r)$$

by (4), we find by (5) that there is a constant $M_2 > 0$ such that

$$(6) \quad \|A^m(x; \xi)\| \leq M_2(1 + |\xi|^l)^\nu (1 + M_1)^m,$$

for any $m \geq 0$ and $(0; \xi_0) \in V$.

Fix the determination of $\log \tau$ in $\operatorname{Re} \tau > \operatorname{Im} \tau$ such that $\log 1 = 0$. We have to estimate the growth of the the matrix-valued holomorphic function $\exp(A(x, \xi) \log \tau)$

as $|\tau| \rightarrow \infty$, τ in a conic neighborhood of $d\tau\infty$. Let $|\tau| > e^\pi$, thus $|\log \tau| \leq \sqrt{2} \log |\tau|$. Using this and (6) we get

$$\begin{aligned} \|\tau^{A(x;\xi)}\| &= \left\| \sum_{m \geq 0} (A(x;\xi) \log \tau)^m / m! \right\| \leq \sum_{m \geq 0} \|A^m(x;\xi)\| |\log \tau|^m / m! \\ &\leq M_2(1 + |\xi|^l)^\nu \sum_{m \geq 0} (1 + M_1)^m 2^{m/2} (\log |\tau|)^m / m! \\ &\leq M_2(1 + |\xi|^l)^\nu |\tau|^{\sqrt{2}(1+M_1)}, \end{aligned}$$

and this holds for $|\tau| > e^\pi$, uniformly for $(x;\xi)$ in a small enough conic neighborhood of $(0;\xi_0)$. The choice of ξ_0 having been made arbitrarily, this proves that $\tau^{A(x;\xi)}$ is a well defined symbol of an operator of $\mathcal{E}_{X,(0;d\tau)}^{\mathbb{R},f}$ that we denote by $D_t^{A(x,D_x)}$ (we omit the notation I_N).

Proof of (ii). Since $[A, D_t] = 0$ one has $[(D_t)^A, t] = A(D_t)^{A-I_N}$. Hence $(D_t)^A(tD_t - A) = t(D_t)^{A+I_N} + A(D_t)^A - (D_t)^A A = tD_t(D_t)^A$. $\square$

Remarks.

1- The theorem should be more generally true when the matrix $A(x, D_x)$ is essentially of nonpositive order, but the proof given above breaks down for this more general case.

2- Distribution solutions of regular operators are investigated in [A-M.F.]; the above theorem provides an alternate proof of proposition 3.1 of that paper.

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ERRATUM to “A conjugacy class of regular operators”

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In the last part of the proof of (i) of theorem 1 a confusion between $(A(x; \xi))^m$ and $A^m(x; \xi)$ occurs. The proof stands when replacing

$$\exp(A(x; \xi) \log \tau)$$

with the following matrix-valued function

$$p(x; \xi) := \sum_{m \geq 0} A^m(x; \xi) (\log \tau)^m / m!,$$

then by defining $(D_t)^A$ as the matrix operator with coefficients in $\mathcal{E}_X^{\mathbf{R}, f}$ given by the symbol $p(x; \xi)$.