

Tridiagonal pairs and the quantum affine algebra $U_q(\widehat{sl_2})$

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Overview

In this talk, I will first recall the notion of a **Leonard pair** and discuss how these objects are related to certain classical orthogonal polynomials.

I will then define a generalization of a Leonard pair called a **Tridiagonal pair**.

I will then show how certain tridiagonal pairs are related to finite dimensional modules for the quantum affine algebra $U_q(\widehat{sl_2})$.

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Leonard pairs

We recall the notion of a Leonard pair. To do this, we first recall what it means for a matrix to be **tridiagonal**.

The following matrices are tridiagonal.

$$\begin{pmatrix} 2 & 3 & 0 & 0 \\ 1 & 4 & 2 & 0 \\ 0 & 5 & 3 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$

Tridiagonal means each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal.

The tridiagonal matrix on the left is **irreducible**. This means each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

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The Definition of a Leonard Pair

We now define a Leonard pair. From now on $\mathbb{K}$ will denote a field.

Definition Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a **Leonard pair** on V , we mean a pair of linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$ which satisfy both conditions below.

1. There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
2. There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

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Example of a Leonard pair

For any nonnegative integer d the pair

$$A = \begin{pmatrix} 0 & d & 0 & & 0 \\ 1 & 0 & d-1 & & \\ & 2 & \cdot & \cdot & \\ & & \cdot & \cdot & 1 \\ 0 & & & d & 0 \end{pmatrix},$$

$$A^* = \text{diag}(d, d-2, d-4, \dots, -d)$$

is a Leonard pair on the vector space $\mathbb{K}^{d+1}$, provided the characteristic of $\mathbb{K}$ is 0 or an odd prime greater than d .

Reason: There exists an invertible matrix P such that $P^{-1}AP = A^*$ and $P^2 = 2^d I$.

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Why Leonard pairs are of interest

There is a natural correspondence between Leonard pairs and a family of orthogonal polynomials consisting of the following types:

q -Racah,
 q -Hahn,
dual q -Hahn,
 q -Krawtchouk,
dual q -Krawtchouk,
quantum q -Krawtchouk,
affine q -Krawtchouk,
Racah,
Hahn,
dual-Hahn,
Krawtchouk,
Bannai/Ito,
orphans ($\text{char}(\mathbb{K}) = 2$ only).

This family coincides with the terminating branch of the Askey scheme of orthogonal polynomials.

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Tridiagonal pairs

We now consider a generalization of a Leonard pair called a **tridiagonal pair**.

A tridiagonal pair is defined as follows.

As before, V will denote a vector space over $\mathbb{K}$ with finite positive dimension.

As before, we consider a pair of linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$.

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Definition of a Tridiagonal pair

We say the pair A, A^* is a **Tridiagonal pair** on V whenever (1)–(4) hold below.

1. Each of A, A^* is diagonalizable on V .
2. There exists an ordering $V_0, V_1, \dots, V_d$ of the eigenspaces of A such that
$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$
where $V_{-1} = 0, V_{d+1} = 0$.
3. There exists an ordering $V_0^*, V_1^*, \dots, V_d^*$ of the eigenspaces of A^* such that
$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq d),$$
where $V_{-1}^* = 0, V_{d+1}^* = 0$.
4. There is no subspace $W \subseteq V$ such that $AW \subseteq W$ and $A^*W \subseteq W$ and $W \neq 0$ and $W \neq V$.

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A comment

Referring to our definition of a tridiagonal pair,

it turns out $d = \delta$; we call this the **diameter** of the pair.

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Leonard pairs and Tridiagonal pairs

We mentioned a tridiagonal pair is a generalization of a Leonard pair.

A Leonard pair is the same thing as a tridiagonal pair for which the eigenspaces V_i and V_i^* all have dimension 1.

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An open problem

Problem Classify the tridiagonal pairs.

For the rest of this talk we focus on a special case of tridiagonal pair said to have **geometric type**.

We will show these tridiagonal pairs are related to $U_q(\widehat{sl_2})$.

We hope this will lead to a classification of the tridiagonal pairs of geometric type.

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Tridiagonal pairs of geometric type

Let A, A^* denote a tridiagonal pair on V with diameter d .

Let the eigenspaces V_i, V_i^* ($0 \leq i \leq d$) be as in the definition.

Let q denote a nonzero scalar in $\mathbb{K}$ which is not a root of unity.

We say A, A^* has **q -geometric type** whenever for $0 \leq i \leq d$, the eigenvalue of A for V_i is q^{2i-d} and the eigenvalue of A^* for V_i^* is q^{d-2i} .

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Tridiagonal pairs of geometric type and $U_q(\widehat{sl_2})$

For the rest of this talk, A, A^* denotes a tridiagonal pair on V of diameter d and q -geometric type.

Using A, A^* we will construct two actions of $U_q(\widehat{sl_2})$ on V .

We use the following notation.

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Decompositions of V

By a **decomposition** of V , we mean a sequence $U_0, U_1, \dots, U_d$ consisting of nonzero subspaces of V such that

$$V = U_0 + U_1 + \dots + U_d \quad (\text{direct sum}).$$

We do not assume each of $U_0, U_1, \dots, U_d$ has dimension 1.

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Six Decompositions of V

We are about to define six decompositions of V .

In order to keep track of these decompositions we will give each of them a name.

Our naming scheme is as follows.

Let Ω denote the set consisting of the four symbols $0, D, 0^*, D^*$.

Each of the six decompositions will get a name $[u]$ where u is a two-element subset of Ω .

We now define the six decompositions.

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Six Decompositions of V , cont.

Lemma For each of the six rows in the table below, and for $0 \leq i \leq d$, let U_i denote the i th subspace described in that row. Then the sequence $U_0, U_1, \dots, U_d$ is a decomposition of V .

name	i th subspace of the decomposition
$[0D]$	V_i
$[0^*D^*]$	V_i^*
$[0^*D]$	$(V_0^* + \dots + V_i^*) \cap (V_i + \dots + V_d)$
$[0^*0]$	$(V_0^* + \dots + V_i^*) \cap (V_0 + \dots + V_{d-i})$
$[D^*0]$	$(V_{d-i}^* + \dots + V_d^*) \cap (V_0 + \dots + V_{d-i})$
$[D^*D]$	$(V_{d-i}^* + \dots + V_d^*) \cap (V_i + \dots + V_d)$

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How the six decompositions are related

Let $U_0, U_1, \dots, U_d$ denote any one of the six decompositions of V . Then for $0 \leq i \leq d$ the sums $U_0 + \dots + U_i$ and $U_i + \dots + U_d$ are given as follows.

name	$U_0 + \dots + U_i$	$U_i + \dots + U_d$
$[0D]$	$V_0 + \dots + V_i$	$V_i + \dots + V_d$
$[0^*D^*]$	$V_0^* + \dots + V_i^*$	$V_i^* + \dots + V_d^*$
$[0^*D]$	$V_0^* + \dots + V_i^*$	$V_i + \dots + V_d$
$[0^*0]$	$V_0^* + \dots + V_i^*$	$V_0 + \dots + V_{d-i}$
$[D^*0]$	$V_{d-i}^* + \dots + V_d^*$	$V_0 + \dots + V_{d-i}$
$[D^*D]$	$V_{d-i}^* + \dots + V_d^*$	$V_i + \dots + V_d$

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The linear transformations B, B^*, K, K^*

We now define four linear transformations from V to V .

We call these B, B^*, K, K^* .

Each of these transformations is diagonalizable.

To define them, we list their eigenspaces and the corresponding eigenvalues as follows.

name	i th sbspace is eigspace	Corresp. eigval
B	$[0^*0]$	q^{2i-d}
B^*	$[D^*D]$	q^{d-2i}
K	$[0^*D]$	q^{2i-d}
K^*	$[D^*0]$	q^{2i-d}

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The q -Weyl Relations

We have

$$\begin{aligned} \frac{qAB - q^{-1}BA}{q - q^{-1}} &= I, \\ \frac{qBA^* - q^{-1}A^*B}{q - q^{-1}} &= I, \\ \frac{qA^*B^* - q^{-1}B^*A^*}{q - q^{-1}} &= I, \\ \frac{qB^*A - q^{-1}AB^*}{q - q^{-1}} &= I. \end{aligned}$$

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The q -Weyl Relations, cont.

We have

$$\begin{aligned} \frac{qK^{-1}A - q^{-1}AK^{-1}}{q - q^{-1}} &= I, \\ \frac{qBK^{-1} - q^{-1}K^{-1}B}{q - q^{-1}} &= I, \\ \frac{qKA^* - q^{-1}A^*K}{q - q^{-1}} &= I, \\ \frac{qB^*K - q^{-1}KB^*}{q - q^{-1}} &= I. \end{aligned}$$

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q -Weyl Relations cont.

We have

$$\begin{aligned}\frac{qAK^* - q^{-1}K^*A}{q - q^{-1}} &= I, \\ \frac{qK^{*-1}B - q^{-1}BK^{*-1}}{q - q^{-1}} &= I, \\ \frac{qA^*K^{*-1} - q^{-1}K^{*-1}A^*}{q - q^{-1}} &= I, \\ \frac{qK^*B^* - q^{-1}B^*K^*}{q - q^{-1}} &= I.\end{aligned}$$

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The q -Serre relations

We have

$$\begin{aligned}A^3A^* - [3]_qA^2A^*A + [3]_qAA^*A^2 - A^*A^3 &= 0, \\ A^{*3}A - [3]_qA^{*2}AA^* + [3]_qA^*AA^{*2} - AA^{*3} &= 0, \\ B^3B^* - [3]_qB^2B^*B + [3]_qBB^*B^2 - B^*B^3 &= 0, \\ B^{*3}B - [3]_qB^{*2}BB^* + [3]_qB^*BB^{*2} - BB^{*3} &= 0,\end{aligned}$$

where

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad n = 0, 1, \dots$$

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Tridiagonal pairs of geometric type and $U_q(\widehat{sl_2})$

We now use A, A^*, B, B^*, K, K^* to get two actions of $U_q(\widehat{sl_2})$ on V .

Before proceeding we recall the definition of $U_q(\widehat{sl_2})$.

The algebra $U_q(\widehat{sl_2})$

Definition The quantum affine algebra $U_q(\widehat{sl_2})$ is the unital associative $\mathbb{K}$ -algebra with generators $e_i^\pm, K_i^{\pm 1}, i \in \{0, 1\}$ and the following relations:

$$\begin{aligned}K_iK_i^{-1} &= K_i^{-1}K_i = 1, \\ K_0K_1 &= K_1K_0, \\ K_ie_i^\pm K_i^{-1} &= q^{\pm 2}e_i^\pm, \\ K_ie_j^\pm K_i^{-1} &= q^{\mp 2}e_j^\pm, \quad i \neq j, \\ [e_i^+, e_i^-] &= \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\ [e_0^\pm, e_1^\mp] &= 0,\end{aligned}$$

$$\begin{aligned}(e_i^\pm)^3e_j^\pm - [3]_q(e_i^\pm)^2e_j^\pm e_i^\pm + [3]_qe_i^\pm e_j^\pm (e_i^\pm)^2 \\ - e_j^\pm (e_i^\pm)^3 = 0, \quad i \neq j.\end{aligned}$$

We call $e_i^\pm, K_i^{\pm 1}, i \in \{0, 1\}$ the **Chevalley generators** for $U_q(\widehat{sl_2})$.

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Another presentation of $U_q(\widehat{sl_2})$

In order to state our main results we introduce a second presentation of $U_q(\widehat{sl_2})$.

Alternate presentation of $U_q(\widehat{sl_2})$

theorem The quantum affine algebra $U_q(\widehat{sl_2})$ is isomorphic to the unital associative $\mathbb{K}$ -algebra with generators $y_i^\pm, k_i^{\pm 1}, i \in \{0, 1\}$ and the following relations:

$$\begin{aligned} k_i k_i^{-1} &= k_i^{-1} k_i = 1, \\ k_0 k_1 &\text{ is central,} \\ \frac{q y_i^+ k_i - q^{-1} k_i y_i^+}{q - q^{-1}} &= 1, \\ \frac{q k_i y_i^- - q^{-1} y_i^- k_i}{q - q^{-1}} &= 1, \\ \frac{q y_i^- y_i^+ - q^{-1} y_i^+ y_i^-}{q - q^{-1}} &= 1, \\ \frac{q y_i^+ y_j^- - q^{-1} y_j^- y_i^+}{q - q^{-1}} &= k_0^{-1} k_1^{-1}, \quad i \neq j, \end{aligned}$$

$$\begin{aligned} (y_i^\pm)^3 y_j^\pm - [3]_q (y_i^\pm)^2 y_j^\pm y_i^\pm + [3]_q y_i^\pm y_j^\pm (y_i^\pm)^2 \\ - y_j^\pm (y_i^\pm)^3 = 0, \quad i \neq j. \end{aligned}$$

We call $y_i^\pm, k_i^{\pm 1}, i \in \{0, 1\}$ the **alternate generators** of $U_q(\widehat{sl_2})$.

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Two actions for $U_q(\widehat{sl_2})$

Theorem Let A, A^* denote a tridiagonal pair on V of geometric type. Let the maps B, B^*, K, K^* be as above.

Then V is an irreducible $U_q(\widehat{sl_2})$ -module on which the alternate generators act as follows.

generator	y_0^+	y_1^+	y_0^-	y_1^-	k_0	k_1	k_0^{-1}	k_1^{-1}
action on V	B^*	B	A^*	A	K	K^{-1}	K^{-1}	K

Also, V is an irreducible $U_q(\widehat{sl_2})$ -module on which the alternate generators act as follows.

generator	y_0^+	y_1^+	y_0^-	y_1^-	k_0	k_1	k_0^{-1}	k_1^{-1}
action on V	A	A^*	B^*	B	K^*	K^{*-1}	K^{*-1}	K^*

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In conclusion

From the previous theorem we readily obtain the following.

Corollary Let V denote a vector space over $\mathbb{K}$ with finite positive dimension.

Let A, A^* denote a tridiagonal pair on V of geometric type.

Then there exists a unique $U_q(\widehat{sl_2})$ -module structure on V such that y_1^- acts as A and y_0^- acts as A^* .

Moreover there exists a unique $U_q(\widehat{sl_2})$ -module structure on V such that y_0^+ acts as A and y_1^+ acts as A^* .

Both $U_q(\widehat{sl_2})$ -module structures are irreducible.

THE END

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