

A COUNTER EXAMPLE OF STRONG BAUM-CONNES CONJECTURES FOR FOLIATED MANIFOLDS

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§1 Introduction Related to a new index theory, Baum and Connes conjectured in [1] that the analytic and topological K-theory for foliations and dynamical systems are isomorphic each other under the K-theoretic index map. Since then, there appeared several papers supporting the conjecture (cf:[1]~[9] except [4]). However, Skandalis recently showed in [4] that there exists a counter example of the strong Connes-Kasparov conjecture for K-theory of C^* -crossed products. In his proof, a central tool is the property T due to Kazhdan in semisimple Lie groups of real rank one.

Modifying his idea, we shall show in this note that there exists a counter example of the strong Baum-Connes conjecture for foliated manifolds.

§2 Preliminaries Let (M, F) be a foliated manifold and G its holonomy groupoid. Taking the source and range maps s, r from G to M respectively, one can define the foliation $\tilde{F}$ of G coming from the tensor product of the pull backs $s^*(F)$ and $r^*(F)$ of F by s and r

respectively. Let $\Omega^{1/2}$ be the half density bundle over G tangential to $\tilde{F}$ and denote by $C_c(\Omega^{1/2})$ the $*$ -algebra of all continuous sections of $\Omega^{1/2}$ with compact support. The $*$ -algebraic operation is defined as follows:

$$(f \cdot g)(\tau) = \int_{\tau=\tau_1\tau_2} f(\tau_1)g(\tau_2)$$

$$f^*(\tau) = \overline{f(\tau^{-1})}$$

for all $f, g \in C_c(\Omega^{1/2})$ where $\overline{f}(\tau) = \overline{f(\tau)}$. Given any $x \in M$, let H_x be the Hilbert space consisting of all L^2 -sections of $\Omega^{1/2}$ over G and π_x the $*$ -representation of $C_c(\Omega^{1/2})$ on H_x defined by

$$(\pi_x(f)\xi)(\tau) = \int_{\tau=\tau_1\tau_2} f(\tau_1)\xi(\tau_2)$$

for all $f \in C_c(\Omega^{1/2})$ and $\xi \in H_x$. The completion $C_r^*(M, F)$ of $C_c(\Omega^{1/2})$ with respect to $\|f\| = \sup_{x \in M} \|\pi_x(f)\|$ is called a foliation C^* -algebra associated to (M, F) .

We now consider the K -theory $K_a(M, F)$ of $C_r^*(M, F)$ which is called the analytic K -theory of (M, F) . We also have another K -theory of (M, F) using a pure topological way. It is called the topological K -theory of (M, F) which is denoted by $K_t(M, F)$. By Baum-Connes[1], if G is torsion free, then $K_t(M, F)$ is nothing more than the twisted K -theory $K^\nu(BG)$ of the classifying space BG of G by the transverse bundle ν of F . More precisely, the latter is the K -theory $K(B\tilde{\nu}/S\tilde{\nu})$ of the Thom space $B\tilde{\nu}/S\tilde{\nu}$ of the ν -bundle $\tilde{\nu}$ over BG . We then state the Baum-Connes conjecture for foliated manifolds as follows:

Baum-Connes Conjecture: Given a smooth foliated manifold (M, F) , the K -index map is an isomorphism from $K_t(M, F)$ to $K_a(M, F)$.

There exists several papers supporting the above conjecture (cf:[1]~[9] except [4]).

Suppose G is torsion free and ν has a spin^C structure, then we know that $K^{\nu}(BG)$ is equal to $K(BG)$ via Thom isomorphism. Related to the conjecture, we may formulate the following conjecture which is a strong version of the Baum-Connes' one:

Strong Baum-Connes Conjecture: Given any smooth foliated manifold (M, F) where G is torsion free and ν is spin^C , then the K -index map is an KK -isomorphism from $K_t(M, F)$ to $K_a(M, F)$ in the sense of Kasparov.

§3 Construction Let G be a connected semisimple linear Lie group which is locally isomorphic to $Sp(n, 1)$ or F_4 ($n \geq 2$). Then it is of real rank one without Kazhdan's property T. Thanks to Skandalis[4], it is non K -nuclear, which means that the K -theory of the reduced group C^* -algebra is no longer KK -isomorphic to that of nuclear C^* -algebras. Using this fact, he found a counter example of the strong Connes-Kasparov conjecture for C^* -crossed products. We now modify his idea in order to find a counter example of the strong Baum-Connes conjecture for foliated manifolds.

Let V be a vector group on which G acts faithfully. By Borel's result [12], there exist torsion free uniform lattices Γ, Δ of G , V respectively such that Δ is Γ -invariant. By Wang's result [10], the semidirect product $V \rtimes_s G$ of V by G has the property T which is no longer of real rank one. Since $\Delta \rtimes_s \Gamma$ is a uniform lattice of $V \rtimes_s G$, it also has the property T. By the similar way as in the proof of Skandalis [4], we have the following lemma which is of

independent interest:

Lemma 1. $C_r^*(\Delta \times_s \Gamma)$ is non KK-nuclear.

Remark Dr. Matsumoto generalized the above lemma in more general setting.

Since Δ is a torsion free uniform lattice of V , the character group $\hat{\Delta}$ of Δ is a torus which may be chosen as of even dimension.

Let H be a maximal compact subgroup of G and $M = (G/H) \times_{\Gamma} \hat{\Delta}$ the orbit space of $(G/H) \times \hat{\Delta}$ by the diagonal Γ -action. As Γ is torsion free, M is a smooth manifold. Let $F = \{ (G/H) \times_{\Gamma} \{t\} \mid t \in \hat{\Delta} \}$. Then we have the following lemma:

Lemma 2. (M, F) is a foliated $\hat{\Delta}$ - bundle over $\Gamma \backslash G/H$.

Let us consider the reduced crossed product $(C(\hat{\Delta}) \times_{\alpha} \Gamma)_r$ of $C(\hat{\Delta})$ by the holonomy action α of Γ on $\hat{\Delta}$ and $BC(\cdot)$ means the C^* -algebra of all compact operators on a Hilbert space. By the joint work with Natsume (cf: [11]), we have by Lemma 1 the following Lemma:

Lemma 3. $C_r^*(M, F)$ is isomorphic to $(C(\hat{\Delta}) \times_{\alpha} \Gamma)_r \otimes BC(L^2(\Gamma \backslash G/H))$.

It then easily follows from Lemma 2 that

Corollary 4. $K_a(M, F) = K_a(\hat{\Delta}, \Gamma) = K((C(\hat{\Delta}) \times_{\alpha} \Gamma)_r)$.

In what follows, we shall determine $K_t(M, F)$. Let us consider the leaf ℓ_{χ} in F passing through $\chi \in \hat{\Delta}$. Then its fundamental group $\pi_1(\ell_{\chi})$ is the stabilizer Γ_{χ} of Γ at χ . Since $\hat{\Delta}$ is the torus of even dimension, the holonomy map h_{χ} of ℓ_{χ} is a homomorphism from $\pi_1(\ell_{\chi})$ into $\pi_1(\hat{\Delta})$. Therefore, $h_{\chi}(\Gamma_{\chi})$ is torsion free for all $\chi \in \hat{\Delta}$, which means that G is also torsion free. By Baum-Connes [1], we then have

the following lemma:

Lemma 5. $K_t(M, F) = K^\nu(BG)$ where $\nu = T(M)/F$.

In our case, as $T(M)/F = T(\hat{\Delta})$, ν has a complex structure. By Thom isomorphism, we deduce the following lemma:

Lemma 6. $K^\nu(BG) = K(BG)$.

Combining all the lemmas cited above, we obtain the following main theorem:

Theorem 7. Let G be a connected semisimple Lie group which is locally isomorphic to $Sp(n, 1)$ or $F_4(n \geq 2)$ acting on a vector group V faithfully. Let Γ, Δ be torsion free uniform lattices of G, V respectively such that Γ is Δ -invariant. If (M, F) is a foliated $\hat{\Delta}$ -bundle over $\Gamma \backslash G/H$, then it is a counter example for the strong Baum-Connes conjecture where $\hat{\Delta}$ is the character group of Δ and H is a maximal compact subgroup of G .

Proof. By definition, the holonomy groupoid G of F has no torsion and $\nu = T(M)/F$ has a complex structure. Suppose the K -index map is a KK -isomorphism from $K_t(M, F)$ to $K_a(M, F)$. This means by Lemma 3, 5 and 6 that there exists an invertible element of $KK((C(\hat{\Delta}) \rtimes_{\alpha} \Gamma)_r, C_0(BG))$ which implement the K -index map where $C_0(BG)$ is the C^* -algebra of all continuous sections of $E\Gamma \times_{\Gamma} C(\hat{\Delta})$ vanishing at infinity. Since $C_0(BG)$ is nuclear and $(C(\hat{\Delta}) \rtimes_{\alpha} \Gamma)_r = C_r^*(\Delta \times_S \Gamma)$ up to isomorphism, this contradicts to Lemma 1. Q.E.D.

Remark. In spite of the above theorem, the original Baum-Connes conjecture still remains open even for the above example.

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