

MAJORIZATION OF SUBORDINATE HARMONIC FUNCTIONS

MAMORU NUNOKAWA, HITOSHI SAITOH,
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ABSTRACT. It is well known that if $f(z)$ and $F(z)$ with $f(0) = F(0)$ are analytic in $|z| < 1$ and if $f(z)$ is subordinate to $F(z)$, then for $0 < p$ and $0 < r < 1$,

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |F(re^{i\theta})|^p d\theta.$$

In this paper, we research the relationship of large and small of the

$$\int_0^{2\pi} |\operatorname{Re} f(re^{i\theta})|^p d\theta \quad \text{and} \quad \int_0^{2\pi} |\operatorname{Re} F(re^{i\theta})|^p d\theta.$$

Suppose that a function $f(z)$ is analytic in the unit disc $E = \{z : |z| < 1\}$ and that a function $F(z)$ is analytic and univalent in E . Suppose that $f(0) = F(0)$. If the image of the disc E under the mapping $w = f(z)$ is contained in the image of that disc under the mapping $w = F(z)$, we say that the function $f(z)$ is subordinate to $F(z)$ in the disc E and that the function $F(z)$ is a univalent majorant of $f(z)$. We denote this by writing $f(z) \prec F(z)$. This is equivalent to regularity of the function $F^{-1}(f(z)) = \varphi(z)$ in E , where $\varphi(0) = 0$ and $|\varphi(z)| \leq 1$ in E .

It follows that the set of all functions $f(z)$ that are subordinate in the disc $|z| < r$ to a given univalent majorant $F(z)$ is defined by the formula

$$f(z) = F(\varphi(z)),$$

where $\varphi(z)$ is an arbitrary function satisfying the conditions of the Schwarz lemma, that is, it is analytic in E , $\varphi(0) = 0$, $|\varphi(z)| < 1$ in E . Then, Rogosinski [3] proved the following theorem.

Theorem A. *If $f(z)$ and $F(z)$ with $f(0) = F(0)$ are analytic in E and if $f(z) \prec F(z)$, then for $0 < p$ and $0 < r < 1$,*

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |F(re^{i\theta})|^p d\theta.$$

On the other hand, Avhadiev and Aksent'ev [1] obtained the following result.

2000 *Mathematics Subject Classification* : Primary 30C45.

Key words : Majorization and Subordination.

Theorem B. *If $f(z)$ and $F(z)$ satisfy the conditions of Theorem A, then it follows that*

$$\int_0^{2\pi} |\operatorname{Re} f(re^{i\theta})| d\theta \leq \int_0^{2\pi} |\operatorname{Re} F(re^{i\theta})| d\theta$$

for $0 < r < 1$.

After the Theorem B was obtained, Nunokawa, Fukui and Saitoh [2] proved the following theorem.

Theorem C. *If $f(z)$ and $F(z)$ satisfy the conditions of Theorem A, then we have*

$$\int_0^{2\pi} |\operatorname{Re} f(re^{i\theta})|^2 d\theta \leq \int_0^{2\pi} |\operatorname{Re} F(re^{i\theta})|^2 d\theta$$

for $0 < r < 1$.

It is the purpose of the present paper to generalize Theorem B.

In this paper, we need Hölder's theorem.

Lemma 1. (Hölder) *Let $f(x)$ and $g(x)$ are continuous on $a \leq x \leq b$, $f(x) \geq 0$ and $g(x) \geq 0$ on $a \leq x \leq b$. Then we have*

$$(1) \quad \int_a^b f(x)g(x) dx \leq \left(\int_a^b f(x)^p dx \right)^{\frac{1}{p}} \left(\int_a^b g(x)^q dx \right)^{\frac{1}{q}}$$

where $1/p + 1/q = 1$, $1 < p$ and $0 < q$ and we have

$$(2) \quad \int_a^b f(x)g(x) dx \geq \left(\int_a^b f(x)^p dx \right)^{\frac{1}{p}} \left(\int_a^b g(x)^q dx \right)^{\frac{1}{q}}$$

where $1/p + 1/q = 1$, $0 < p < 1$ and $q < 0$.

Proof. (1) is very popular and applying the same method as the proof of (1), we can obtain (2). □

Theorem 1. *If $f(z)$ and $F(z)$ with $f(0) = F(0)$ are analytic in E and if $f(z) \prec F(z)$, then we have for $0 < r < 1$ and for the case $1 \leq p$,*

$$(3) \quad \int_0^{2\pi} |\operatorname{Re} f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |\operatorname{Re} F(re^{i\theta})|^p d\theta.$$

and (3) does not hold for the case $0 < p < 1$ or (3) is not always true for the case $0 < p < 1$.

Proof. For the case $1 \leq p$, from the hypothesis of Theorem 1, we have

$$f(z) = F(\varphi(z))$$

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where $\varphi(z)$ is analytic in E , $\varphi(0) = 0$ and $|\varphi(z)| < 1$ in E . Applying the same reason as the proof of Theorem B [1, p.934] and by using Lemma 1, we have for $0 < r < \rho < 1$,

$$\begin{aligned}
& \int_0^{2\pi} |\operatorname{Re} f(re^{i\theta})|^p d\theta \\
& \leq \int_0^{2\pi} \left(\frac{1}{2\pi} \int_0^{2\pi} |\operatorname{Re} F(\rho e^{i\nu})| \operatorname{Re} \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})} d\nu \right)^p d\theta \\
& = \int_0^{2\pi} \left(\frac{1}{2\pi} \int_0^{2\pi} |\operatorname{Re} F(\rho e^{i\nu})| \left(\operatorname{Re} \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})} \right)^{\frac{1}{p} + \frac{p-1}{p}} d\nu \right)^p d\theta \\
& \leq \int_0^{2\pi} \left\{ \left(\frac{1}{2\pi} \int_0^{2\pi} |\operatorname{Re} F(\rho e^{i\nu})|^p \operatorname{Re} \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})} d\nu \right)^{\frac{1}{p}} \left(\frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})} d\nu \right) \right\}^p d\theta \\
& = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} |\operatorname{Re} F(\rho e^{i\nu})|^p \operatorname{Re} \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})} d\nu d\theta \\
& = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} |\operatorname{Re} F(\rho e^{i\nu})|^p \operatorname{Re} \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})} d\theta d\nu \\
& = \frac{1}{2\pi} \int_0^{2\pi} |\operatorname{Re} F(\rho e^{i\nu})|^p d\nu.
\end{aligned}$$

Putting $r \rightarrow \rho$, we obtain (3).

For the case $0 < p < 1$, let us take a function $F(z)$ whose real part is positive in E , then we have (e.g. [4, p.227])

$$\begin{aligned}
& \operatorname{Re} f(re^{i\theta}) = |\operatorname{Re} f(re^{i\theta})| = \operatorname{Re} F(\varphi(re^{i\theta})) \\
& = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} F(\rho e^{i\nu}) \operatorname{Re} \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})} d\nu \\
(4) \quad & = \frac{1}{2\pi} \int_0^{2\pi} |\operatorname{Re} F(\rho e^{i\nu})| \operatorname{Re} \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})} d\nu.
\end{aligned}$$

Applying the same method as the proof of (3), Lemma 1 and equation (4), we have

$$\int_0^{2\pi} |\operatorname{Re} f(re^{i\theta})|^p d\theta \geq \int_0^{2\pi} |\operatorname{Re} F(re^{i\theta})|^p d\theta$$

where $\operatorname{Re} F(z) > 0$ in E and so $\operatorname{Re} f(z) > 0$ in E . This completes the proof of Theorem 1. $\square$

Remark.

$$0 < \operatorname{Re} \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})}$$

for $0 < r < \rho < 1$.

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