

Generalization and randomization of some number-theoretic special functions

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1 Introduction

There are many special number-theoretic functions around the Riemann zeta function $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$, $\text{Re } s > 1$, such as

$$\begin{aligned}\zeta(s, x) &= \sum_{k=1}^{\infty} (k+x)^{-s}, \quad \text{Re } s > 1, \quad x > -1, \quad (\text{Hurwitz zeta function}),^1 \\ \Gamma(1+x)^{-1} &= \exp(\zeta'(0) - \zeta'(0, x)) \quad \left[\zeta'(0, x) := \frac{\partial}{\partial s} \zeta(0, x) \right]^2 \\ &= e^{\gamma x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{k} \right) e^{-x/k} \quad \left[\text{where } \gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right) \right] \\ &= \lim_{n \rightarrow \infty} n^x \prod_{k=1}^n \left(1 + \frac{x}{k} \right), \\ \psi(x+1) &= (\log \Gamma(x+1))' = \frac{\Gamma'(x+1)}{\Gamma(x+1)} \\ &= - \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k+x} - \log n \right). \quad (\text{digamma function})\end{aligned} \tag{1}$$

As we see in the above infinite sum or infinite product formulas, these special functions are related to the sequence of natural numbers $\{k\}_{k=1}^{\infty}$. In this article, we study what we get when $\{k\}_{k=1}^{\infty}$ is replaced with other positive increasing sequences, including random sequences.

The most popular method for generalization of number-theoretic special functions is the so-called *zeta regularization*.

Definition 1 ([4, 6]) Let a positive sequence $a = \{a_k\}_{k=1}^{\infty}$ satisfy $\sum_{k=1}^{\infty} a_k^{-\alpha} < \infty$ for some $\alpha > 0$. Then we define the zeta function

$$z(s) := \sum_{k=1}^{\infty} a_k^{-s},$$

¹Slightly different from the traditional definition.

²This notation will be used for any functions of two variables in this article.

which is holomorphic in $\operatorname{Re} s > \alpha$. If $z(s)$ is analytically continued to a meromorphic function which is holomorphic at $s = 0$, a is said to be *zeta regularizable*. Then we write

$$z\text{-}\prod_{k=1}^{\infty} a_k := \exp(-z'(0))$$

and call it the *zeta regularized product* of $\prod_{k=1}^{\infty} a_k$.

But, for our purpose, this notion is too strong, indeed, it is quite unlikely that random sequences become zeta regularizable. We therefore assume a rather mild condition (Assumption 1 below) which random sequences can satisfy.

This work is somewhat an experimental one. We are not sure that it is a promising research. However, we think that some of results, such as Example 2, Theorem 6, Theorem 7, and their extensions in § 4.1 are fully interesting by themselves.

2 Deterministic generalization

2.1 Zeta regularized product

In this article, we consider real sequences which satisfy the following condition.

Assumption 1 (i) $a = \{a_k\}_{k=1}^{\infty}$ is a positive non-decreasing sequence diverging to ∞ .
(ii) a is uniformly distributed in the half line $(0, \infty)$ with the same density as $\mathbb{N}$ in the following sense: Setting

$$F(x) := \#\{k \in \mathbb{N}; a_k \leq x\},$$

there exists some $\delta > 0$ such that

$$F(x)x^{-1} = 1 + O(x^{-\delta}), \quad x \rightarrow \infty. \quad (3)$$

Remark 1 As we will see later, Assumption 1 alone does not assure $a = \{a_k\}_{k=1}^{\infty}$ to be zeta regularizable.

Throughout this section § 2 (except Remark 4), we consider everything under Assumption 1.

Lemma 1 For any $\varepsilon > 0$, we have $\sum_{k=1}^{\infty} a_k^{-1-\varepsilon} < \infty$.

Proof. Since $k \leq F(a_k)$, we see that $ka_k^{-1} \leq F(a_k)a_k^{-1} \rightarrow 1$ as $k \rightarrow \infty$, which implies³ $\limsup_{k \rightarrow \infty} ka_k^{-1} \leq 1$. From this, the assertion of the lemma easily follows. Q.E.D.

Lemma 2 *The following limit exists:*

$$\lim_{x \rightarrow \infty} \left[\sum_{a_k \leq x} a_k^{-1} - \log x \right] = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n a_k^{-1} - \log n \right] =: q. \quad (4)$$

³In fact, we have $\lim_{k \rightarrow \infty} ka_k^{-1} = 1$ ([5]).

Proof. Take $0 < \varepsilon < a_1$, and note that $F(\varepsilon) = 0$. By integration by parts formula,

$$\begin{aligned} \int_{\varepsilon}^x (F(t)t^{-1} - 1)t^{-1}dt &= \int_{\varepsilon}^x F(t)t^{-2}dt - \int_{\varepsilon}^x t^{-1}dt \\ &= -F(x)x^{-1} + F(\varepsilon)\varepsilon^{-1} + \int_{\varepsilon}^x t^{-1}dF(t) - (\log x - \log \varepsilon) \\ &= -F(x)x^{-1} + \left(\sum_{a_k \leq x} a_k^{-1} - \log x \right) + \log \varepsilon. \end{aligned}$$

Since Assumption 1 implies $\int_{\varepsilon}^{\infty} |F(t)t^{-1} - 1|t^{-1}dt < \infty$ and that $\lim_{x \rightarrow \infty} F(x)x^{-1} = 1$, the term $\lim_{x \rightarrow \infty} [\sum_{a_k \leq x} a_k^{-1} - \log x]$ of the last right-hand side of the above also has a limit as $x \rightarrow \infty$. We thus have

$$\int_{\varepsilon}^{\infty} (F(t)t^{-1} - 1)t^{-1}dt = -1 + \log \varepsilon + \lim_{x \rightarrow \infty} \left[\sum_{a_k \leq x} a_k^{-1} - \log x \right]. \quad (5)$$

Since we have

$$\begin{aligned} \liminf_{x \rightarrow \infty} \left[\sum_{a_k \leq x} a_k^{-1} - \log F(x) \right] &\leq \liminf_{n \rightarrow \infty} \left[\sum_{k=1}^n a_k^{-1} - \log n \right] \\ &\leq \limsup_{n \rightarrow \infty} \left[\sum_{k=1}^n a_k^{-1} - \log n \right] \\ &\leq \limsup_{x \rightarrow \infty} \left[\sum_{a_k \leq x} a_k^{-1} - \log F(x) \right], \end{aligned}$$

and since (3) implies

$$\begin{aligned} \liminf_{x \rightarrow \infty} \left[\sum_{a_k \leq x} a_k^{-1} - \log F(x) \right] &= \limsup_{x \rightarrow \infty} \left[\sum_{a_k \leq x} a_k^{-1} - \log F(x) \right] \\ &= \lim_{x \rightarrow \infty} \left[\sum_{a_k \leq x} a_k^{-1} - \log x \right], \end{aligned}$$

we know (4) is valid. Q.E.D.

Proposition 1 (cf. [4] Theorem 2) *$z(s)$ is analytically continued to a meromorphic function in $\operatorname{Re} s > 1 - \delta$ with a unique single pole at $s = 1$, whose residue is 1. In addition, the 'finite part' of $z(s)$ at the pole is equal to q , i.e.,*

$$\lim_{s \rightarrow 1} \left[z(s) - \frac{1}{s-1} \right] = q. \quad (6)$$

Proof. Let $\sigma := \operatorname{Re} s > 1$ and let $0 < \varepsilon < a_1$. By integration by parts,

$$\begin{aligned} \sum_{a_n \leq x} a_n^{-s} &= \int_{\varepsilon}^x t^{-s}dF(t) = F(x)x^{-s} + s \int_{\varepsilon}^x F(t)t^{-s-1}dt \\ &= O(x^{1-\sigma}) + s \int_{\varepsilon}^x (F(t) - t)t^{-s-1}dt + s \int_{\varepsilon}^x t^{-s}dt \\ &= O(x^{1-\sigma}) + s \int_{\varepsilon}^x (F(t) - t)t^{-s-1}dt + \frac{s\varepsilon^{-s+1}}{s-1} - \frac{sx^{-s+1}}{s-1}. \end{aligned}$$

Letting $x \rightarrow \infty$, we have

$$\begin{aligned} z(s) &= \frac{s\varepsilon^{-s+1}}{s-1} + s \int_{\varepsilon}^{\infty} (F(t) - t)t^{-s-1} dt \\ &= \frac{1}{s-1} + \frac{s\varepsilon^{-s+1} - 1}{s-1} + s \int_{\varepsilon}^{\infty} (F(t)t^{-1} - 1)t^{-s} dt. \end{aligned}$$

This expression and Assumption 1 implies that $z(s)$ is analytically continued to a meromorphic function in $\operatorname{Re} s > 1 - \delta$ with a unique single pole at $s = 1$, whose residue is 1. Moreover

$$\lim_{s \rightarrow 1} \left[z(s) - \frac{1}{s-1} \right] = 1 - \log \varepsilon + \int_{\varepsilon}^{\infty} (F(t)t^{-1} - 1)t^{-1} dt.$$

Then (5) shows that

$$\lim_{s \rightarrow 1} \left[z(s) - \frac{1}{s-1} \right] = \lim_{x \rightarrow \infty} \left[\sum_{a_k \leq x} a_k^{-1} - \log x \right] = q.$$

Q.E.D.

It is easy to see that the corresponding Hurwitz zeta function

$$z(s, x) := \sum_{k=1}^{\infty} (a_k + x)^{-s}, \quad x > -a_1,$$

is also analytically continued to a meromorphic function in $\operatorname{Re} s > 1 - \delta$ with a unique single pole at $s = 1$, whose residue is 1 (cf. [4] Theorem 1).

However, in general, $z(s)$ and $z(s, x)$ do not necessarily become holomorphic at $s = 0$. Indeed, for the existence of $z'(0)$, the integral $\int_{\varepsilon}^{\infty} (F(t) - t)t^{-1} dt$ should be convergent, which Assumption 1 does not assure. Nevertheless their difference becomes holomorphic at $s = 0$.

Proposition 2 *For each $x > -a_1$, the difference function $g(s, x) := z(s) - z(s, x)$ is analytically continued to a holomorphic function in $\operatorname{Re} s > -\delta$.*

Proof. Since Proposition 1 implies that $sz(s+1)$ is holomorphic in $\operatorname{Re} s > -\delta$, it is enough to show that

$$h(s) := g(s, x) - sz(s+1)x$$

is holomorphic in $\operatorname{Re} s > -\delta$.

First, $h(s)$ is expressed in the following series in $\operatorname{Re} s > 1$.

$$h(s) = \sum_{k=1}^{\infty} a_k^{-s} - \sum_{k=1}^{\infty} (a_k + x)^{-s} - s \sum_{k=1}^{\infty} a_k^{-s-1} x.$$

Suppose $|x| < a_{k_0}$. Then applying the Taylor expansion (negative binomial theorem)

$$\begin{aligned} (a_k + x)^{-s} &= a_k^{-s} \sum_{j=0}^{\infty} \binom{s+j-1}{j} \left(\frac{-x}{a_k} \right)^j \\ &= a_k^{-s} + \lambda s a_k^{-s-1} + a_k^{-s} \sum_{j=2}^{\infty} \binom{s+j-1}{j} \left(\frac{-x}{a_k} \right)^j, \quad k \geq k_0, \end{aligned} \quad (7)$$

which converges absolutely, we see

$$h(s) = -s \sum_{k=1}^{k_0-1} (a_k + x)^{-s} - s \sum_{k=k_0}^{\infty} a_k^{-s} \sum_{j=2}^{\infty} \frac{(s+1)(s+2)\cdots(s+j-1)}{j!} \left(\frac{-x}{a_1}\right)^j \left(\frac{a_1}{a_k}\right)^j. \quad (8)$$

Since

$$\begin{aligned} & \sum_{k=k_0}^{\infty} \left| a_k^{-s} \sum_{j=2}^{\infty} \frac{(s+1)(s+2)\cdots(s+j-1)}{j!} \left(\frac{-x}{a_1}\right)^j \left(\frac{a_1}{a_k}\right)^j \right| \\ & \leq \sum_{k=k_0}^{\infty} a_k^{-\operatorname{Re} s} \left(\frac{a_1}{a_k}\right)^2 \sum_{j=2}^{\infty} \left| \frac{(s+1)(s+2)\cdots(s+j-1)}{j!} \right| \left(\frac{-x}{a_1}\right)^j \\ & = a_1^2 \sum_{k=k_0}^{\infty} a_k^{-\operatorname{Re} s-2} \sum_{j=2}^{\infty} \left| \frac{(s+1)(s+2)\cdots(s+j-1)}{j!} \right| \left(\frac{-x}{a_1}\right)^j \end{aligned}$$

is finite in $\operatorname{Re} s > -1$ by Lemma 1, $h(s)$ becomes holomorphic in $\operatorname{Re} s > -1$. Q.E.D.

Definition 2 We define the zeta regularized product of $\prod_{k=1}^{\infty} (1 + \frac{x}{a_k})$ by

$$z\text{-}\prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right) := \exp(g'(0, x)). \quad (9)$$

Remark 2 If $a = \{a_k\}_{k=1}^{\infty}$ is zeta regularizable, we have

$$z\text{-}\prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right) = \frac{z\text{-}\prod_{k=1}^{\infty} (a_k + x)}{z\text{-}\prod_{k=1}^{\infty} a_k} = \exp(z'(0) - z'(0, x)).$$

2.2 Generalized Wallis formula

Proposition 3 (Weierstrass' infinite product formula, [4] Theorem 2, [6])

$$z\text{-}\prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right) = e^{qx} \prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right) \exp\left(-\frac{x}{a_k}\right).$$

Proof. Noting $\lim_{s \rightarrow 0} sz(s+1) = 1$, we first calculate $h'(0)$.

$$\begin{aligned} h'(0) &= g'(0, x) - \lim_{s \rightarrow 0} \frac{sz(s+1) - 1}{x} \lambda \\ &= g'(0, x) - \lim_{s \rightarrow 0} \left[z(s+1) - \frac{1}{s} \right] x \\ &= g'(0, x) - qx \quad (\text{cf. (6)}). \end{aligned} \quad (10)$$

On the other hand, (8) implies $h(0) = 0$ and so that $h'(0) = \lim_{s \rightarrow 0} h(s)/s$. Therefore

$$\begin{aligned} h'(0) &= - \sum_{k=1}^{\infty} \sum_{j=2}^{\infty} \frac{(j-1)!}{j!} \left(\frac{-x}{a_k}\right)^j = \sum_{k=1}^{\infty} \left[- \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{-x}{a_k}\right)^j - \frac{x}{a_k} \right] \\ &= \sum_{k=1}^{\infty} \left[\log \left(1 + \frac{x}{a_k}\right) - \frac{x}{a_k} \right]. \end{aligned}$$

This and (10) imply that

$$g'(0, x) = qx + \sum_{k=1}^{\infty} \left[\log \left(1 + \frac{x}{a_k} \right) - \frac{x}{a_k} \right]. \quad (11)$$

Plugging this into the exponential function, we finally obtain

$$z\text{-}\prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k} \right) = e^{qx} \prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k} \right) e^{-\frac{x}{a_k}}.$$

Q.E.D.

Theorem 1 (Generalized Wallis formula)

$$z\text{-}\prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k} \right) = \lim_{n \rightarrow \infty} n^{-x} \prod_{k=1}^n \left(1 + \frac{x}{a_k} \right). \quad (12)$$

Remark 3 For the special case where $a_k = k$, $k = 1, 2, \dots$, and $x = -1/2$, we have

$$\begin{aligned} z\text{-}\prod_{k=1}^{\infty} \left(1 - \frac{1}{2k} \right) &= \Gamma(1/2)^{-1} = \pi^{-1/2}, \\ n^{1/2} \prod_{k=1}^n \left(1 - \frac{1}{2k} \right) &= n^{1/2} \binom{2n}{n} 2^{-2n}. \end{aligned}$$

So (12) implies now the classical Wallis formula.

Proof of Theorem 1. From (4) and Proposition 3, it follows that

$$\begin{aligned} z\text{-}\prod_{k=1}^{\infty} \left(1 + \frac{\lambda}{a_k} \right) &= \lim_{n \rightarrow \infty} \exp \left((a_1^{-1} + a_2^{-1} + \dots + a_n^{-1} - \log n)x \right) \prod_{k=1}^n \left(1 + \frac{x}{a_k} \right) e^{-\frac{x}{a_k}} \\ &= \lim_{n \rightarrow \infty} n^{-x} \prod_{k=1}^n \left(1 + \frac{x}{a_k} \right). \end{aligned}$$

Q.E.D.

By definition, $z\text{-}\prod(1 + \frac{x}{a_k})$ is neither 0 nor infinite. Consequently, Proposition 3 and Theorem 1 have substantial meaning.

Example 1 The square of the classical Wallis formula is in fact a zeta regularized product:

$$\pi^{-1} = z\text{-}\prod_{k=1}^{\infty} \left(1 - \frac{1}{2k} \right)^2 = z\text{-}\prod_{k=1}^{\infty} \left(1 - \frac{1}{a_k} \right),$$

where $\frac{1}{a_k} = \frac{1}{k} - \frac{1}{4k^2}$ or

$$a_k = \frac{1}{\frac{1}{k} - \frac{1}{4k^2}} = k + \frac{1}{4} + \frac{1}{4(4k-1)}, \quad k = 1, 2, \dots,$$

which satisfies Assumption 1. Then let us show that

$$q = \lim_{s \rightarrow 1} \left(z(s) - \frac{1}{s-1} \right) = \gamma - \frac{\pi^2}{24}.$$

Since

$$\begin{aligned}\pi^{-1} &= \lim_{n \rightarrow \infty} n \prod_{k=1}^n \left(1 - \frac{1}{2k}\right)^2 \\ &= \prod_{k=1}^{\infty} \left(1 - \frac{1}{2k}\right)^2 \exp\left(\frac{1}{k} - \frac{1}{4k^2}\right) \cdot \lim_{n \rightarrow \infty} n \prod_{k=1}^n \exp\left(-\frac{1}{k} + \frac{1}{4k^2}\right),\end{aligned}$$

we must have

$$e^{-q} = \lim_{n \rightarrow \infty} n \prod_{k=1}^n \exp\left(-\frac{1}{k} + \frac{1}{4k^2}\right),$$

namely,

$$q = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{4k^2}\right) - \log n \right] = \gamma - \sum_{k=1}^{\infty} \frac{1}{4k^2} = \gamma - \frac{\pi^2}{24}.$$

Remark 4 In case $\sum_{k=1}^{\infty} a_k^{-1} < \infty$, $z(s)$ becomes finite at $s = 1$, so that its 'finite part' q at $s = 1$ is, of course, $\sum_{k=1}^{\infty} a_k^{-1}$. Then it holds that $z \cdot \prod_{k=1}^{\infty} (1 + \frac{x}{a_k}) = \prod_{k=1}^{\infty} (1 + \frac{x}{a_k})$. Let us show it.

(i) For a finite sequence $a = \{a_k\}_{k=1}^N$,

$$z(s) := \sum_{k=1}^N a_k^{-s}, \quad z(s, x) := \sum_{k=1}^N (a_k + x)^{-s}, \quad 0 \leq \lambda < a_1,$$

which are entire functions, it is easy to see that $\exp(z'(0) - z'(0, x)) = \prod_{k=1}^N (1 + \frac{x}{a_k})$.

(ii) For an infinite sequence $a = \{a_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} a_k^{-1} < \infty$,

$$z(s) := \sum_{k=1}^{\infty} a_k^{-s}, \quad z(s, x) := \sum_{k=1}^{\infty} (a_k + x)^{-s}, \quad 0 \leq \lambda < a_1,$$

are finite at $s = 1$, but we do not know whether they are analytically continued beyond $\operatorname{Re} s > 1$. Nevertheless their difference $g(s, x) := z(s) - z(s, x)$ is analytically continued to a holomorphic function in $\operatorname{Re} s > -1$, which is shown in a similar way as Proposition 3. Indeed, by (7),

$$g(s, x) = - \sum_{k=1}^{\infty} a_k^{-s} \sum_{j=1}^{\infty} \binom{s+j-1}{j} \left(\frac{-x}{a_k}\right)^j,$$

from which it follows that

$$g'(0, x) = - \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{(j-1)!}{j!} \left(\frac{-x}{a_k}\right)^j = - \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{-x}{a_k}\right)^j = \sum_{k=1}^{\infty} \log \left(1 + \frac{x}{a_k}\right).$$

Thus

$$\exp(g'(0, x)) = \prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right).$$

2.3 Generalized digamma function

If $a = \{a_k\}_{k=1}^{\infty}$ satisfies Assumption 1, so does $\{a_k + x\}_{k=1}^{\infty}$ for each $x > 0$, and hence we can define

$$q(x) := \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{x + a_k} - \log n \right), \quad x > 0.$$

Comparing with (2), we can say that $-q(x)$ is a generalized digamma function.

Suppose that unit electric charges are located at each point of $\{a_k\}_{k=1}^{\infty}$ on the real line $\mathbf{R}$. Then $q(x)$ can be regarded as the renormalized Coulomb potential at $-x$ caused by those electric charges. Indeed, we see

$$\begin{aligned} q'(x) &= \frac{d}{dx} \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{x + a_k} - \log n \right) \\ &= \frac{d}{dx} \sum_{k=1}^{\infty} \left(\frac{1}{x + a_k} - \log \frac{k+1}{k} \right) \\ &= \sum_{k=1}^{\infty} \frac{d}{dx} \left(\frac{1}{x + a_k} - \log \frac{k+1}{k} \right) \\ &= - \sum_{k=1}^{\infty} \frac{1}{(x + a_k)^2} = -z(2, x). \end{aligned}$$

By (11), we have

$$q(x) = - \frac{d}{ds} (z(s, x) - z(s, x-1)) \Big|_{s=0} + \sum_{k=1}^{\infty} \left[\frac{1}{x + a_k} + \log \left(1 - \frac{1}{x + a_k} \right) \right]. \quad (13)$$

Applying this formula to the sequence $\{a_k = k\}_{k=1}^{\infty}$, we have

$$-\psi(x+1) = -\log x + \sum_{k=1}^{\infty} \left[\frac{1}{x+k} + \log \left(1 - \frac{1}{x+k} \right) \right], \quad (x > 0), \quad (14)$$

because

$$\frac{d}{ds} (\zeta(s, x) - \zeta(s, x-1)) \Big|_{s=0} = \frac{d}{ds} (-x^{-s}) \Big|_{s=0} = \log x.$$

Theorem 2 For any sequence $a = \{a_k\}_{k=1}^{\infty}$ satisfying Assumption 1 and

$$a_k k^{-1} = 1 + O(k^{-\delta'}), \quad k \rightarrow \infty, \quad \delta' > 0,$$

we have

$$q(x) = -\log x + O(x^{-1}), \quad x \rightarrow \infty.$$

Proof. From (14) it follows that

$$-\psi(x+1) = -\log x + O(x^{-\min(1, \delta')}), \quad x \rightarrow \infty.$$

On the other hand, for $x > 0$, we have

$$\begin{aligned} q(x) + \psi(x+1) &= \sum_{k=1}^{\infty} \left(\frac{1}{x+k} - \frac{1}{x+a_k} \right) \\ &= \sum_{k=1}^{\infty} \frac{a_k - k}{(x+k)(x+a_k)} \\ &= \sum_{k=1}^{\infty} \frac{O(k^{1-\delta'})}{(x+k)(x+a_k)} = O(x^{-\delta'}), \quad x \rightarrow \infty. \end{aligned}$$

Q.E.D.

2.4 Generalized Gamma functions

The following lemma is easily derived from Theorem 1.

Lemma 3 For each $n \in \mathbb{N}$,

$$z\text{-}\prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right) = \prod_{k=1}^n \left(1 + \frac{x}{a_k}\right) z\text{-}\prod_{k=n+1}^{\infty} \left(1 + \frac{x}{a_k}\right).$$

Now, recalling $z\text{-}\prod_{k=1}^{\infty} (1 + \frac{x}{k}) = \Gamma(1+x)^{-1}$, Lemma 3 implies

$$\begin{aligned} \Gamma(n+1+x) &= \Gamma(1+x) \prod_{k=1}^n (k+x) = \frac{\prod_{k=1}^n (k+x)}{z\text{-}\prod_{k=1}^{\infty} \left(1 + \frac{x}{k}\right)} \\ &= \frac{n! \prod_{k=1}^n \left(1 + \frac{x}{k}\right)}{z\text{-}\prod_{k=1}^{\infty} \left(1 + \frac{x}{k}\right)} = \frac{n!}{z\text{-}\prod_{k=n+1}^{\infty} \left(1 + \frac{x}{k}\right)}. \end{aligned}$$

Therefore

$$\Gamma(x) = \frac{n!}{z\text{-}\prod_{k=n+1}^{\infty} \left(1 + \frac{x-n-1}{k}\right)}. \quad (15)$$

We consider an analogy of this.

Definition 3 For each $n \in \mathbb{N}$, we define

$$G^{(n+1)}(x) := \frac{\prod_{k=1}^n a_k}{z\text{-}\prod_{k=n+1}^{\infty} \left(1 + \frac{x-a_{n+1}}{a_k}\right)}. \quad (16)$$

Obviously, we have

$$G^{(n+1)}(a_{n+1}) = \prod_{k=1}^n a_k \quad (17)$$

$$z\text{-}\prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right) = \frac{\prod_{k=1}^n (a_k + x)}{G^{(n+1)}(a_{n+1} + x)}, \quad n = 1, 2, \dots \quad (18)$$

By (15), when $a_k = k$ for each $k \in \mathbb{N}$, $G^{(n+1)}(x) = \Gamma(x)$ holds for any $n \in \mathbb{N}$. In general, for $a = \{a_k\}_{k=1}^{\infty}$ satisfying the following assumption, the corresponding $G^{(n+1)}$ has a Gamma function-like property.

Assumption 2 There exists some $\alpha > 0$ such that $a_{k+1} - a_k = 1 + O(k^{-\alpha})$, $k \rightarrow \infty$.

Theorem 3 If $a = \{a_k\}_{k=1}^{\infty}$ satisfies Assumption 1 and Assumption 2, it holds for any $j \in \mathbb{N}$ that

$$G^{(n+1)}(a_{n+1-j}) \sim \prod_{k=1}^{n-j} a_k, \quad n \rightarrow \infty. \quad (19)$$

Here “ $\sim$ ” indicates that the ratio of the both hand sides tends to 1 in the specified limit.

Proof. For $j < n$,

$$\begin{aligned} G^{(n+1)}(a_{n+1-j}) &= \frac{\prod_{k=1}^n a_k}{z^{-} \prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right)} \\ &= \frac{\prod_{k=1}^{n-j} a_k}{\prod_{k=n+1-j}^n a_k^{-1} z^{-} \prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right)}. \end{aligned}$$

Therefore it is sufficient to show that

$$\lim_{n \rightarrow \infty} \prod_{k=n+1-j}^n a_k^{-1} z^{-} \prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right) = 1. \quad (20)$$

By Proposition 3, we have

$$\begin{aligned} z^{-} \prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right) &= \exp(-q_{n+1}(a_{n+1} - a_{n+1-j})) \\ &\quad \times \prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right) \exp\left(\frac{a_{n+1} - a_{n+1-j}}{a_k}\right), \end{aligned}$$

where

$$\begin{aligned} q_{n+1} &:= \lim_{N \rightarrow \infty} \left[\sum_{k=n+1}^N a_k^{-1} - \log(N - n + 1) \right] \\ &= q - \sum_{k=1}^n a_k^{-1} = -\log n + o(1), \quad n \rightarrow \infty. \end{aligned}$$

Then Assumption 2 implies that

$$\begin{aligned} \exp(-q_{n+1}(a_{n+1} - a_{n+1-j})) &= n^{j-O(n^{-\alpha})} e^{o(1)(-j+O(n^{-\alpha}))} \\ &\sim n^j, \quad n \rightarrow \infty. \end{aligned}$$

The following is obvious.

$$\prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right) \exp\left(\frac{a_{n+1} - a_{n+1-j}}{a_k}\right) \rightarrow 1, \quad n \rightarrow \infty.$$

From these, it follows that

$$z^{-\prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right)} \sim n^j, \quad n \rightarrow \infty.$$

And hence

$$\begin{aligned} & \lim_{n \rightarrow \infty} \prod_{k=n+1-j}^n a_k z^{-\prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right)} \\ &= \lim_{n \rightarrow \infty} \left(\prod_{k=n+1-j}^n a_k \right) \cdot n^j = \prod_{k=0}^{j-1} \left(\lim_{n \rightarrow \infty} a_{n-k}^{-1} n \right) = 1. \end{aligned}$$

Q.E.D.

If $a = \{a_k\}_{k=1}^{\infty}$ satisfies Assumption 1 and Assumption 2, the expression (18) and Theorem 3 can be used for numerical evaluation of $z^{-\prod_{k=1}^{\infty} (1 + \frac{x}{a_k})}$ in some cases. The method is as follows: First, for a suitably large n and $j_0 < n$, construct a Lagrange's polynomial $h_a^{(n, j_0)}(x)$ of degree $(j_0 - 1)$ that interpolates the points

$$(x_j, y_j) = \left(a_{n+1-j}, \sum_{k=1}^{n-j} \log a_k \right), \quad j = 0, 1, \dots, j_0 - 1.$$

Substituting $h_a^{(n, j_0)}(x)$ for $\log G^{(n+1)}(x)$ in (18), we calculate

$$c^{(n, n', j_0)}(x) := \frac{\prod_{k=1}^{n'-1} (a_k + x)}{\exp \left(h_a^{(n, j_0)}(a_{n'} + x) \right)}$$

as an approximated value of $z^{-\prod_{k=1}^{\infty} (1 + \frac{x}{a_k})}$. (In doing this, to prevent overflow or underflow, all calculations should be done by taking logarithm, i.e., we calculate

$$\sum_{k=1}^{n'-1} \log(a_k + x) - h_a^{(n, j_0)}(a_{n'} + x)$$

then plug the result into the exponential function.) Here n' is a suitable integer between $n - j_0$ and n . Probably, it is better to pick up n' from the middle of the interval $[n + 1 - j_0, n]$.

Example 2 Let us consider the square of the Wallis formula again. The sequence dealt in Example 1, i.e., $a_k = k + \frac{1}{4} + \frac{1}{4(4k-1)}$ satisfies Assumption 2 for $\alpha = 2$, so that we can apply the above method to get an approximated value of $z^{-\prod_{k=1}^{\infty} (1 - \frac{1}{a_k})}$.

For $n = 30, 300, 3000$, we constructed Lagrange polynomials $h_a^{(n, 5)}$, and calculated $c^{(n, n-2, 5)}(1)$, which are listed in the table below. Since the true value is

$$1/\pi = 1/3.14159265\dots,$$

roughly speaking, the error decreases at the rate of $O(n^{-2})$.

For comparison, we also calculated $w(n) := n \prod_{k=1}^n (1 - \frac{1}{a_k})$ as approximated values due to the Wallis formula. This time, the error decreases at the rate of $O(n^{-1})$.

n	$c^{(n,n-2,5)}(1)$	$w(n)$
30	1/3.14059	1/3.16788
300	1/3.1415789	1/3.14421
3000	1/3.141592468	1/3.14185

In this way, $c^{(n,n-2,5)}(1)$ is much better than $w(n)$. But this example may be a special case, and since we have not established a precise error estimate, we do not know if our method is valid for general cases.

3 Randomized special functions

By randomizing the objects in the previous sections, we can find a new type of limit theorems in probability theory.

3.1 In the case of Poisson process

Let $\{\xi_i\}_{i=0}^{\infty}$ be a positive i.i.d. random variables whose common distribution is the exponential distribution with parameter 1, i.e.,⁴

$$P(\xi_i \leq x) = \int_0^x e^{-t} dt = 1 - e^{-x}, \quad x \geq 0,$$

and set

$$X = \{X_k\}_{k=1}^{\infty} := \{\xi_1 + \cdots + \xi_k\}_{k=1}^{\infty}.$$

Then by virtue of the strong law of large numbers, the sequence $\{b + X_k\}_{k=1}^{\infty}$, $b > 0$, satisfies Assumption 1 almost surely. Note that

$$\eta(t) := F_X(t) = \#\{k | X_k \leq t\}, \quad t \geq 0,$$

is a standard Poisson process.

3.1.1 Randomization of the Wallis formula

First let us calculate the distribution of the following random variable.

$$W(b, \lambda) := z \cdot \prod_{k=1}^{\infty} \left(1 + \frac{\lambda}{b + X_k}\right), \quad \lambda \geq -b, b > 0.$$

Theorem 4 The n -th moment of $z(b, \lambda)$ is calculated as follows.⁵

$$\mathbf{E}[W(b, \lambda)^n] = \begin{cases} b^{-\lambda}, & n = 1, \\ b^{-n\lambda} \exp\left(\sum_{r=2}^n \binom{n}{r} \frac{\lambda^r}{(r-1)b^{r-1}}\right), & n = 2, 3, \dots \end{cases}$$

Lemma 4 (Durrett[1], (5.1) Theorem, Chapt.3.) *Under the conditional probability measure $P(\cdot | \eta(t) = N)$, $t > 0$, the distribution of $\{X_k\}_{k=1}^N$ coincides with that of the order statistics of N independent uniformly distributed random variables in $[0, t]$.*

⁴ P stands for *probability*.

⁵ $\mathbf{E}$ stands for *expectation*.

Sketch of Proof of Theorem 4.

Step 1. Let $\{X_{t,k}\}_{k=1}^{\infty}$ be independent uniformly distributed random variables in $[0, t]$. First, we define a random variable

$$W_{t,N,\lambda} := \prod_{k=1}^N \left(1 + \frac{\lambda}{b + X_{t,k}}\right), \quad N \in \mathbb{N}.$$

and calculate its moments.

$$\begin{aligned} \mathbf{E}[(W_{t,N,\lambda})^n] &= \prod_{k=1}^N \mathbf{E} \left[\left(1 + \frac{\lambda}{b + X_{t,k}}\right)^n \right] = \left(\int_0^t \left(1 + \frac{\lambda}{b + y}\right)^n \frac{dy}{t} \right)^N \\ &= \left(\sum_{r=0}^n \binom{n}{r} \lambda^r \frac{1}{t} \int_0^t \frac{dy}{(b + y)^{r+1}} \right)^N \\ &= \left(1 + \frac{n\lambda}{t} [\log(b + y)]_0^t + \sum_{r=2}^n \binom{n}{r} \lambda^r \frac{1}{t} \left[\frac{-1}{(r-1)(b + y)^{r-1}} \right]_0^t \right)^N \\ &= \left(1 + \frac{n\lambda}{t} (\log(b + t) - \log b) + \frac{1}{t} \sum_{r=2}^n \binom{n}{r} \frac{\lambda^r}{r-1} \left[\frac{1}{b^{r-1}} - \frac{1}{(b + t)^{r-1}} \right] \right)^N \\ &= \left(1 + \frac{n\lambda}{t} (\log(b + t) - \log b) + \frac{1}{t} \sum_{r=2}^n \binom{n}{r} \frac{\lambda^r}{(r-1)b^{r-1}} + O(t^{-2}) \right)^N \\ &= \left(1 + \frac{n\lambda \log(b + t)}{t} - \frac{n\lambda \log b + C}{t} + O(t^{-2}) \right)^N, \quad t \rightarrow \infty, \end{aligned}$$

where

$$C := \sum_{r=2}^n \binom{n}{r} \frac{\lambda^r}{(r-1)b^{r-1}}.$$

Step 2. Since $P(\eta(t) = N) = t^N e^{-t} / N!$, Lemma 4 implies that

$$\begin{aligned} \mathbf{E} \left[\eta(t)^{-n\lambda} \prod_{k=1}^{\eta(t)} \left(1 + \frac{\lambda}{1 + X_k}\right)^n \right] &= \sum_{N=0}^{\infty} \mathbf{E} \left[N^{-n\lambda} \prod_{k=1}^N \left(1 + \frac{\lambda}{1 + X_k}\right)^n ; \eta(t) = N \right] \\ &= \sum_{N=0}^{\infty} \mathbf{E} \left[N^{-n\lambda} \prod_{k=1}^N \left(1 + \frac{\lambda}{1 + X_k}\right)^n \middle| \eta(t) = N \right] \frac{t^N e^{-t}}{N!} \\ &= \sum_{N=0}^{\infty} N^{-n\lambda} \mathbf{E}[(W_{t,N,\lambda})^n] \frac{t^N e^{-t}}{N!}. \end{aligned}$$

A change of variables $u := N/t$ shows

$$\mathbf{E} \left[\eta(t)^{-n\lambda} \prod_{k=1}^{\eta(t)} \left(1 + \frac{\lambda}{1 + X_k}\right)^n \right] = \sum_{u \in \frac{1}{t}\mathbb{N}} (tu)^{-n\lambda} \mathbf{E}[(W_{t,tu,\lambda})^n] \frac{t^{tu} e^{-t}}{(tu)!}.$$

Since the distribution of $\eta(t)/t$ is convergent to the Dirac measure $\delta_1(du)$, we see

$$\mathbf{E}[W(b, \lambda)^n] = \mathbf{E} \left[\lim_{t \rightarrow \infty} \eta(t)^{-n\lambda} \prod_{k=1}^{\eta(t)} \left(1 + \frac{\lambda}{1 + X_k}\right)^n \right]$$

$$\begin{aligned}
&= \lim_{t \rightarrow \infty} \mathbf{E} \left[\eta(t)^{-n\lambda} \prod_{k=1}^{\eta(t)} \left(1 + \frac{\lambda}{1 + X_k} \right)^n \right] \\
&= \lim_{t \rightarrow \infty} \sum_{u \in \frac{1}{t}\mathbf{N}} (tu)^{-n\lambda} \mathbf{E}[(W_{t,tu,\lambda})^n] \frac{t^{tu} e^{-t}}{(tu)!} \\
&= \lim_{t \rightarrow \infty} t^{-n\lambda} \mathbf{E}[(W_{t,t,\lambda})^n] = b^{-n\lambda} e^C.
\end{aligned}$$

The above calculation can be rigorously justified.

Q.E.D.

In case $b = 1$, if $|\lambda| \ll 1$, then the n -th moments of $W(1, \lambda)$ is approximately equal to $\exp(\frac{n(n-1)}{2} \lambda^2)$, which is nothing but the n -th moment of a log normal distribution, more precisely, the distribution of the random variable e^Y where Y is distributed as $\mathcal{N}(-\frac{\lambda^2}{2}, \lambda^2)$. Hence when $|\lambda| \ll 1$, the distribution of $W(1, \lambda)$ is close to that of e^Y .⁶

3.1.2 Random digamma function

Proposition 3 implies that

$$\log W(1, \lambda) = Q\lambda + \sum_{k=1}^{\infty} \left[\log \left(1 + \frac{\lambda}{1 + X_k} \right) - \frac{\lambda}{1 + X_k} \right],$$

where

$$Q = \lim_{N \rightarrow \infty} \left[\sum_{k=1}^N \frac{1}{1 + X_k} - \log N \right].$$

This limit exists a.s.

Now, we have

$$\mathbf{E}[Q] = 0, \quad \mathbf{E}[Q^2] = 1,$$

which is shown in the following way. First, it is easy to see that

$$\sum_{k=1}^{\infty} \left[\log \left(1 + \frac{\lambda}{1 + X_k} \right) - \frac{\lambda}{1 + X_k} \right] = O(\lambda^2), \quad \lambda \rightarrow 0.$$

Hence we see

$$\mathbf{E}[\log W(1, \lambda)] = -\mathbf{E}[Q]\lambda + O(\lambda^2), \quad \lambda \rightarrow 0.$$

On the other hand, since the mean of $\log W(1, \lambda)$ is approximately equal to $-\lambda^2/2$, we see $\mathbf{E}[Q] = 0$. And the 2-nd moment of $\log W(1, \lambda)$ is approximately equal to λ^2 , so we see $\mathbf{E}[Q^2] = 1$.

Suppose $|\lambda| \ll 1$. Since $-Q\lambda$ is the main part of $\log W(1, \lambda)$, and $\log W(1, \lambda)$ is approximately distributed as $\mathcal{N}(-\frac{\lambda^2}{2}, \lambda^2)$, one may well expect that Q is distributed as $\mathcal{N}(0, 1)$. But although its distribution is close to $\mathcal{N}(0, 1)$, it is not exactly distribute as $\mathcal{N}(0, 1)$. In deed we have $\mathbf{E}[Q^3] = 1/2$.

Let us investigate a little bit more general case. Let

$$Q(x) := \lim_{N \rightarrow \infty} \left[\sum_{k=1}^N \frac{1}{x + X_k} - \log N \right], \quad x > 0.$$

Comparing with (2), we can say that $-Q(x)$ is a random digamma function.

⁶This may hold in any case where $\{X_k\}_{k=1}^{\infty}$ is the partial sum of positive i.i.d. random variables with mean 1.

Theorem 5 The mean of $Q(x)$ is $\mathbf{E}[Q(x)] = -\log x$, and the centered moments of $Q(x)$ are

$$\mathbf{E}[(Q(x) + \log x)^n] = n! \sum_{p=1}^{\lfloor \sqrt{2n} \rfloor} \sum_{\substack{2 \leq n_1 < \dots < n_p \\ k_1 n_1 + \dots + k_p n_p = n}} \frac{1}{x^{n-(k_1+\dots+k_p)}} \times \prod_{j=1}^p \frac{1}{k_j! (n_j!)^{k_j} (n_j - 1)^{k_j}}, \quad n = 2, 3, \dots \quad (21)$$

More concretely, the centered moments of order $1, 2, \dots, 6$ are

$$0, \quad \frac{1}{x}, \quad \frac{1}{2x^2}, \quad \frac{3}{x^2} + \frac{1}{3x^3}, \quad \frac{5}{x^3} + \frac{1}{4x^4}, \quad \frac{15}{x^3} + \frac{15}{2x^4} + \frac{1}{5x^5}.$$

Sketch of Proof. As in the previous section, let $\{X_{t,k}\}_{k=1}^N$ be i.i.d. uniform random variables in $[0, t]$. Define

$$Y_{t,N} := \sum_{k=1}^N \left(\frac{1}{x + X_{t,k}} - c(t) \right), \quad c(t) = \frac{1}{t} (\log(x+t) - \log x).$$

Then, when $t \rightarrow \infty$, we have

$$\mathbf{E} \left[\left(\frac{1}{x + X_{t,k}} - c(t) \right)^n \right] = \begin{cases} 0 & (n = 1) \\ \frac{1}{x^{n-1}(n-1)t} + O(t^{-2+\varepsilon}) & (n \geq 2) \end{cases} \quad (22)$$

Here $\varepsilon > 0$ can be arbitrarily small. Indeed, if we write the L.H.S. of the above expression by integrals,

$$\begin{aligned} &= \int_0^t \left(\frac{1}{x+y} - c(t) \right)^n \frac{dy}{t} \\ &= \sum_{r=0}^n \binom{n}{r} (-c(t))^{n-r} \frac{1}{t} \int_0^t (x+y)^{-r} dy \\ &= \sum_{r=0}^n \binom{n}{r} (-c(t))^{n-r} \times \begin{cases} 1 & (r=0) \\ c(t) & (r=1) \\ \frac{1}{(r-1)t} (x^{-r+1} - (x+t)^{-r+1}) & (r \geq 2) \end{cases} \\ &= c(t)^n - nc(t)^n + \sum_{r=2}^n \binom{n}{r} (-c(t))^{n-r} \frac{1}{(r-1)t} \left(\frac{1}{x^{r-1}} - \frac{1}{(x+t)^{r-1}} \right). \end{aligned}$$

Here we have $c(t) = O(t^{-1+\varepsilon})$, $t \rightarrow \infty$, and hence we obtain (22).

Under these preparations, we see

$$\begin{aligned} &\mathbf{E} [Y_{t,N}^n] \\ &= \sum_{n_1+n_2+\dots+n_N=n} \mathbf{E} \left[\prod_{k=1}^N \left(\frac{1}{x + X_{t,k}} - c(t) \right)^{n_k} \right] \\ &= \sum_{n_1+n_2+\dots+n_N=n} \prod_{k=1}^N \mathbf{E} \left[\left(\frac{1}{x + X_{t,k}} - c(t) \right)^{n_k} \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{p=1}^n \frac{N!}{(N-p)!} \sum_{\substack{2 \leq n_1 < \dots < n_p \\ k_1 n_1 + k_2 n_2 + \dots + k_p n_p = n}} \frac{1}{k_1! k_2! \dots k_p!} \cdot \frac{n!}{(n_1!)^{k_1} (n_2!)^{k_2} \dots (n_p!)^{k_p}} \times \\
&\quad \prod_{j=1}^p \mathbf{E} \left[\left(\frac{1}{x + X_{t,1}} - c(t) \right)^{n_j} \right]^{k_j} \\
&= \sum_{p=1}^n N(N-1) \dots (N-p+1) \times \\
&\quad \sum_{\substack{2 \leq n_1 < \dots < n_p \\ k_1 n_1 + k_2 n_2 + \dots + k_p n_p = n}} \frac{1}{k_1! k_2! \dots k_p!} \cdot \frac{n!}{(n_1!)^{k_1} (n_2!)^{k_2} \dots (n_p!)^{k_p}} \times \\
&\quad \prod_{j=1}^p \left(\frac{1}{x^{n_j-1} (n_j-1)t} + O(t^{-2+\varepsilon}) \right)^{k_j} \\
&= \sum_{p=1}^n \left(1 - \frac{1}{N} \right) \dots \left(1 - \frac{p-1}{N} \right) \cdot \left(\frac{N}{t} \right)^p \times \\
&\quad \sum_{\substack{2 \leq n_1 < \dots < n_p \\ k_1 n_1 + k_2 n_2 + \dots + k_p n_p = n}} \frac{1}{x^{n-(k_1+k_2+\dots+k_p)}} \cdot \frac{1}{k_1! k_2! \dots k_p!} \cdot \frac{n!}{(n_1!)^{k_1} (n_2!)^{k_2} \dots (n_p!)^{k_p}} \times \\
&\quad \prod_{j=1}^p \left(\frac{1}{n_j-1} + O(t^{-1+\varepsilon}) \right)^{k_j}.
\end{aligned}$$

From this, it follows similarly to the previous theorem that

$$\begin{aligned}
\mathbf{E}[(Q(x) + \log x)^n] &= \mathbf{E} \left[\lim_{t \rightarrow \infty} \left(\sum_{k=1}^{\eta(t)} \left(\frac{1}{x + X_k} - c(t) \right) \right)^n \right] \\
&= \lim_{t \rightarrow \infty} \mathbf{E} \left[\left(\sum_{k=1}^{\eta(t)} \left(\frac{1}{x + X_k} - c(t) \right) \right)^n \right] \\
&= \lim_{t \rightarrow \infty} \sum_{N=0}^{\infty} \mathbf{E} [Y_{t,N}^n] \frac{t^N e^{-t}}{N!} \\
&= \lim_{t \rightarrow \infty} \sum_{u \in \frac{1}{t} \mathbf{N}} \mathbf{E} [Y_{t,tu}^n] \frac{t^{tu} e^{-t}}{(tu)!} \left(= \lim_{t \rightarrow \infty} \mathbf{E} [Y_{t,t}^n] \right) \\
&= \sum_{p=1}^n \sum_{\substack{2 \leq n_1 < \dots < n_p \\ k_1 n_1 + k_2 n_2 + \dots + k_p n_p = n}} \frac{1}{x^{n-(k_1+k_2+\dots+k_p)}} \times \\
&\quad \frac{1}{k_1! k_2! \dots k_p!} \cdot \frac{n!}{(n_1!)^{k_1} (n_2!)^{k_2} \dots (n_p!)^{k_p}} \prod_{j=1}^p \frac{1}{(n_j-1)^{k_j}}.
\end{aligned}$$

Now it is easy to get (21).

Q.E.D.

The following theorem is an easy consequence of the law of iterated logarithm and Theorem 2.

Theorem 6 For any $\varepsilon > 0$, $Q(x) = -\log x + O(x^{-(1/2)+\varepsilon})$, $x \rightarrow \infty$, a.s.

Regarding this theorem as a law of large numbers, the following corresponds to the central limit theorem.

Theorem 7 *The distribution of $\sqrt{x}(Q(x) + \log x)$ converges to $\mathcal{N}(0, 1)$ as $x \rightarrow \infty$.*

Proof. The n -th moment of $\sqrt{x}(Q(x) + \log x)$ is

$$\mathbf{E} \left[(\sqrt{x}(Q(x) + \log x))^n \right] = n! \sum_{p=1}^{\lfloor \sqrt{2n} \rfloor} \sum_{\substack{2 \leq n_1 < \dots < n_p \\ k_1 n_1 + \dots + k_p n_p = n}} \frac{1}{x^{n/2 - (k_1 + \dots + k_p)}} \prod_{j=1}^p \frac{1}{k_j! (n_j!)^{k_j} (n_j - 1)^{k_j}}.$$

In the sum $\sum_{p=1}^{\lfloor \sqrt{2n} \rfloor}$ of the R.H.S., the sum for p such that $n/2 > k_1 + \dots + k_p$ converge to 0 as $x \rightarrow \infty$. Therefore as $x \rightarrow \infty$, what survive are the terms for p such that $n/2 = k_1 + \dots + k_p$ (from this n must be even), that is, for $p = 1$, $k_1 = n/2$, $n_1 = 2$. Hence we see that

$$\lim_{x \rightarrow \infty} \mathbf{E} \left[(\sqrt{x}(Q(x) + \log x))^n \right] = \frac{n!}{2^{n/2} (n/2)!}, \quad (n: \text{even}).$$

This is nothing but the n -th moment of $\mathcal{N}(0, 1)$.

Q.E.D.

Suppose that unit electric charges are located at each random point of $\{X_k\}_{k=1}^\infty$, the renormalized Coulomb potential $Q(x)$ at the location $-x$ is distributed approximately as $\mathcal{N}(-\log x, 1/x)$ if x is large.

Another proof of Theorem 7. Since the distribution function $F_{\{x+X_k\}}(t)$ corresponding to the sequence $\{x+X_k\}_{k=1}^\infty$ is exactly $\eta(t-x)$, (5) implies that

$$\begin{aligned} Q(x) &= -\log x + 1 + \int_x^\infty (\eta(t-x) - t) \frac{dt}{t^2} \\ &= -\log x + \int_0^\infty (\eta(t) - t) \frac{dt}{(t+x)^2}. \end{aligned}$$

Since $\mathbf{E}[\eta(t) - t] = 0$, we readily see $\mathbf{E}[Q(x)] = -\log x$. From the following expression

$$\sqrt{x}(Q(x) + \log x) = \int_0^\infty (\eta(t) - t) \frac{\sqrt{x}}{(t+x)^2} dt, \quad (23)$$

let us derive Theorem 7 by using the Lindeberg - Feller theorem ([2] Chapt.2 (4.5) Theorem).

First, note that $\tilde{\eta}(t) := \eta(t) - t$ is a martingale with mean 0. The Fubini theorem (or integration by parts formula) implies

$$\begin{aligned} \int_S^T \tilde{\eta}(t) \frac{\sqrt{x}}{(t+x)^2} dt &= \int_S^T \left(\tilde{\eta}(S) + \int_S^t d\tilde{\eta}(s) \right) \frac{\sqrt{x}}{(t+x)^2} dt \\ &= \int_S^T \left(\int_s^T \frac{\sqrt{x}}{(t+x)^2} dt \right) d\tilde{\eta}(s) + \tilde{\eta}(S) \int_n^T \frac{\sqrt{x}}{(t+x)^2} dt \\ &= \int_S^T \left(\frac{\sqrt{x}}{s+x} - \frac{\sqrt{x}}{T+x} \right) d\tilde{\eta}(s) + \left(\frac{\sqrt{x}}{S+x} - \frac{\sqrt{x}}{T+x} \right) \tilde{\eta}(S) \\ &= \int_S^T \frac{\sqrt{x}}{s+x} d\tilde{\eta}(s) - \frac{\sqrt{x}}{T+x} \tilde{\eta}(T) + \frac{\sqrt{x}}{S+x} \tilde{\eta}(S). \end{aligned}$$

Letting $S \rightarrow 0$, $T \rightarrow \infty$, we have the following expression.

$$\sqrt{x}(Q(x) + \log x) = \int_0^\infty \tilde{\eta}(t) \frac{\sqrt{x}}{(t+x)^2} dt = \int_0^\infty \frac{\sqrt{x}}{t+x} d\tilde{\eta}(t), \quad \text{a.s.}$$

Now put

$$\begin{cases} U_{n,m} := \int_{(m-1)^2}^{m^2} \frac{\sqrt{n}}{t+n} d\tilde{\eta}(t), & m = 1, 2, \dots, n-1, \\ U_{n,n} := \int_{(n-1)^2}^\infty \frac{\sqrt{n}}{t+n} d\tilde{\eta}(t). \end{cases}$$

Let us the triangular array $\{U_{n,m}\}_{1 \leq m \leq n, 1 \leq n}$ satisfies the Lindeberg - Feller's conditions.

Step 1. Since $\{\eta(t)\}_{t \geq 0}$ is an independent increment process, $\{U_{n,m}\}_{m=1}^n$ is independent sequence for each n .

Step 2. It holds (without letting $n \rightarrow \infty$) that

$$\begin{aligned} \sum_{m=1}^n \mathbf{E}[U_{n,m}^2] &= \sum_{m=1}^{n-1} \int_{(m-1)^2}^{m^2} \left(\frac{\sqrt{n}}{t+n} \right)^2 dt + \int_{(n-1)^2}^\infty \left(\frac{\sqrt{n}}{t+n} \right)^2 dt \\ &= \int_0^\infty \frac{n}{(t+n)^2} dt = 1. \end{aligned}$$

Step 3. Now, to prove the theorem, it is sufficient to show the last Lindeberg - Feller's condition: for any $\varepsilon > 0$ we have

$$\lim_{n \rightarrow \infty} \sum_{m=1}^n \mathbf{E}[U_{n,m}^2; |U_{n,m}| > \varepsilon] = 0. \quad (24)$$

Lemma 5 *If a random variable U has the 4-th moment, then*

$$\mathbf{E}[U^2; |U| > \varepsilon] \leq \varepsilon^{-2} \mathbf{E}[U^4].$$

Proof.

$$\mathbf{E}[U^2; |U| > \varepsilon] \leq \mathbf{E}\left[U^2 \cdot \frac{U^2}{\varepsilon^2}; |U| > \varepsilon\right] \leq \frac{1}{\varepsilon^2} \mathbf{E}[U^4].$$

Q.E.D.

Since

$$\mathbf{E}[U_{n,n}^2] = \int_{(n-1)^2}^\infty \left(\frac{\sqrt{n}}{t+n} \right)^2 dt = \frac{n}{(n-1)^2 + n} \rightarrow 0, \quad n \rightarrow \infty,$$

it is sufficient to prove

$$\lim_{n \rightarrow \infty} \sum_{m=1}^{n-1} \mathbf{E}[U_{n,m}^4] = 0 \quad (25)$$

to show (24) by Lemma 5.

Let us estimate each of $\mathbf{E}[U_{n,m}^4]$. Putting

$$U_m(t) := \int_{(m-1)^2}^t \frac{\sqrt{n}}{s+n} d\tilde{\eta}(s), \quad m \leq t,$$

and applying the Itô formula, we have

$$\begin{aligned}
\mathbf{E}[U_{n,m}^4] &= \mathbf{E} \left[\int_{(m-1)^2}^{m^2} \left(\left(U_m(t) + \frac{\sqrt{n}}{t+n} \right)^4 - U_m(t)^4 - 4U_m(t)^3 \frac{\sqrt{n}}{t+n} \right) dt \right] \\
&= \int_{(m-1)^2}^{m^2} \left(6 \mathbf{E}[U_m(t)^2] \left(\frac{\sqrt{n}}{t+n} \right)^2 + \left(\frac{\sqrt{n}}{t+n} \right)^4 \right) dt \\
&= \int_{(m-1)^2}^{m^2} \left(6 \left(\frac{n}{(m-1)^2+n} - \frac{n}{t+n} \right) \frac{n}{(t+n)^2} + \frac{n^2}{(t+n)^4} \right) dt \\
&= n^2 \int_{(m-1)^2}^{m^2} \left(6 \cdot \frac{t - (m-1)^2}{((m-1)^2+n)(t+n)^3} + \frac{1}{(t+n)^4} \right) dt \\
&< n^2 \int_{(m-1)^2}^{m^2} \left(6 \cdot \frac{2m-1}{((m-1)^2+n)^4} + \frac{1}{((m-1)^2+n)^4} \right) dt \\
&= 6n^2 \cdot \frac{(2m-1)^2}{((m-1)^2+n)^4} + n^2 \cdot \frac{2m-1}{((m-1)^2+n)^4} \\
&= n^2 \cdot \frac{6(2m-1)^2 + 2m-1}{((m-1)^2+n)^4}.
\end{aligned}$$

From this, we derive

$$\sum_{m=1}^{n-1} \mathbf{E}[U_{n,m}^4] < n^2 \sum_{m=1}^{n-1} \frac{6(2m-1)^2 + 2m-1}{((m-1)^2+n)^4} = O(n^{-1/2}), \quad n \rightarrow \infty,^7$$

thus (25) holds. Now the proof of Theorem 7 is complete.

Q.E.D.

3.2 In the case of random walk

We next consider the case where $\{\xi_i\}_{i=1}^\infty$ is a Bernoulli sequence with $P(\xi_i = 0) = P(\xi_i = 2) = 1/2$, and $X_n := \sum_{i=1}^n \xi_i$. Again by the strong law of large numbers, $\{x + X_k\}_{k=1}^\infty$, $x > 0$, satisfies Assumption 1 almost surely.

3.2.1 Random digamma function

Defining G_k by

$$G_k := \#\{n \in \mathbf{N}; X_n = 2k\}, \quad k = 1, 2, \dots$$

$\{G_k\}_{k=1}^\infty$ is an i.i.d. sequence with a geometric distribution $P(G_k = n) = 2^{-n}$, $n \in \mathbf{N}$, and we have

$$\sum_{k=1}^{G_1+\dots+G_N} \frac{1}{x + X_k} = \sum_{k=0}^N \frac{G_k}{x + 2k}.$$

Let us first look at the law of large numbers. Since $\mathbf{E}[G_k] = 2$, for sufficiently large N ,

$$\log(G_1 + G_2 + \dots + G_N) = \log \left(\frac{1}{N} \sum_{k=1}^N G_k \right) + \log N \approx \log 2 + \log N.$$

⁷For this estimate, use $n^2 \int_0^\infty x^2/(x^2+n)^4 dx = \pi/(32\sqrt{n})$.

Therefore, with probability 1,

$$Q(x) = \lim_{N \rightarrow \infty} \left[\sum_{k=0}^N \frac{G_k}{x+2k} - (\log 2 + \log N) \right]$$

converges. Theorem 2 implies that with probability 1,

$$Q(x) = -\log x + O(x^{-1}), \quad x \rightarrow \infty.$$

The mean of $Q(x)$ is computed as follows.

$$\begin{aligned} \mathbf{E}[Q(x)] &= \lim_{N \rightarrow \infty} \mathbf{E} \left[\sum_{k=0}^N \frac{G_k}{x+2k} - (\log 2 + \log N) \right] \\ &= \lim_{N \rightarrow \infty} \left[\sum_{k=0}^N \frac{2}{x+2k} - (\log 2 + \log N) \right] \\ &= \lim_{N \rightarrow \infty} \left[\sum_{k=0}^N \frac{1}{(x/2) + k} - (\log 2 + \log N) \right] \\ &= -\psi((x/2) + 1) - \log 2 \\ &= -\log x + \sum_{k=0}^{\infty} \left[\frac{1}{(x/2) + k} + \log \left(1 - \frac{1}{(x/2) + k} \right) \right]. \end{aligned}$$

Next, let us look at the central limit theorem. We put

$$Q_N(x) := \sum_{k=0}^N \frac{G_k}{x+2k} - (\log 2 + \log N),$$

and calculate its characteristic function (Fourier transform). Noting that

$$\mathbf{E}[\exp(itG_k)] = \frac{\frac{1}{2}e^{it}}{1 - \frac{1}{2}e^{it}},$$

we have

$$\begin{aligned} \mathbf{E}[e^{itQ_N(x)}] &= \prod_{k=0}^N \mathbf{E} \left[\exp \left(it \cdot \frac{G_k}{x+2k} \right) \right] \exp(-it(\log 2 + \log N)) \\ &= \prod_{k=0}^N \left[\frac{\frac{1}{2} \exp \left(it \cdot \frac{1}{x+2k} \right)}{1 - \frac{1}{2} \exp \left(it \cdot \frac{1}{x+2k} \right)} \right] \exp(-it(\log 2 + \log N)) \\ &= \prod_{k=0}^N \left[\frac{\frac{1}{2} \exp \left(-it \cdot \frac{1}{x+2k} \right)}{1 - \frac{1}{2} \exp \left(it \cdot \frac{1}{x+2k} \right)} \right] \exp \left[-it \left(-\sum_{k=0}^N \frac{2}{x+2k} + \log 2 + \log N \right) \right]. \end{aligned}$$

Thus

$$\mathbf{E}[e^{itQ(x)}] = \prod_{k=0}^{\infty} \left[\frac{\exp \left(-it \cdot \frac{1}{x+2k} \right)}{2 - \exp \left(it \cdot \frac{1}{x+2k} \right)} \right] \exp[-it(-\psi((x/2) + 1) + \log 2)].$$

Let us take the limit $x \rightarrow \infty$ of the infinite product

$$\prod_{k=0}^{\infty} \left[\frac{\exp\left(-it \cdot \frac{1}{x+2k}\right)}{2 - \exp\left(it \cdot \frac{1}{x+2k}\right)} \right] = \prod_{k=0}^{\infty} \left[\frac{1}{2 \exp\left(it \cdot \frac{1}{x+2k}\right) - \exp\left(it \cdot \frac{2}{x+2k}\right)} \right].$$

Developing the denominator, we see

$$\begin{aligned} & 2 \exp\left(it \cdot \frac{1}{x+2k}\right) - \exp\left(it \cdot \frac{2}{x+2k}\right) \\ &= 2 \left(1 + \frac{it}{x+2k} - \frac{1}{2} \cdot \frac{t^2}{(x+2k)^2} + \dots \right) - \left(1 + \frac{2it}{x+2k} - \frac{1}{2} \cdot \frac{4t^2}{(x+2k)^2} + \dots \right) \\ &= 1 + \frac{t^2}{(x+2k)^2} + \dots, \end{aligned}$$

and hence

$$\begin{aligned} \prod_{k=0}^{\infty} \left[\frac{\exp\left(-it \cdot \frac{1}{x+2k}\right)}{2 - \exp\left(it \cdot \frac{1}{x+2k}\right)} \right] &= \prod_{k=0}^{\infty} \left(1 + \frac{t^2}{(x+2k)^2} + \dots \right)^{-1} \\ &\sim \prod_{k=0}^{\infty} \exp\left(-\frac{t^2}{(x+2k)^2}\right) \\ &= \exp\left(-\frac{t^2}{2} \cdot \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{((x/2)+k)^2}\right), \quad x \rightarrow \infty. \end{aligned}$$

From the above, when $x \gg 1$, the distribution of $Q(x)$ is close to the normal distribution with mean $-\psi((x/2)+1) + \log 2$ and variance $\frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{((x/2)+k)^2}$. Since we have

$$\begin{aligned} -\psi((x/2)+1) + \log 2 &= -\log x + O(x^{-1}), \\ \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{((x/2)+k)^2} &= x^{-1} + O(x^{-2}), \end{aligned}$$

as $x \rightarrow \infty$, in the long run, we proved the following convergence in distribution.

$$\sqrt{x}(Q(x) + \log x) \rightarrow \mathcal{N}(1, 0), \quad x \rightarrow \infty.$$

That is, the assertion of Theorem 7 holds in this case, too.

4 Further discussions

4.1 Extension of Theorem 6 and Theorem 7

Recently, Theorem 6 and Theorem 7 have been much more generalized by S.Takanobu.

Theorem 8 ([5]) *Let $\{\xi_i\}_{i=1}^{\infty}$ be an i.i.d. sequence with $\xi_i > 0$, $\mathbf{E}[\xi_i] = 1$, and $\mathbf{E}[\xi_i^{\beta}] < \infty$ for some $\beta > 1$. Then we have $Q(x) = -\log x + O(x^{-1})$, $x \rightarrow \infty$, a.s..*

Theorem 9 ([5]) *Let $\{\xi_i\}_{i=1}^{\infty}$ be an i.i.d. sequence with $\xi_i > 0$, $\mathbf{E}[\xi_i] = 1$, and $v := \mathbf{V}[\xi_i^2] < \infty$. Then, for $X = \{X_k\}_{k=1}^{\infty} = \{\xi_1 + \dots + \xi_k\}_{k=1}^{\infty}$, it holds that the distribution of $\sqrt{x}(Q_X(x) + \log x)$ converges to $\mathcal{N}(0, v)$ as $x \rightarrow \infty$.*

For i.i.d. sequences $\{\xi_i\}_{i=1}^\infty$ with $\xi_i > 0$, $\mathbf{E}[\xi_i] = 1$, but $\mathbf{V}[\xi_i^2] = \infty$, we have following limit theorem.

Theorem 10 ([5]) (i) If $[0, \infty) \ni s \mapsto \mathbf{E}[\xi_1^2; \xi_1 \leq s] \in [0, \infty)$ is slowly varying at ∞ , there exists a positive sequence $\{B_n\}_{n=1}^\infty$ such that

$$\frac{x}{B_{[x]}} (Q_X(x) + \log x) \longrightarrow \mathcal{N}(0, 1), \quad x \rightarrow \infty, \quad \text{in distribution.}$$

(ii) If there exist a $\beta \in (1, 2)$ and an $L(\cdot)$ which is slowly varying at ∞ such that

$$P(\xi_1 > x) \sim L(x)x^{-\beta}, \quad x \rightarrow \infty,$$

then there exists $\{B_n\}_{n=1}^\infty$ such that

$$\begin{aligned} & \lim_{x \rightarrow \infty} \mathbf{E} \left[\exp \left(\sqrt{-1} t (\beta - 1)^{1/\beta} \frac{x}{B_{[x]}} (Q_X(x) + \log x) \right) \right] \\ &= \exp \left(\beta \int_0^\infty (e^{\sqrt{-1} b y} - 1 - \sqrt{-1} b y) \frac{dy}{y^{\beta+1}} \right). \end{aligned}$$

These results with proofs will be written in a paper in near future.

4.2 The case of two dimensional random array of electric charges

We mentioned about the electro-static interpretation of random digamma function in § 3.1.2. In this context, a natural question arises: *Suppose that unit electrical charges are located at random in an unbounded domain of $\mathbf{R}^2$. Then, can we define a renormalized Coulomb potential as a random variable?*

Example 3 Suppose that the distribution of the unit electrical charges are described by a Poisson random measure on the out side of centered circle $B(O, x)^c$ with the Lebesgue measure as the intensity. Then the Coulomb potential at O will be expressed as

$$\int_x^\infty \frac{dN(\pi t^2)}{t} = \sqrt{\pi} \int_{x^2}^\infty \frac{dN(t)}{\sqrt{t}}$$

by a standard Poisson process $N(t)$, which is of course divergent. The renormalized potential would be

$$\sqrt{\pi} \int_{x^2}^\infty \frac{d\tilde{N}(t)}{\sqrt{t}}, \quad \tilde{N}(t) := N(t) - t,$$

but it is not well-defined because

$$\mathbf{E} \left[\left(\int_{x^2}^\infty \frac{d\tilde{N}(t)}{\sqrt{t}} \right)^2 \right] = \int_{x^2}^\infty \frac{dt}{t} = \infty.$$

To look at the situation closely, let us observe the following deterministic case.: The sequence $a = \{\sqrt{k}\}_{k=1}^\infty$ is zeta regularizable, because the corresponding zeta function is $z(s) = \zeta(s/2)$. Hence by Theorem 2 in [4] and Theorem 1.8 in [3], we have

$$z\text{-}\prod_{k=1}^\infty \left(1 + \frac{x}{\sqrt{k}} \right) = \exp \left(\zeta \left(\frac{1}{2} \right) x - \frac{\gamma x^2}{2} \right) \prod_{k=1}^\infty \left(1 + \frac{x}{\sqrt{k}} \right) \exp \left(-\frac{x}{\sqrt{k}} + \frac{x^2}{2k} \right),$$

where

$$\zeta\left(\frac{1}{2}\right) = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \frac{1}{\sqrt{k}} - 2\sqrt{n} \right] = -1.46035 \dots$$

For $X = \{X_k\}_{k=1}^{\infty} := \{\xi_1 + \dots + \xi_k\}_{k=1}^{\infty}$, partial sums of i.i.d. random variables, Assumption 1 is satisfied with $\delta < 1/2$, and then the corresponding zeta function

$$Z(s) = \sum_{k=1}^{\infty} X_k^{-s}$$

will become meromorphic in $\operatorname{Re} s > 1/2$, but $Z(1/2)$ may not be defined. This fact has something to do with the non-existence of the limit

$$\int_{x^2}^{\infty} \frac{d\tilde{N}(t)}{\sqrt{t}} = \lim_{y \rightarrow \infty} \left[\sum_{x^2 \leq X_k \leq y} \frac{1}{\sqrt{X_k}} - 2\sqrt{y} + 2x \right].$$

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