

Prawitz の定理の周辺

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1変数の単位円内の正則単葉函数の境界挙動については、種々なことがあり、それらのいくつかは Pommerenke [5] にまとめられている。特に、非有界な単葉函数についての境界値を調べるには、Prawitz の定理が基本的な役割を演ずる。本講は、この定理が $\mathbb{C}^n$ の場合にどうなるかを考える一つの試みである。

§1. Prawitz の定理

$U := \{z \in \mathbb{C} \mid |z| < 1\}$: 単位円

$S :=$ U で定義された、正則、単葉、正規された函数の集合

$$M_p(r, f) := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p} \quad \begin{matrix} (0 < p < \infty) \\ (0 < r < 1) \end{matrix}$$

$$M_\infty(r, f) := \max_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|$$

本節では, Duren[1], p.61. の Prawitz の定理 (Theorem 2.22) の別証を与える。

Lemma: [1, p.62], C_1, C_2 を原点をかこむ, 互めらわぬ Jordan 曲線とし, C_1 は C_2 の内部にあるものとす。このとき, 次の式が成り立つ:

$$\int_{C_1} r^p d\theta \leq \int_{C_2} r^p d\theta \quad (0 < p < \infty)$$

(ただし, $z = r e^{i\theta}$)。

□

証明は Green の定理による。

定理 [Prawitz の定理, [1], p.61]. $f \in S$, $0 < p < \infty$ に対して, 次の式が成り立つ:

$$M_p^p(r, f) \leq p \int_0^r \frac{1}{t} M_\infty^p(t, f) dt, \quad (0 < r < 1) \quad \square$$

(証明)

$$z = r e^{i\theta}, \quad \bar{z} = r e^{-i\theta} \quad \text{に対して,}$$

$$dz = e^{i\theta} dr + i r e^{i\theta} d\theta, \quad d\bar{z} = e^{-i\theta} dr - i r e^{-i\theta} d\theta$$

$$\bar{z} dz = r dr + i r^2 d\theta, \quad z d\bar{z} = r dr - i r^2 d\theta$$

$$\therefore d\theta = \frac{\bar{z} dz - z d\bar{z}}{2 r^2 i} \quad \dots \dots (1)$$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial z} \frac{\partial z}{\partial r} + \frac{\partial}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial r} = \frac{1}{r} \left(z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} \right) \quad \dots \dots (2)$$

$w = f(z) \in S$ に対して, $f(z) = R e^{i\theta}$ とおくと

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial w} \left(\frac{\partial w}{\partial \bar{z}} \right), \quad \frac{\partial R}{\partial w} = \frac{1}{2R} \bar{w}, \quad \frac{\partial R}{\partial \bar{w}} = \frac{1}{2R} w.$$

と (2) より,

$$\frac{\partial R}{\partial \bar{z}} = \frac{1}{2R} \left\{ \bar{w} \left(\frac{\partial w}{\partial \bar{z}} \right) \bar{z} + \bar{\bar{z}} \left(\frac{\partial w}{\partial \bar{z}} \right) w \right\}$$

より,

$$2\pi \frac{d}{d\bar{z}} M_p^p(z, f) = \int_0^{2\pi} \frac{\partial R^p}{\partial \bar{z}} d\theta$$

$$= p \int_0^{2\pi} R^{p-1} \frac{\partial R}{\partial \bar{z}} d\theta = (*)$$

$z = z'$, $dz = \frac{\partial z}{\partial w} dw$, $d\bar{z} = \overline{\left(\frac{\partial z}{\partial w} \right)} d\bar{w}$ より

$$d\theta = \frac{\bar{z} \left(\frac{\partial z}{\partial w} \right) dw - z \overline{\left(\frac{\partial z}{\partial w} \right)} d\bar{w}}{2R^2 i}$$

$$\therefore \frac{\partial R}{\partial \bar{z}} d\theta = \frac{1}{2R^3 i} \left\{ \bar{w} \left(\frac{\partial w}{\partial \bar{z}} \right) \bar{z} + \bar{\bar{z}} \left(\frac{\partial w}{\partial \bar{z}} \right) w \right\}.$$

$$- \left\{ \bar{z} \left(\frac{\partial z}{\partial w} \right) dw - z \overline{\left(\frac{\partial z}{\partial w} \right)} d\bar{w} \right\}$$

$$= \frac{1}{2R^3 i} \left[|z|^2 \{ \bar{w} dw - w d\bar{w} \} \right.$$

$$\left. + \left\{ \bar{z}^2 \overline{\left(\frac{\partial w}{\partial \bar{z}} \right)} \left(\frac{\partial z}{\partial w} \right) w dw - z^2 \left(\frac{\partial w}{\partial \bar{z}} \right) \overline{\left(\frac{\partial z}{\partial w} \right)} \bar{w} d\bar{w} \right\} \right]$$

ここで, [] 内の第 2 項は $\bar{z}^2 \overline{\left(\frac{\partial w}{\partial \bar{z}} \right)} \left(\frac{\partial z}{\partial w} \right) w dw$ の虚部であるから

$$\int_{\Gamma} \frac{R^{p-2}}{4Y^3 i} \left\{ \bar{z}^2 \left(\frac{\partial \bar{w}}{\partial \bar{z}} \right) \left(\frac{\partial z}{\partial w} \right) w dw - z^2 \left(\frac{\partial w}{\partial z} \right) \left(\frac{\partial \bar{z}}{\partial \bar{w}} \right) \bar{w} d\bar{w} \right\} = 0$$

$$\therefore (*) = \frac{p}{Y} \int_{\Gamma} R^p \frac{\bar{w} dw - w d\bar{w}}{2 R^2 i} = \frac{p}{Y} \int_{\Gamma} R^p d(\mathbb{H})$$

$C_1 = \Gamma$, $C_2 := \{ |w| = M_\infty(x, f) + \varepsilon, \varepsilon > 0 \}$ とし, lemma を用いて,

$$\int_{C_1} R^p d(\mathbb{H}) \leq \int_{C_2} (M_\infty(x, f) + \varepsilon)^p d(\mathbb{H}) = 2\pi (M_\infty(x, f) + \varepsilon)^p$$

そこで, $\varepsilon \rightarrow 0$ とし

$$\frac{d}{dY} M_p(x, f) \leq \frac{p}{Y} M_\infty^p(x, f)$$

これを積分して, 定理の式が得られる。

Prawitz の定理として,

系. $H^p \supset S$ ($p < \frac{1}{2}$).

証明は, Growth Theorem を用いる。

この系より, 函数 $f(z) \in S$ の radial limit

$$\lim_{r \rightarrow 1-0} f(re^{i\theta}) = f(e^{i\theta})$$

はほとんど到る所存在する。

また, 境界値 $f(e^{i\theta})$ はほとんど到る所 0 に収束する。

([5], p. 70).

§2. $\mathbb{C}^n$ の Poincaré の定理の拡張

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{C}^n \text{ に対して, } z^* = (\bar{z}_1, \dots, \bar{z}_n), \quad I := \|z\| := \sqrt{z^* z},$$

$$\partial z = \begin{pmatrix} dz_1 \\ \vdots \\ dz_n \end{pmatrix}, \quad \frac{\partial}{\partial z} = \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right),$$

$$\frac{\partial}{\partial z^*} = \begin{pmatrix} \frac{\partial}{\partial \bar{z}_1} \\ \vdots \\ \frac{\partial}{\partial \bar{z}_n} \end{pmatrix}, \quad (* \text{ は 転置共役})$$

$$B_I := \{z \in \mathbb{C}^n \mid \|z\| < I\}, \quad B := B_1, \quad S_I := \text{bdry } B_I = \{z \in \mathbb{C}^n \mid \|z\| = I\}, \quad S := S_1$$

$$\text{いま, } w = f(z) = \begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix} \text{ を holomorphic map in } B,$$

単葉とすると, $z = f^{-1}(w) =: g(w)$ が存在し,

$$\frac{\partial w}{\partial z} = \begin{pmatrix} \frac{\partial w_1}{\partial z_1} & \dots & \frac{\partial w_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial w_n}{\partial z_1} & \dots & \frac{\partial w_n}{\partial z_n} \end{pmatrix} : \text{ nonsingular}$$

$$\therefore \frac{\partial z}{\partial w} = \frac{\partial g}{\partial w} = \begin{pmatrix} \frac{\partial z_1}{\partial w_1} & \dots & \frac{\partial z_1}{\partial w_n} \\ \vdots & & \vdots \\ \frac{\partial z_n}{\partial w_1} & \dots & \frac{\partial z_n}{\partial w_n} \end{pmatrix} = \left(\frac{\partial w}{\partial z} \right)^{-1}$$

$$\text{また, } \frac{\partial}{\partial z} = \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right) =$$

$$= \left(\frac{\partial}{\partial w_1} \cdots \frac{\partial}{\partial w_n} \right) \begin{pmatrix} \frac{\partial w_1}{\partial z_1} & \cdots & \frac{\partial w_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial w_n}{\partial z_1} & \cdots & \frac{\partial w_n}{\partial z_n} \end{pmatrix} = \left(\frac{\partial}{\partial w} \right) \left(\frac{\partial w}{\partial z} \right)$$

また, $R := \|f(z)\|$ とおくと,

$$\frac{\partial R^2}{\partial w} \left(= 2R \frac{\partial R}{\partial w} \right) = \frac{\partial (w^* w)}{\partial w} = w^* \quad \therefore \frac{\partial R}{\partial w} = \frac{1}{2R} w^*$$

また, $z = \gamma a$, $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ ($\|a\| = 1$) とおくと,

$$\frac{\partial}{\partial \gamma} = \frac{\partial}{\partial z} \frac{\partial z}{\partial \gamma} + \frac{\partial z^*}{\partial \gamma} \frac{\partial}{\partial z^*} = \frac{\partial}{\partial z} a + a^* \frac{\partial}{\partial z^*} = \frac{1}{\gamma} \left(\frac{\partial}{\partial z} z + z^* \frac{\partial}{\partial z^*} \right)$$

$$\therefore \frac{\partial R}{\partial \gamma} = \frac{1}{\gamma} \left(\frac{\partial R}{\partial z} z + z^* \frac{\partial R}{\partial z^*} \right) = \frac{1}{2\gamma R} \left\{ w^* \left(\frac{\partial w}{\partial z} \right) z + z^* \left(\frac{\partial w^*}{\partial z^*} \right) w \right\}$$

また,

--- (1)

$$dz = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n = \frac{1}{n!} |dz_1, \dots, dz_n|$$

$$= \frac{1}{n!} \begin{vmatrix} dz_1 & dz_1 & \cdots & dz_1 \\ \vdots & \vdots & & \vdots \\ dz_n & dz_n & \cdots & dz_n \end{vmatrix}$$

$$dz[j] := dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \cdots \wedge dz_n \quad (dz_j \text{ を除く } n \text{ 個の})$$

$$\eta(z) := \sum_{j=1}^n (-1)^{j+1} z_j dz[j] = \frac{1}{(n-1)!} |z dz_1 \cdots dz_n|$$

(これを Heray form とする)

$$\partial z = \begin{pmatrix} dz_1 \\ \vdots \\ dz_m \end{pmatrix} = \begin{pmatrix} \frac{\partial z_1}{\partial w_1} & \cdots & \frac{\partial z_1}{\partial w_n} \\ \vdots & & \vdots \\ \frac{\partial z_m}{\partial w_1} & \cdots & \frac{\partial z_m}{\partial w_n} \end{pmatrix} \begin{pmatrix} dw_1 \\ \vdots \\ dw_n \end{pmatrix} = \left(\frac{\partial z}{\partial w} \right) \partial w$$

$$\therefore d\bar{z} = \frac{1}{n!} \left| \left(\frac{\partial z}{\partial w} \right) (\partial w \cdots \partial w) \right| = \frac{1}{n!} \left| \frac{\partial z}{\partial w} \right| dw \cdots (2)$$

$$\eta(z) = \frac{1}{(n-1)!} \left| z, \left(\frac{\partial z}{\partial w} \right) \partial w, \cdots, \left(\frac{\partial z}{\partial w} \right) \partial w \right|$$

$$= \frac{1}{(n-1)!} \left| \frac{\partial z}{\partial w} \right| \left| \left(\frac{\partial z}{\partial w} \right)^T z, \partial w, \cdots, \partial w \right| \cdots (3)$$

$d\bar{z} := d\bar{z}_1 \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_m$, $\eta(\bar{z})$: z を $\bar{z}$ にあきかえたもの, とし,

$$\Theta_n(z) := \frac{1}{(2R^2i)^n} \left\{ \eta(\bar{z}) \wedge d\bar{z} + (-1)^{\frac{n(n-1)}{2}} \eta(z) \wedge d\bar{z} \right\} \cdots (4)$$

とおくと, $\Theta_n(z)$ は S 上の面素である。

(1), (2), (3), (4)より,

$$\begin{aligned} \frac{\partial R}{\partial r} \Theta_n(z) &= \frac{1}{2Ri(2R^2i)^n} \left\{ w^* \left(\frac{\partial z}{\partial w} \right)^T z + \bar{z}^* \left(\frac{\partial z}{\partial w} \right)^{*-T} w \right\} \\ &\quad - \frac{1}{(n-1)!} \left\{ \left| \left(\frac{\partial z}{\partial w} \right)^T z, \partial w \cdots \partial w \right| + (-1)^{\frac{n(n-1)}{2}} \left| \left(\frac{\partial z}{\partial w} \right)^T \bar{z}, \partial w \cdots \partial w \right| \wedge d\bar{w} \right\} \end{aligned}$$

この form の球面 S への正射影を作ると

$$= \frac{r}{2R(2R^2i)^n} \left\{ \eta(\bar{w}) \wedge dw + (-1)^{\frac{n(n-1)}{2}} \eta(w) \wedge d\bar{w} \right\}$$

(1変数の場合と同様にして)

$M_p(r, f)$, $M_\infty(r, f)$ を1変数の場合と同様に定義すると

($\text{mes}(S)$ を S の体積とすると)

$$\text{mes}(S) \frac{d}{dY} M_p^p(r, f) = \int_S \frac{\partial R^p}{\partial Y} \Theta_n(x)$$

$$= p \int_{\delta_r} \frac{R^{p+2n-2}}{r^{2n-1}} \Theta_n(w) = \frac{p}{r^{2n-1}} \int_{\delta_r} R^{p+2n-2} \Theta_n(w)$$

$$\leq \frac{p}{r^{2n-1}} (M_\infty(r, f))^{p+2n-2} \cdot \text{mes}(S).$$

よって、次の定理が得られる:

定理. $f(z)$ を B から $\mathbb{C}^n$ への単葉, 正則, 正規化された写像とすると, 次の式が得られる.

$$M_p^p(r, f) \leq p \int_0^r \frac{1}{t^{2n-1}} M_\infty^{p+2n-2}(t, f) dt$$

$$(0 < p < \infty, 0 < r < 1)$$

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