

Limit formulas for dimensions of spaces of automorphic forms

Werner Hoffmann

The problem

The purpose of this contribution is to give a short introduction to the recent papers of the author and A. Deitmar [3], [4] on limit multiplicities and to comment on the relations with the similar problem of Weyl asymptotics for arithmetic quotients.

Let G be a connected reductive $\mathbb{Q}$ -group. We denote by A_G the greatest $\mathbb{Q}$ -split torus in the center of G . In the group $G(\mathbb{R})$ of $\mathbb{R}$ -rational points, which we also denote by G for simplicity of notation, we choose a maximal compact subgroup K . The center of the universal enveloping algebra of the complexification of the Lie algebra $\mathfrak{g}$ of G will be denoted by $\mathfrak{Z}$.

Given an arithmetic subgroup Γ of $G(\mathbb{Q})$, a finite-dimensional unitary representation τ of K on a hermitian vector space W_τ and a $\mathbb{C}$ -valued linear functional λ on the Lie algebra $\mathfrak{a}_G$ of $A_G(\mathbb{R})$, we consider the corresponding space $\mathcal{A}(\Gamma, \tau, \lambda)$ of automorphic forms. These are the smooth slowly increasing $\mathfrak{Z}$ -finite functions $\phi : \Gamma \backslash G \rightarrow W_\tau$ satisfying

$$\phi(\exp(Z)gk) = e^{\lambda(Z)}\tau(k)\phi(g) \quad \text{for all } k \in K, Z \in \mathfrak{a}_G.$$

For simplicity, we usually assume that $A_G = \{1\}$ and omit the argument λ . We denote by $\mathcal{A}^2(\Gamma, \tau)$ the subspace of square-integrable automorphic forms, i.e., those satisfying

$$\int_{\Gamma \backslash G/K} |\phi(g)|^2 dg < \infty.$$

Due to Chevalley's structure theorem about $\mathfrak{Z}$, the set of all infinitesimal characters

$$V_{\mathbb{C}} := \text{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}, \mathbb{C})$$

can be endowed with a non-canonical structure of an affine space (whose dimension is the absolute rank of G). We have an $\mathbb{R}$ -structure (complex conjugation) on $\mathfrak{g}_{\mathbb{C}}$, which induces $\mathbb{R}$ -structures (involutions) on $\mathfrak{Z}$ and on $V_{\mathbb{C}}$, and a suitable chosen affine structure on $V_{\mathbb{C}}$ is preserved by this involution. In any case, the real subspace V of $V_{\mathbb{C}}$ is well-defined. As a very special case of a theorem of Langlands [6], we have a direct sum decomposition

$$\mathcal{A}^2(\Gamma, \tau) = \bigoplus_{\chi \in V} \mathcal{A}^2(\Gamma, \tau, \chi),$$

where each summand on the right-hand side has finite dimension and is characterised by the condition $T\phi = \chi(T)\phi$ for all $T \in \mathfrak{Z}$. One is interested in the dimensions

$$N(\Gamma, \tau, S) := \sum_{\chi \in S} \dim \mathcal{A}^2(\Gamma, \tau, \chi),$$

where S is a subset of V . Since a precise formula cannot be obtained in general, one studies the asymptotic of this function

(A) as Γ tends to the trivial subgroup $\{1\}$ or

(B) as S tends to the full space V .

Problem (B) asks for a generalization of the *Weyl asymptotic law* as applied to the locally-symmetric space $\Gamma \backslash G/K$. In fact, the elements of $\mathcal{A}(\Gamma, \tau)$ can be identified with smooth sections of the bundle $\Gamma \backslash G \times_K W_\tau$ over this space, and the Casimir element $\omega \in \mathfrak{Z}$ acts as the Laplacian shifted by a multiple of the identity. Now

$$N(\Gamma, \tau, t) := N(\Gamma, \tau, S_t) \quad \text{with} \quad S_t = \{\chi \in V \mid -\chi(\omega) \leq t\}, \quad t \geq 0,$$

is just the spectral counting function of this Laplacian.

Problem (A) is contained in the question about *limit multiplicities* in the following setting. The group G acts by right translations in the Hilbert space $\mathcal{H}(\Gamma) = L^2(\Gamma \backslash G)$. Let $\mathcal{H}_{\text{dis}}(\Gamma)$ be the closure of the sum of all irreducible closed invariant subspaces and $\mathcal{H}_{\text{con}}(\Gamma)$ its orthogonal complement. Then $\mathcal{A}^2(\Gamma, \tau)$ is dense in the subspace of K -invariant elements of $\mathcal{H}_{\text{dis}}(\Gamma) \otimes W_\tau$. The theorem of Langlands cited above in fact implies that

$$\mathcal{H}_{\text{dis}}(\Gamma) \cong \bigoplus_{\pi \in \Pi(G)} H_\pi^{N(\Gamma, \pi)} \quad (\text{Hilbertian direct sum})$$

as unitary G -modules, where $\Pi(G)$ denotes the set of equivalence classes of irreducible unitary representations of G , each π is realised on a Hilbert space H_π , and the $N(\Gamma, \pi)$ are certain natural numbers called multiplicities. Since

$$N(\Gamma, \tau, S) = \sum_{\substack{\pi \in \Pi(G) \\ \chi_\pi \in S}} [\pi|_K : \check{\tau}] N(\Gamma, \pi),$$

where χ_π is the infinitesimal character of π and $\check{\tau}$ the contragredient of τ , our problems are special cases of the question about the asymptotics of

$$N(\Gamma, \Omega) := \sum_{\pi \in \Omega} N(\Gamma, \pi)$$

as the set $\Omega \subset \Pi(G)$ exhausts (a connected component of) $\Pi(G)$ or as Γ shrinks to $\{1\}$.

Results for $\Gamma \backslash G$ compact

In the case when $\Gamma \backslash G$ is compact, one has the most complete results, which we recall in this section. First we consider problem (A). Usually, one restricts attention to decreasing sequences $\Gamma_1 \supset \Gamma_2 \supset \dots$ of normal subgroups of finite index in Γ such that the intersection of all Γ_j , $j \in \mathbb{N}$, is the trivial subgroup $\{1\}$. In this situation, DeGeorge and Wallach [2] have proved that

$$\lim_{j \rightarrow \infty} \frac{N(\Gamma_j, \pi)}{\text{vol}(\Gamma_j \backslash G)} = d_\pi, \quad (1)$$

where d_π denotes the formal dimension of $\pi \in \Pi(G)$ corresponding to the choice of Haar measure which is implicit in the volume appearing in the denominator. Note that $d_\pi \neq 0$ iff π belongs to the discrete series $\Pi_{\text{dis}}(G)$. If π is integrable (i.e., if its K -finite matrix coefficients are integrable over G) and Γ is torsion-free then, by a well-known result of Langlands, the formula is true without passing to the limit.

More generally, for $\Omega \subset \Pi(G)$ open, relatively compact and with regular boundary, Delorme [5] has proved that

$$\lim_{j \rightarrow \infty} \frac{N(\Gamma_j, \Omega)}{\text{vol}(\Gamma_j \backslash G)} = \mu(\Omega),$$

where μ is the Plancherel measure corresponding to the above choice of Haar measure. Of course, the preceding result is a special case, since $d_\pi = \mu(\{\pi\})$. Note that the support of μ is the set $\Pi_{\text{temp}}(G)$ of tempered $\pi \in \Pi(G)$.

As indicated in the first section, these results about limit multiplicities easily imply asymptotic formulas for the dimensions of spaces of automorphic forms. If we define a measure μ_τ on V by

$$\mu_\tau(S) = \int_{\Omega_S} [\pi|_K : \tilde{\tau}] d\mu(\pi), \quad \text{where } \Omega_S = \{\pi \in \Pi(G) \mid \chi_\pi \in S\},$$

then, for $S \subset V$ subject to the same requirements as Ω above,

$$\lim_{j \rightarrow \infty} \frac{N(\Gamma_j, \tau, S)}{\text{vol}(\Gamma_j \backslash G)} = \mu_\tau(S). \quad (2)$$

Now we come to problem (B). In the same paper [5], Delorme has shown that, if G has a single conjugacy class of Cartan subgroups and Γ is torsion-free, then

$$\frac{N(\Gamma, \Omega_t)}{\text{vol}(\Gamma \backslash G)} \sim \mu(\Omega_t) \quad \text{as } t \rightarrow \infty$$

for suitable expanding families of subsets $\Omega_t \subset \Pi(G)$, $t > 0$, like the following one. Let M be a minimal Levi subgroup, A the greatest $\mathbb{R}$ -split torus in its center and $\sigma \in \Pi(M)$ trivial on $\exp \mathfrak{a}$. Then one can take

$$\Omega_t = \{\text{Ind}_P^G(\sigma_\lambda) \mid \lambda \subset t\Omega\}$$

for any open, relatively compact set $\Omega \subset i\mathfrak{a}^*$ with regular boundary. Here P is a parabolic subgroup of G with Levi component M and σ_λ is the usual twist of σ by the character $\exp H \mapsto e^{\lambda(H)}$ of $\exp \mathfrak{a}$ considered as a character of M . In this case, $\mu(\Omega_t) \sim Ct^n$, where $n = \dim(G/K)$. It should be noted that all results in the present section do not use the assumption that Γ is arithmetic.

Results for $\Gamma \backslash G$ noncompact

In the case when $\Gamma \backslash G$ is non-compact, all results on problem (A) are proved only under an additional assumption about the tower of subgroups Γ_j . Let $\rho : G \rightarrow \mathrm{GL}(n)$ be a faithful $\mathbb{Q}$ -rational representation, and let $\Gamma_n(N)$ be the principal congruent subgroup of level $N \in \mathbb{N}$ consisting of all $\gamma \in \mathrm{GL}(n, \mathbb{Z})$ such that the matrix entries of $\gamma - \mathrm{Id}$ are divisible by N . Under the condition that there exist numbers N_j such that $\rho(\Gamma_j) = \rho(G) \cap \Gamma_n(N_j)$ for all $j \in \mathbb{N}$, Savin [10] has proved that the limit formula (1) still holds. The assumption depends on the choice of ρ , but it should be possible to replace it by the condition that the tower has bounded depth in the following sense (cf. [3]).

Definition *The tower of normal subgroups Γ_j is called a tower of bounded depth if there are natural numbers N_j, D such that $\rho(\Gamma_j) \subset \Gamma(N_j)$ and*

$$[\rho(G) \cap \Gamma(N_j) : \rho(\Gamma_j)] \leq D$$

for all $j \in \mathbb{N}$ and for some (hence for any) faithful $\mathbb{Q}$ -rational representation ρ .

Note that $\Gamma \backslash G$ is compact iff the $\mathbb{Q}$ -rank of G is zero. The results we are going to describe generalise formula (2) to the case when the $\mathbb{Q}$ -rank of G is one. First let us recall the decomposition of $\mathcal{H}_{\mathrm{con}}(\Gamma)$ in this situation. If M is a Levi component of a parabolic subgroup P of G (everything defined over $\mathbb{Q}$), we obtain an arithmetic subgroup Γ_P of M by projecting $\Gamma \cap P$ along the unipotent radical of P . Denoting by $\mathfrak{Z}_M, V_M$ the analogues of $\mathfrak{Z}$ and V for M , we have the Harish-Chandra homomorphism $\mathfrak{Z} \rightarrow \mathfrak{Z}_M$, which induces a map $p_M : V_M \rightarrow V$. The direct product decomposition $M = M^1 \times \exp \mathfrak{a}_M$, where M^1 is the intersection of the kernels of all $\mathbb{R}_+$ -valued characters of M , induces a decomposition $V_M = V_M^1 \times i\mathfrak{a}_M^*$. Note that $\dim \mathfrak{a}_M = 1$ according to our assumption on the rank of G .

Langlands' spectral theory [6] provides an isomorphism of unitary G -modules

$$\mathcal{H}_{\mathrm{con}}(\Gamma) \cong \frac{1}{2\pi} \bigoplus_{\chi \in V_M^1} \int_{i\mathfrak{a}_M^*/\pm 1} \bigoplus_{\{P\}} \mathrm{Ind}_P^G \mathcal{H}(\Gamma_P, \chi, \lambda) d\lambda,$$

where the right-hand side is a Hilbertian direct integral over $(\chi, \lambda) \in V_M$ modulo the action of the (two-element) Weyl group of $(\mathfrak{g}, \mathfrak{a}_M)$ and the inner sum is over all Γ -conjugacy classes of proper parabolic $\mathbb{Q}$ -subgroups P of G , whose Levi components M can be identified with each other by G -conjugation. The justification for taking the integral over the quotient by the Weyl group is provided by the intertwining operator

$$M(\chi, \lambda) : \bigoplus_{\{P\}} \mathrm{Ind}_P^G \mathcal{H}(\Gamma_P, \chi, \lambda) \rightarrow \bigoplus_{\{P\}} \mathrm{Ind}_P^G \mathcal{H}(\Gamma_P, \chi, -\lambda),$$

which comes from the constant terms of Eisenstein series and may also be interpreted as a scattering matrix. This operator is unitary for $\lambda \in i\mathfrak{a}_M^*$, and after restriction to K -finite vectors it extends meromorphically to $\mathfrak{a}_{M, \mathbb{C}}^*$ satisfying $M(\chi, \lambda)M(\chi, -\lambda) = \mathrm{Id}$. If we tensor both domain and range with W_τ and extend the operator $M(\chi, \lambda)$ by the identity, it restricts to a map $M(\tau, \chi, \lambda)$ between the corresponding (finite-dimensional) subspaces of K -invariants.

Our main object of interest is the measure $N(\Gamma, \tau, S)$ on V , which counts the multiplicities of infinitesimal characters in the discrete subspace. However, we would also like to define an analogous measure for the continuous subspace by

$$N_{\text{con}}(\Gamma, \tau, S) = \frac{1}{4\pi} \sum_{\chi \in V_M^1} \int_{\{\lambda \in i\mathfrak{a}_M^* \mid p_M(\chi, \lambda) \in S\}} \text{tr} (M(\tau, \chi, \lambda)^{-1} M'(\tau, \chi, \lambda)) d\lambda$$

and a total measure by

$$\tilde{N}(\Gamma, \tau, S) = N(\Gamma, \tau, S) + N_{\text{con}}(\Gamma, \tau, S).$$

Unless the real rank of G is one, we can only show at present that the latter is defined as a tempered distribution of order 1, i.e., the characteristic function of S has to be replaced by a function which, together with its first derivatives, is rapidly decreasing on V .

The inclusion of the continuous subspace can be best motivated in the setting of problem (B). For $G = \text{SL}(2, \mathbb{R})$, trivial τ , and Γ not necessarily arithmetic, Selberg ([11], p. 668) has proved the generalisation of the Weyl asymptotic law

$$\frac{\tilde{N}(\Gamma, \tau, t)}{\text{vol}(\Gamma \backslash G)} \sim \frac{t}{4\pi} \quad \text{as } t \rightarrow \infty.$$

If Γ is a congruence subgroup, one can calculate $M(\tau, \lambda)$ (note that $V_M^1 = \{1\}$ in this case) in terms of Dirichlet L -series, and it follows that this asymptotic is also true with $\tilde{N}$ replaced by N . For nonarithmetic Γ , however, $N(\Gamma, \tau, t)$ is conjecturally bounded, see [8].

Now we return to problem (A). The main result of [4] is as follows.

Theorem 1 *Let G be of $\mathbb{Q}$ -rank one and (Γ_j) a tower of bounded depth. Then*

$$\lim_{j \rightarrow \infty} \frac{\tilde{N}(\Gamma_j, \tau)}{\text{vol}(\Gamma_j \backslash G)} = \mu_\tau$$

as a tempered distribution on V of order one.

This result raises several questions. One may ask, e.g., whether the restriction to towers of bounded depth is necessary. If the answer is affirmative, then one stays in the range of arithmetic lattices, and in analogy to problem (B) one might hope that the result is true with $\tilde{N}$ replaced by N (see [4], Conjecture 3.10). For a tower of principal congruence subgroups $\Gamma_j = \Gamma(N_j)$ in $\text{SL}(2, \mathbb{Z})$, this is indeed the case, as was already shown in [9] (cf. also [4], section 5). In general, however, it seems to be very difficult to show that the continuous contribution is of smaller order than the discrete one.

Of course, one would expect that a similar result holds for limit multiplicities of representations, not only infinitesimal characters. This would require the application of more elaborate Paley-Wiener theorems as in [5].

Our main tool is the trace formula, as it was in [11], [5] and [9]. The central problem is to show that the spectral side extends to a continuous linear functional on a larger space of test functions than those for which the trace formula is valid. This is done by means of a spectral estimate (see [3]) for the cut-off Laplacian uniform in towers. It has the following special case, where N_{cus} refers to the space of cusp forms $\mathcal{A}_{\text{cus}} \subset \mathcal{A}^2$.

Theorem 2 *Let the $\mathbb{Q}$ -rank of G be arbitrary and (Γ_j) a tower of bounded depth. Then there exists $C > 0$ such that*

$$N_{\text{cus}}(\Gamma_j, \tau, t) \leq C \text{vol}(\Gamma_j \backslash G) (1+t)^{n/2} \quad \text{for all } t \geq 0, j \in \mathbb{N}.$$

For a fixed Γ_j , this is due to Müller [7]. The method to estimate N_{con} in terms of the spectral counting function of the cut-off Laplacian also goes back to the same paper. It yields a bound linear in $\text{vol}(\Gamma_j \backslash G)$ only for G of $\mathbb{Q}$ -rank one, whence the restriction in Theorem 1, and also imposes the restriction to distributions instead of measures.

References

- [1] D. DeGeorge, N. Wallach, Limit formulas for multiplicities in $L^2(\Gamma \backslash G)$. *Ann. Math.* 107 (1978), 133–150.
- [2] D. DeGeorge, N. Wallach, Limit formulas for multiplicities in $L^2(\Gamma \backslash G)$ II. *Ann. Math.* 109 (1979), 477–495.
- [3] A. Deitmar, W. Hoffmann, Spectral estimates of towers of noncompact quotients, *Canad. J. Math.* 51, no. 2 (1999), 266–293.
- [4] A. Deitmar, W. Hoffmann, On limit multiplicities for spaces of automorphic forms, *Canad. J. Math.* 51, no. 5 (1999), 952–976.
- [5] P. Delorme, Formules limites et formules asymptotiques pour les multiplicites dans $L^2(\Gamma \backslash G)$. *Duke Math. J.* 53 (1986), 691–731.
- [6] R. Langlands, On the functional equations satisfied by Eisenstein series, *Lecture Notes in Math.*, vol. 544, Springer, 1976.
- [7] W. Müller, The trace class conjecture in the theory of automorphic forms. *Ann. Math.* 130 (1989), 473–529.
- [8] P. Sarnak, On cusp forms, in: *The Selberg trace formula and related topics*, *Contemporary Math.*, vol. 53, Amer. Math. Soc., 1986, pp. 393–407.
- [9] P. Sarnak, A note on the spectrum of cusp forms for congruence subgroups, preprint, 1983.
- [10] G. Savin, Limit multiplicities of cusp forms. *Invent. Math.* 95 (1989), 149–162.
- [11] A. Selberg, Harmonic analysis, in: *Collected Papers*, vol. 1, Springer, 1989, pp. 626–674.

Humboldt-Universität zu Berlin
 Institut für Mathematik
 Rudower Chaussee 25
 10099 Berlin
 Germany
 hoffmann@mathematik.hu-berlin.de