

Anomalous enstrophy dissipation via triple collapse of point vortices in a Euler-Poincaré system

Takeshi Gotoda *

*Department of mathematics, Kyoto University, Kitashirakawa Oiwake-cho, Sakyo-ku, Kyoto,
JAPAN email : gotoda@math.kyoto-u.ac.jp

Acknowledgements

I am grateful to my supervisor Professor Takashi Sakajo in Kyoto university for his worthwhile advice, spending much time for our discussion, successive educational encouragements and helps throughout my research. The contents of Part [III](#) in this paper are based on the joint work with him. I also thank him for giving me many precious opportunities to talk about my research to the other researchers.

I would also like to express my sincere gratitude to Professor Robert Krasny in University of Michigan for his gentle hospitality during my stay in Ann Arbor, which was supported by KTGU program of top global university project in Kyoto university. I spent a very precious time to discuss mathematical topics with him. His suggestions were always productive and the contents of Part [II](#) are actually motivated by his advice.

I had great experience in my PhD course in Kyoto university. I would like to thank all staffs in the department of mathematics in Kyoto university for supporting me in various situations.

Contents

I	Introduction	5
1	Overview of the theory of turbulence	5
2	Construction of enstrophy dissipating solutions	7
2.1	Review of the two-dimensional Euler equations	7
2.2	The Euler-Poincaré model and point vortex solutions	9
II	The Euler-Poincaré equations	10
3	General framework	10
3.1	Derivation of the Euler-Poincaré equations	10
3.2	Vortex blob model	11
3.3	Euler- α model	12
4	Global solvability and convergence theorems	13
4.1	Statements of theorems	13
4.2	Properties of integral kernel	15
4.3	Global solvability in the space of Radon measure	24
4.4	Convergence to the Euler equations	28
4.4.1	Proof of Theorem 4.3	28
4.4.2	Proof of Corollary 4.6	29
5	Point vortex problem	31
5.1	The Euler-Poincaré point vortex system	31
5.2	Variations of the enstrophy and energy	34
III	Enstrophy dissipation and triple collapse of point vortices	37
6	Review of the point vortex system	37
6.1	The point vortex system and its self-similar solutions	37
6.2	Self-similar collapsing solutions in the three point vortex system . . .	39

7	The α-point vortex system	44
7.1	The N vortex problem and the enstrophy dissipation	44
7.2	The three vortex problem and general properties	47
8	Enstrophy dissipation in the three point vortex problem	51
8.1	Statements of theorems	51
8.2	Existence of enstrophy dissipating solutions	54
8.2.1	$M = 0$ case	54
8.2.2	$M \neq 0$ case, general results	64
8.2.3	$M < 0$ case	67
8.2.4	$M > 0$ case	78
8.3	Convergence to self-similar collapsing solutions	84
9	Concluding Remarks	86
A	Quasi-Lipschitz estimates for the singular kernel	87
B	Fundamental lemmas related to ordinary differential problems	89
C	Properties of some auxiliary functions	93

Part I

Introduction

1 Overview of the theory of turbulence

The turbulent phenomena are caused by the complex interaction of flows in the wide range of scales from macro scales where the energy is supplied, to micro scales where the energy dissipates by the viscosity. The interaction in turbulent flows transfers the energy sequentially from larger scales gradually to smaller ones. This process is known as the *energy cascade*. The theory of Kolmogorov [43] gives a mathematical description to the three-dimensional (3D) homogeneous isotropic turbulence (cf. Frisch [28]). In his theory, it is assumed that the energy dissipation rate should converge to a finite positive value in the limit of vanishing viscosity. In other words, we can expect that solutions of the 3D Euler equations, which are formally equivalent to the zero viscous limit of the 3D Navier-Stokes equations, with the energy dissipation describe turbulent flows. However, it is easily confirmed that smooth solutions of the 3D Euler equations conserve the energy and the energy dissipation rate is always zero. That is to say, such an anomalous energy dissipation occurs only for weak solutions of the 3D Euler equations.

Onsager conjectured in [57], without any mathematically rigorous proof, that weak solutions of the 3D Euler equations acquiring a Hölder continuity with the order greater than $1/3$ should conserve the energy, and the energy dissipation could occur when the order is less than $1/3$, which is known as *Onsager's conjecture* (refer to Eyink & Székelyhidi [25] or Shvydkoy [63] for a survey of Onsager's conjecture). The former part of this conjecture, the energy conservation has been settled recently. A slightly weaker statement was first proved by Eyink [22] and, then, Constantin *et al.* [13] gave a simple proof of the energy conservation under the assumption that the solution belongs to $L^3(0, T; B_{3\infty}^\alpha(\mathbb{T}^3) \cap C([0, T]; L^2(\mathbb{T}^3)))$ with $\alpha > 1/3$, where $\mathbb{T}^3$ denotes three dimensional flat torus. Since the index α in the Besov space corresponds to the order of a Hölder continuity of functions, their result asserts that the energy conservation holds in the critical space of Onsager's conjecture. Subsequently, the result was proven under a slightly weaker assumption by Duchon and Robert [20], and Cheskidov *et al.* [10] also sharpened it. In [20], they defined *dissipative weak solutions* for the Euler equations as solutions whose corresponding energy dissipation rate is greater than or equal to zero in the sense of distribution, and then showed that the dissipation rate is zero when the Hölder continuity of weak solutions is greater than $1/3$. Moreover, they showed the remarkable statistical property in

the turbulence theory, which is predicted by Kolmogorov, called the 4/3-law. Another important statistical property, the 4/5-law, has also been shown in terms of the dissipative weak solutions by Eyink [24] as well. The latter half of Onsager's conjecture, the energy dissipation, has been studied in stages. The first example of a weak solution that breaks the energy conservation was constructed by Scheffer [59], which gives a non-trivial weak solution with a compact support in space and time $\mathbb{R}^2 \times \mathbb{R}$. A simpler construction was later shown by Shnirelman [61]. He also proved the existence of a weak solution whose energy is monotone decreasing [62]. Then, DeLellis & Székelyhidi [14, 15] extended these results to general dimension $n \geq 2$. We note that these constructed solutions are just square integrable functions in space and time, and thus not continuous in general. DeLellis and Székelyhidi have successfully constructed continuous weak solutions whose energy is not conserved [16]. Subsequently, through several contributions to Onsager's conjecture (cf. Buckmaster [5], Buckmaster *et al.* [6], De Lellis & Székelyhidi [15], and Isett [36, 37]), Buckmaster, DeLellis & Székelyhidi [7] have finally proven the existence of a weak solution of the 3D Euler equations that does not conserve the energy and belongs to the space $L^1(0, 1; C^{1/3-\varepsilon}(\mathbb{T}^3))$ for any $\varepsilon > 0$ (Although Onsager claimed that such solutions exist even in the class $L_t^\infty(C_x^{1/3-\varepsilon})$).

As we see from the Kolmogorov-Onsager theory, non-smooth weak solutions of the 3D Euler equations with the energy dissipation may play a vital role in understanding of 3D turbulence. However, the notion of weak solutions is so abstract that it is uncertain what kind of fluid evolution belongs to the class of dissipative weak solutions. A motivation of the present study is describing such an anomalous dissipation in terms of the physical dynamics of fluids without losing mathematical rigor, which could help us understand the essential mechanism of turbulent flows. Especially, we investigate the dynamics of ideal vorticity, called *singular vortices*, such as point vortices, vortex sheets, vortex dipoles, etc., and try to make it clear what kind of vortex dynamics cause the energy dissipation. But, since the existence of weak solutions with singular vorticity has not been established for the 3D Euler equations, it is hard to analyze the evolution of singular vortices mathematically. In order to overcome this difficulty, we regularize the Euler equations in a certain manner so that they have global solutions with singular vorticity, and then take the limit of their global solution with respect to the regularization parameter. Specifically, we consider the Euler-Poincaré equations as a regularized model, which are proposed by Holm, Marsden & Ratiu [32]. For instance, the Euler- α equations are one of the regularized Euler equations derived from the Euler-Poincaré framework, and the 3D Euler- α equations and their viscous extension, the 3D Navier-Stokes- α equations, are regarded as physically relevant models of turbulent flows (cf. Chen *et*

al. [8, 9], Foias, Holm & Titi [26, 27] and Mohseni *et al.* [53]). In spite of its physical significance, there are still difficulties to apply our strategy to the three dimensional problem mathematically, since, for example, the existence of global solutions for the 3D Euler- α equations is as yet unknown (cf. Hou & Li [35]).

For these kinds of difficulties, in the present paper, we focus on the 2D turbulence flow. Many investigations (cf. Batchelor [3], Kraichnan [39] and Leith [44]) indicate that the 2D turbulence is characterized by the conservation of the energy and the dissipation of the enstrophy that is the L^2 norm of the vorticity, since there appear two inertial ranges corresponding to the backward energy cascade and the forward enstrophy cascade in the energy density spectrum at sufficiently large Reynolds number. Hence, similarly to the three dimensional problem, we expect that an enstrophy dissipating solution of the 2D Euler equations describes the 2D turbulent phenomena. On the other hand, since the enstrophy is preserved for smooth solutions, such dissipating solutions need to be non-smooth as well as three dimensional problem. For the 2D Euler equations, the global well-posedness has already been established in the appropriate functional space and, furthermore, the existence of weak solutions are also known in the class of less regular functions. However, it is still difficult to consider the enstrophy dissipation in 2D Euler flows. Although the fluid dynamics in two dimensional flows are different from that in the three dimension, the 2D Euler-Poincaré equations provide us with more theoretical advantages mathematically than the three dimensional ones in dealing with the present problem. In the following section, we review the theory of the 2D Euler equations and explain our strategy to construct enstrophy dissipating solutions.

2 Construction of enstrophy dissipating solutions

2.1 Review of the two-dimensional Euler equations

The motion of inviscid incompressible 2D flows is described by the 2D Euler equations,

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = 0, \quad \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad (2.1)$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is the fluid velocity field and $p = p(\mathbf{x}, t)$ is the scalar pressure. Taking the curl of (2.1) and defining the vorticity field by $\omega = \operatorname{curl} \mathbf{u}$, we obtain the transport equation for ω ,

$$\partial_t \omega + (\mathbf{u} \cdot \nabla) \omega = 0, \quad \omega(\mathbf{x}, 0) = \omega_0(\mathbf{x}), \quad (2.2)$$

and $\omega_0 = \text{curl } \mathbf{u}_0$. Then, the velocity $\mathbf{u}$ is recovered from the vorticity ω via the Biot-Savart law,

$$\mathbf{u}(\mathbf{x}) = (\mathbf{K} * \omega)(\mathbf{x}) = \int_{\mathbb{R}^2} \mathbf{K}(\mathbf{x} - \mathbf{y}) \omega(\mathbf{y}) d\mathbf{y}, \quad (2.3)$$

in which $\mathbf{K}$ denotes a singular integral kernel defined by

$$\mathbf{K}(\mathbf{x}) = \nabla^\perp G(\mathbf{x}) = -\frac{\mathbf{x}^\perp}{2\pi |\mathbf{x}|^2}, \quad G(\mathbf{x}) = -\frac{1}{2\pi} \log |\mathbf{x}|,$$

with $\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})$ and $\mathbf{x}^\perp = (x_2, -x_1)$. Here, the function G is the fundamental solution to the 2D Laplacian. If a passive particle is initially placed at the position $\mathbf{x}$, its evolution is governed by the following equation,

$$\partial_t \boldsymbol{\eta}(\mathbf{x}, t) = \mathbf{u}(\boldsymbol{\eta}(\mathbf{x}, t), t), \quad \boldsymbol{\eta}(\mathbf{x}, 0) = \mathbf{x}, \quad (2.4)$$

where $\boldsymbol{\eta}(\mathbf{x}, t)$ is called the Lagrangian flow map describing the position of the passive point $\mathbf{x}$ at time t . The solution of (2.4) yields that of (2.2) with the initial vorticity ω_0 by

$$\omega(\mathbf{x}, t) = \omega_0(\boldsymbol{\eta}^{-1}(\mathbf{x}, t)). \quad (2.5)$$

Hence, finding a solution to (2.4) with (2.3) and (2.5) is equivalent to solving the problem (2.2) with (2.3). Regarding the initial value problem of the 2D Euler equations, the global existence of a unique weak solution has been established for $\omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ [49, 66]. We also have global weak solutions with $\omega_0 \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ and $1 < p < \infty$, though the uniqueness is still open [19]. Moreover, the existence theorem can be extended to the case that ω_0 belongs to the space of finite Radon measure on $\mathbb{R}^2$, denoted by $\mathcal{M}(\mathbb{R}^2)$, with a distinguished sign and its induced velocity $\mathbf{u}_0 \in L^2_{loc}(\mathbb{R}^2)$ [18, 21, 48]. Note that this class includes vortex sheets but not point vortices. Regarding the enstrophy dissipation, it has been shown by Eyink [23] that weak solutions of the 2D Euler equations in $L^p(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, $p > 2$ can not dissipate the enstrophy in the sense of distribution. Hence, it is necessary to deal with the initial vorticity in a less regular space such as $\mathcal{M}(\mathbb{R}^2)$ to obtain weak solutions dissipating the enstrophy. However, in the 2D Euler equations, the global existence of solutions for $\omega_0 \in \mathcal{M}(\mathbb{R}^2)$ has not yet been established in general. This makes it difficult to construct enstrophy dissipating solutions directly in the 2D Euler equations. Accordingly, if we could construct a singular weak solution of the 2D Euler-Poincaré equations that dissipates the enstrophy by taking the limit of regularized parameter, it may shed some light upon the relation between the 2D turbulence and singular weak solutions of the 2D Euler equations.

2.2 The Euler-Poincaré model and point vortex solutions

In accordance with the strategy we mentioned above, we consider the 2D Euler-Poincaré equations to construct enstrophy dissipating solutions. As we discuss the detail of those equations in Part II, it is noteworthy that two well-known regularizations of the Euler equations, the Euler- α equations and the vortex blob method, can be described in the Euler-Poincaré framework [34], see Section 3, though both models are originally derived in different backgrounds and are usually applied to the different problem. The α -regularized models for the Euler equations and the Navier-Stokes equations are effective models for fluid turbulence even in two dimensional flows. Indeed, Lunasin *et al.* [47] numerically confirmed that the 2D Navier-Stokes- α flow acquired the inertial ranges in the energy density spectrum that correspond to the backward energy cascade and the forward enstrophy cascade for scales larger than the filtering level α . On the other hand, the vortex blob method was originally introduced by Chorin [11] and Chorin & Bernard [12] to compute the evolution of vortex sheets, also see [1, 40]. Moreover, this method has been applied to numerical simulations of incompressible flows [45], especially the dynamics of singular vorticity distributions such as point vortices or vortex sheets. Thus, we expect that the 2D Euler-Poincaré equations are also a relevant model of 2D turbulent flows and effective to investigate the dynamics of the singular vorticity. Actually, in Section 4, we prove the 2D Euler-Poincaré equations have a unique global weak solution with initial vorticity in $\mathcal{M}(\mathbb{R}^2)$, which is one of the remarkable mathematical advantages to treat the Euler-Poincaré system. In Section 5, we especially consider point vortex solutions of the 2D Euler-Poincaré equations, which are the solutions for point vortex initial data. Then, the equations are reduced to the ordinary differential equations that describe as dynamics of point vortices, we call them *the Euler-Poincaré point-vortex system*. Owing to the global solvability of the equations, point vortices in the Euler-Poincaré flow never collapse. In the preceding research by Sakajo [58], he investigated the α -point vortex system that is derived from the Euler- α equations, see Subsection 7, and numerically suggested that solutions of three vortex problem in the α -point vortex system dissipate the enstrophy in the limit of $\alpha \rightarrow 0$ under certain conditions. Moreover, numerical simulations imply that such a dissipation could occur at the time of the triple collapse of point vortices. Taking into account this work, we show mathematically that the existence of the enstrophy dissipating solutions and the triple collapse of point vortices causes the dissipation. In the three point vortex system, it is known that self-similar collapsing solutions exist in a certain circumstance [2, 38, 54], which is discussed in Section 6. Our theorems in Section 8 assert that there exist point vortex solutions of three α point vortex system

that converge to self-similar collapsing solutions of the three point vortex system and dissipate the enstrophy at the time of the triple collapse. We finally summarize our research and discuss future works in Section 9.

Part II

The Euler-Poincaré equations

3 General framework

3.1 Derivation of the Euler-Poincaré equations

In this section, we derive a regularized 2D Euler equations based on the framework given in [26, 34]. Let $\mathbf{v}$ be an incompressible velocity field that we call *singular velocity*. We define a *regularized velocity* by

$$\mathbf{u}_h(\mathbf{x}) = (h * \mathbf{v})(\mathbf{x}) = \int_{\mathbb{R}^2} h(\mathbf{x} - \mathbf{y}) \mathbf{v}(\mathbf{y}) d\mathbf{y}, \quad (3.1)$$

where h is a scalar valued function on $\mathbb{R}^2$. In this paper, we assume that h is a integrable function and may have a singularity at the origin. Owing to the convolution with h , $\mathbf{u}_h$ is smoother than $\mathbf{v}$. Similarly, we define the *singular vorticity* q and *regularized vorticity* ω_h by

$$q = \text{curl } \mathbf{v}, \quad \omega_h = \text{curl } \mathbf{u}_h.$$

Let us remark that it follows that $\text{div } \mathbf{u}_h = 0$ and $\omega_h = h * q$ when the convolution commutes with the differential operator.

Now, we consider the Hamiltonian structure with the Hamiltonian,

$$\mathcal{H} = \frac{1}{2} \int_{\mathbb{R}^2} \mathbf{v}(\mathbf{x}) \cdot \mathbf{u}_h(\mathbf{x}) d\mathbf{x},$$

and the Lagrangian flow map $\boldsymbol{\eta}_h$ induced by regularized velocity,

$$\partial_t \boldsymbol{\eta}_h(\mathbf{x}, t) = \mathbf{u}_h(\boldsymbol{\eta}_h(\mathbf{x}, t), t), \quad \boldsymbol{\eta}_h(\mathbf{x}, 0) = \mathbf{x}. \quad (3.2)$$

Then, the Euler-Poincaré equations arise from an application of Hamilton's principle as follows.

$$\partial_t \mathbf{v} + (\mathbf{u}_h \cdot \nabla) \mathbf{v} - (\nabla \mathbf{v})^T \cdot \mathbf{u}_h - \nabla \Pi = 0, \quad (3.3)$$

where Π is a generalized pressure. Taking the curl of (3.3) with the divergence-free condition, we obtain the transport equation for the singular vorticity convected by the regularized velocity,

$$\partial_t q + (\mathbf{u}_h \cdot \nabla) q = 0.$$

Note that the Biot-Savart law gives $\mathbf{u}_h = \mathbf{K} * \omega_h$. Then, we finally obtain the vorticity form of the Euler-Poincaré equations.

$$\partial_t q + (\mathbf{u}_h \cdot \nabla) q = 0, \quad \mathbf{u}_h = \mathbf{K}_h * q, \quad \mathbf{K}_h = \mathbf{K} * h. \quad (3.4)$$

It is important to remark that the regularized Kernel $\mathbf{K}_h$ satisfies

$$\mathbf{K}_h = \nabla^\perp G_h, \quad -\Delta G_h = h, \quad (3.5)$$

where G_h is a solution for the Poisson equation and thus we have $G_h = G * h$. Considering the definition of the singular integral kernel, we can see that a feature of the regularized equations (3.4) comes from how to regularize the delta function by the smoothing function h .

3.2 Vortex blob model

The vortex blob method is one of the most common regularizations of the Euler flow. The method was introduced to compute the vortex sheets' evolutions numerically [1, 11, 12, 40] and has been applied to a variety of fluid dynamics, for instance see [45]. In this method, the regularized integral kernel $\mathbf{K}_\delta$ is given by

$$\mathbf{K}^\delta(\mathbf{x}) = -\frac{1}{2\pi} \frac{\mathbf{x}^\perp}{|\mathbf{x}|^2 + \delta^2}.$$

This vortex blob regularization can be derived from the Euler-Poincaré framework. Indeed, replacing the function h in (3.1) by

$$h^\delta(\mathbf{x}) = \frac{1}{2\pi\delta^2} \psi\left(\frac{|\mathbf{x}|}{\delta}\right), \quad \psi(r) = \frac{2}{(r^2 + 1)^2},$$

we find that $\mathbf{K}^\delta$ comes through the solution for the following Poisson equation.

$$\mathbf{K}^\delta = \nabla^\perp G^\delta, \quad -\Delta G^\delta = h^\delta.$$

It is easily confirmed that the solution G^δ is given by

$$G^\delta(\mathbf{x}) = G_r^\delta(|\mathbf{x}|) = -\frac{1}{2\pi} \log \sqrt{|\mathbf{x}|^2 + \delta^2},$$

and $\mathbf{K}^\delta$ is rewritten by

$$\mathbf{K}^\delta(\mathbf{x}) = \mathbf{K}(\mathbf{x})\Psi\left(\frac{|\mathbf{x}|}{\delta}\right), \quad \Psi(r) = -2\pi r \frac{dG_r^{\delta=1}}{dr}(r) = \frac{r^2}{r^2 + 1}.$$

Note that we have $\Psi'(r) = r\psi(r)$. Hence, both G^δ and $\mathbf{K}^\delta$ have no singularity, which is the essence of the regularization. A remarkable feature of the vortex blob regularization is that the smoothing function h^δ is bounded and decays algebraically. Regarding the relation with the Euler equations in the $\delta \rightarrow 0$ limit, it has been shown in [46] that the vortex blob methods converge to weak solutions of the 2D Euler equations under the conditions that the vorticity is in $\mathcal{M}(\mathbb{R}^2)$ with a distinguished sign and its induced velocity is locally square integrable.

3.3 Euler- α model

The Euler- α model was first introduced in [32, 33, 31] as the higher dimensional Camassa-Holm equations. We can also derive it by applying Lagrangian averaging to the Euler equations. The spatial information of the flow at scales smaller than α is averaged in the Euler- α model [50]. A remarkable property of this model is that the regularized velocity $\mathbf{u}^\alpha$ is written by $\mathbf{u}^\alpha = (1 - \alpha^2 \Delta)^{-1} \mathbf{v}$, see [26], and thus the smoothing function h^α in (3.1) is defined by a fundamental solution for the Helmholtz operator $1 - \alpha^2 \Delta$, namely

$$h^\alpha(\mathbf{x}) = \frac{1}{2\pi\alpha^2} K_0\left(\frac{|\mathbf{x}|}{\alpha}\right),$$

in which K_0 denotes the modified Bessel function of the second kind. It is important to remark that K_0 has a singularity at the origin like $K_0(r) \sim -\log r$ as $r \rightarrow 0$ and decays exponentially. Since the singular velocity $\mathbf{v}$ is explicitly expressed by the regularized velocity $\mathbf{u}^\alpha$ with the Helmholtz operator, we find the Euler-Poincaré equations for $\mathbf{u}^\alpha$ as follows.

$$(1 - \alpha^2 \Delta) \partial_t \mathbf{u}^\alpha + \mathbf{u}^\alpha \cdot \nabla (1 - \alpha^2 \Delta) \mathbf{u}^\alpha + (\nabla \mathbf{u}^\alpha)^T \cdot (1 - \alpha^2 \Delta) \mathbf{u}^\alpha = \nabla \Pi.$$

This equations are known as the Euler- α equations. Regarding the evolution of the singular vorticity described by the equation corresponding to (3.4), there exist a unique global weak solution for the initial vorticity $q_0 \in \mathcal{M}(\mathbb{R}^2)$ in contrast to the 2D Euler equations [56]. Moreover, it is also shown that a weak solution for $q_0 \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ converges to that of the vorticity form of the 2D Euler equations (2.2). In terms of the formulation in (3.5), we have

$$\mathbf{K}^\alpha = \nabla^\perp G^\alpha, \quad -\Delta G^\alpha(\mathbf{x}) = h^\alpha(\mathbf{x}),$$

and the solution G^α for the Poisson equation is concretely given by

$$G^\alpha(\mathbf{x}) = G_r^\alpha(|\mathbf{x}|) = -\frac{1}{2\pi} \left[\log |\mathbf{x}| + K_0 \left(\frac{|\mathbf{x}|}{\alpha} \right) \right].$$

Hence, we obtain the regularized integral kernel $\mathbf{K}^\alpha$ as follows.

$$\mathbf{K}^\alpha(\mathbf{x}) = -\frac{1}{2\pi} \frac{\mathbf{x}^\perp}{|\mathbf{x}|^2} B_K \left(\frac{|\mathbf{x}|}{\alpha} \right) = \mathbf{K}(\mathbf{x}) B_K \left(\frac{|\mathbf{x}|}{\alpha} \right), \quad (3.6)$$

where $B_K(r) = 1 - rK_1(r)$ and K_1 is the first order modified Bessel function of the second kind. Similar to the vortex blob case, we have the relation,

$$B_K(r) = -2\pi r \frac{dG_r^{\alpha=1}}{dr}, \quad B'_K(r) = rK_0(r).$$

Especially, it follows from the properties of the Bessel function that G^α is bounded and $\mathbf{K}^\alpha(0) = 0$, namely they have no singularity.

4 Global solvability and convergence theorems

4.1 Statements of theorems

Our main results in this section consist of two theorems; The existence of a unique global weak solution for (3.4) in the space of Radon measure and its convergence to the Euler equations with the integrable and bounded initial vorticity. Recall that $\mathcal{M}(\mathbb{R}^2)$ denotes the space of finite Radon measures on $\mathbb{R}^2$ with the norm,

$$\|\mu\|_{\mathcal{M}} = \sup \left\{ \left| \int_{\mathbb{R}^2} f d\mu \right| \mid f \in C_0(\mathbb{R}^2), \|f\|_{L^\infty} \leq 1 \right\},$$

where $C_0(\mathbb{R}^2)$ is the space of continuous functions vanishing at infinity. For later use, we define functions as follows.

$$\chi_{\log}^-(\mathbf{x}) = \begin{cases} (1 - \log |\mathbf{x}|)^{-1} & , |\mathbf{x}| \leq 1, \\ 0 & , |\mathbf{x}| > 1, \end{cases} \quad \chi_{\log}^+(\mathbf{x}) = \begin{cases} 0 & , |\mathbf{x}| \leq 1, \\ 1 + \log |\mathbf{x}| & , |\mathbf{x}| > 1, \end{cases}$$

and

$$\chi_\alpha^-(\mathbf{x}) = \begin{cases} |\mathbf{x}|^\alpha & , |\mathbf{x}| \leq 1, \\ 0 & , |\mathbf{x}| > 1, \end{cases} \quad \chi_\alpha^+(\mathbf{x}) = \begin{cases} 0 & , |\mathbf{x}| \leq 1, \\ |\mathbf{x}|^\alpha & , |\mathbf{x}| > 1. \end{cases}$$

We also set $\chi_\alpha(\mathbf{x}) = \chi_\alpha^-(\mathbf{x}) + \chi_\alpha^+(\mathbf{x}) = |\mathbf{x}|^\alpha$ for $\mathbf{x} \in \mathbb{R}^2$. These functions are used to characterize a singularity and a decay rate of functions. Then, we have the following theorem.

Theorem 4.1. Suppose that $h \in C^1(\mathbb{R}^2) \cap W_1^1(\mathbb{R}^2)$ satisfies $\chi_1^+ h \in L^1(\mathbb{R}^2)$ and

$$\chi_{\log}^- h \in L^\infty(\mathbb{R}^2), \quad \chi_1 \nabla h \in L^\infty(\mathbb{R}^2). \quad (4.1)$$

Then, for any initial vorticity $q_0 \in \mathcal{M}(\mathbb{R}^2)$, there exists a unique global weak solution of (3.4) such that

$$\boldsymbol{\eta}_h \in C^1(\mathbb{R}; \mathcal{G}), \quad \mathbf{u}_h \in C(\mathbb{R}; C(\mathbb{R}^2; \mathbb{R}^2)), \quad q \in C(\mathbb{R}; \mathcal{M}(\mathbb{R}^2)),$$

where $\mathcal{G}$ denotes the group of all homeomorphism of $\mathbb{R}^2$ that preserves the Lebesgue measure.

Remark 4.2. The assumption (4.1) allows the smoothing function h and ∇h to have singularities and behave

$$h(\mathbf{x}) \sim \mathcal{O}(-\log |\mathbf{x}|), \quad \nabla h(\mathbf{x}) \sim \mathcal{O}(|\mathbf{x}|^{-1}),$$

as $|\mathbf{x}| \rightarrow 0$. The decay rates of them are estimated by the conditions $\chi_1 h, \nabla h \in L^1$.

In order to consider the convergence of solutions in (3.4) to those in the Euler equations, we consider the parameterized function h^ε defined by

$$h^\varepsilon(\mathbf{x}) = \frac{1}{2\pi\varepsilon^2} h\left(\frac{\mathbf{x}}{\varepsilon}\right).$$

Moreover, we assume that h is a radial function, namely it depends only on $r = |\mathbf{x}|$, and define $h_r(|\mathbf{x}|) = h(\mathbf{x})$ that is a function on $\mathbb{R}$. Then, we have the following parameterized solutions $(\boldsymbol{\eta}^\varepsilon, \mathbf{u}^\varepsilon, q)$ such that

$$\begin{aligned} \partial_t \boldsymbol{\eta}^\varepsilon(\mathbf{x}, t) &= \mathbf{u}^\varepsilon(\boldsymbol{\eta}^\varepsilon(\mathbf{x}, t), t), \quad \boldsymbol{\eta}^\varepsilon(\mathbf{x}, 0) = \mathbf{x}, \\ q(\mathbf{x}, t) &= q_0(\boldsymbol{\eta}^\varepsilon(\mathbf{x}, -t)), \\ \mathbf{u}^\varepsilon &= \mathbf{K}^\varepsilon * q, \quad \mathbf{K}^\varepsilon = \mathbf{K} * h^\varepsilon, \end{aligned}$$

and the following theorem about the convergence to the Euler equations in the $\varepsilon \rightarrow 0$ limit holds.

Theorem 4.3. In addition to the assumptions in Theorem 4.1, suppose that h is radial and satisfies

$$\int_0^\infty k h_r(k) dk = 1.$$

Let $q_0 = \omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Then, for any $T > 0$, there exists $C(T) > 0$ such that

$$\sup_{t \in [0, T]} \sup_{\mathbf{x} \in \mathbb{R}^2} |\boldsymbol{\eta}^\varepsilon(\mathbf{x}, t) - \boldsymbol{\eta}(\mathbf{x}, t)| \leq C(T) \varepsilon^{e^{-T}},$$

where $\boldsymbol{\eta}$ is the flow map in the Euler system with the initial vorticity ω_0 .

Remark 4.4. These theorems are a generalization of the results shown by Oliver & Shkoller [56], in which they proved the global solvability and convergence theorems for the 2D Euler- α equations.

Remark 4.5. We remark that both the smoothing functions h^δ and h^α satisfy the assumptions of Theorem 4.1 and Theorem 4.3. Especially, the Euler- α model is a critical regularization in terms of the singularity since we know $K_0(r) \sim -\log r$ as $r \rightarrow 0$ in the smoothing function h^α . On the other hand, it is easily confirmed that h^δ has a slowest algebraic decay to satisfy the assumptions of the theorems.

We easily find the following corollary from Theorem 4.3, which is the same result shown in [56] for the Euler- α equations.

Corollary 4.6. *Suppose that $q_0 = \omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is approximated by a sequence $\{q_0^n\}_{n \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^2)$ with $q_0^n \rightharpoonup q_0$ weakly in $\mathcal{M}(\mathbb{R}^2)$ and $\|q_0^n\|_{\mathcal{M}} \rightarrow \|q_0\|_{L^1}$. Then, for any $T > 0$ and sequence $\{\varepsilon^m\}_{m \in \mathbb{N}}$ that converges to zero, there exists a subsequence $\{q_0^{n^m}\}_{m \in \mathbb{N}}$ such that*

$$\lim_{m \rightarrow \infty} \sup_{t \in [0, T]} \sup_{\mathbf{x} \in \mathbb{R}^2} |\boldsymbol{\eta}^{n^m, \varepsilon^m}(\mathbf{x}, t) - \boldsymbol{\eta}(\mathbf{x}, t)| = 0,$$

where $\boldsymbol{\eta}^{n^m, \varepsilon^m}$ denotes the flow map in the parameterized Euler-Poincaré flows with the parameter ε^m and the initial vorticity $q_0^{n^m}$.

4.2 Properties of integral kernel

In order to show the main theorems, it is important to investigate the properties of the integral kernel $\mathbf{K}_h$ and $\mathbf{K}^\varepsilon$ in the regularized Biot-Savart law. Let us first define the function $\varphi(r)$ for $r \geq 0$ by

$$\varphi(r) = \begin{cases} r(1 - \log r) & , \quad r < 1, \\ 1 & , \quad r \geq 1. \end{cases}$$

We now show that $\mathbf{K}_h$ is well-defined and especially $\mathbf{K}_h$ vanishes at the origin. Here, the smoothing function h may have a singularity only at the origin. We actually assume that $h \in L^\infty(\mathbb{R}^2 \setminus B_\delta)$ for sufficiently small $\delta > 0$, where $B_R = \{\mathbf{y} \in \mathbb{R}^2 \mid |\mathbf{y}| < R\}$.

Lemma 4.7. *Suppose that $h \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2 \setminus B_\delta)$, $\delta > 0$ and $\chi_\alpha^- h \in L^\infty(\mathbb{R}^2)$, $\alpha \in [0, 1)$. Then, $\mathbf{K}_h$ is uniformly bounded, namely $\mathbf{K}_h \in L^\infty(\mathbb{R}^2)$. Moreover, if h is radial, then $\mathbf{K}_h(0) = 0$.*

Proof. We have

$$\begin{aligned}
|\mathbf{K}_h(\mathbf{x})| &\leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{|\mathbf{x} - \mathbf{y}|} |h(\mathbf{y})| d\mathbf{y} \\
&\leq \frac{1}{2\pi} \int_{|\mathbf{x} - \mathbf{y}| < \delta} \frac{1}{|\mathbf{x} - \mathbf{y}|} |h(\mathbf{y})| d\mathbf{y} + \frac{1}{2\pi} \int_{|\mathbf{x} - \mathbf{y}| \geq \delta} \frac{1}{|\mathbf{x} - \mathbf{y}|} |h(\mathbf{y})| d\mathbf{y} \\
&\leq \frac{1}{2\pi} \|\chi_\alpha^- h\|_{L^\infty} \left[\int_{|\mathbf{y}| < |\mathbf{x} - \mathbf{y}| < \delta} \frac{1}{|\mathbf{y}|^{1+\alpha}} d\mathbf{y} + \int_{|\mathbf{x} - \mathbf{y}| \leq |\mathbf{y}| < \delta} \frac{1}{|\mathbf{x} - \mathbf{y}|^{1+\alpha}} d\mathbf{y} \right] \\
&\quad + \frac{1}{2\pi} \|h\|_{L^\infty(\mathbb{R}^2 \setminus B_\delta)} \int_{|\mathbf{x} - \mathbf{y}| < \delta \cap |\mathbf{y}| \geq \delta} \frac{1}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} + \frac{1}{2\pi\delta} \int_{\mathbb{R}^2} |h(\mathbf{y})| d\mathbf{y} \\
&\leq \frac{2\delta^{1-\alpha}}{1-\alpha} \|\chi_\alpha^- h\|_{L^\infty} + \delta \|h\|_{L^\infty(\mathbb{R}^2 \setminus B_\delta)} + \frac{1}{2\pi\delta} \|h\|_{L^1},
\end{aligned}$$

in which $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(\mathbb{R}^2)}$ and this notation is used throughout this paper. Thus, $\mathbf{K}_h$ is uniformly bounded. In terms of the polar coordinates, the value of $\mathbf{K}_h$ at the origin is calculated as follows.

$$\begin{aligned}
\mathbf{K}_h(0) &= -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(y_2, -y_1)}{|\mathbf{y}|^2} h(\mathbf{y}) d\mathbf{y} \\
&= -\frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \frac{(k \sin \theta, -k \cos \theta)}{k^2} h_r(k) k d\theta dk \\
&= -\frac{1}{2\pi} \int_0^\infty h_r(k) dk \int_0^{2\pi} (\sin \theta, -\cos \theta) d\theta,
\end{aligned}$$

Since h_r is integrable, we conclude that $\mathbf{K}_h(0) = 0$. □

Remark 4.8. When we show that $\mathbf{K}_h$ is bounded outside of a neighborhood of the origin, it is enough to suppose that $h \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2 \setminus B_\delta)$.

Next, we see the asymptotic behavior of $\mathbf{K}_h$. The following lemma asserts that it decays at least as fast as the singular kernel $\mathbf{K}$.

Lemma 4.9. *For $h \in L^1(\mathbb{R}^2)$ satisfying $\chi_1^+ h \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, we have*

$$\limsup_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}| |\mathbf{K}_h(\mathbf{x})| \leq \frac{1}{2\pi} \|h\|_{L^1}.$$

Proof. We divide the integral in $\mathbf{K}_h$ as follows.

$$\begin{aligned} |\mathbf{x}| |\mathbf{K}_h(\mathbf{x})| &\leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(1 + \frac{|\mathbf{y}|}{|\mathbf{x} - \mathbf{y}|}\right) |h(\mathbf{y})| d\mathbf{y} \\ &= \frac{1}{2\pi} \left[\|h\|_{L^1} + \int_{B_R} \frac{|\mathbf{y}|}{|\mathbf{x} - \mathbf{y}|} |h(\mathbf{y})| d\mathbf{y} + \int_{\mathbb{R}^2 \setminus B_R} \frac{|\mathbf{y}|}{|\mathbf{x} - \mathbf{y}|} |h(\mathbf{y})| d\mathbf{y} \right]. \end{aligned}$$

The second term in the right-hand side vanishes in the $|\mathbf{x}| \rightarrow \infty$ limit, since we have

$$\int_{B_R} \frac{|\mathbf{y}|}{|\mathbf{x} - \mathbf{y}|} |h(\mathbf{y})| d\mathbf{y} \leq \frac{R}{|\mathbf{x}| - R} \int_{B_R} |h(\mathbf{y})| d\mathbf{y},$$

for sufficiently large $|\mathbf{x}|$. The third term is estimated as follows.

$$\begin{aligned} \int_{\mathbb{R}^2 \setminus B_R} \frac{|\mathbf{y}|}{|\mathbf{x} - \mathbf{y}|} |h(\mathbf{y})| d\mathbf{y} &= \int_{(\mathbb{R}^2 \setminus B_R) \cap |\mathbf{x} - \mathbf{y}| < \delta} \frac{|\mathbf{y}|}{|\mathbf{x} - \mathbf{y}|} |h(\mathbf{y})| d\mathbf{y} \\ &\quad + \int_{(\mathbb{R}^2 \setminus B_R) \cap |\mathbf{x} - \mathbf{y}| \geq \delta} \frac{|\mathbf{y}|}{|\mathbf{x} - \mathbf{y}|} |h(\mathbf{y})| d\mathbf{y} \\ &\leq 2\pi\delta \|\chi_1^+ h\|_{L^\infty(\mathbb{R}^2 \setminus B_R)} + \frac{1}{\delta} \|\chi_1^+ h\|_{L^1(\mathbb{R}^2 \setminus B_R)}. \end{aligned}$$

Summarizing the above estimates, we find

$$\limsup_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}| |\mathbf{K}_h(\mathbf{x})| \leq \frac{1}{2\pi} \|h\|_{L^1} + \delta \|\chi_1^+ h\|_{L^\infty(\mathbb{R}^2 \setminus B_1)} + \frac{1}{2\pi\delta} \|\chi_1^+ h\|_{L^1(\mathbb{R}^2 \setminus B_R)}.$$

Since $\chi_1^+ h$ is integrable, the third term vanishes in the $R \rightarrow \infty$ limit. Finally, taking the $\delta \rightarrow 0$ limit, we obtain the desired result. $\square$

In the following lemma, we show that $\mathbf{K}_h$ is quasi-Lipschitz continuous.

Lemma 4.10. *Let $h \in C^1(\mathbb{R}^2) \cap W_1^1(\mathbb{R}^2)$ satisfies $\chi_{\log}^- h \in L^\infty(\mathbb{R}^2)$ and $\chi_1 \nabla h \in L^\infty(\mathbb{R}^2)$. Then,*

$$|\mathbf{K}_h(\mathbf{x}) - \mathbf{K}_h(\mathbf{x}')| \leq c\varphi(|\mathbf{x} - \mathbf{x}'|).$$

Proof. Let us set $r = |\mathbf{x} - \mathbf{x}'|$ and note that it is enough to prove the lemma for

sufficiently small $r > 0$. We have

$$\begin{aligned}
|\mathbf{K}_h(\mathbf{x}) - \mathbf{K}_h(\mathbf{x}')| &= \left| \int_{\mathbb{R}^2} \mathbf{K}(\mathbf{y}) (h(\mathbf{x} - \mathbf{y}) - h(\mathbf{x}' - \mathbf{y})) d\mathbf{y} \right| \\
&\leq \int_{|\mathbf{x} - \mathbf{y}| < 2r} |\mathbf{K}(\mathbf{y})| (|h(\mathbf{x} - \mathbf{y})| + |h(\mathbf{x}' - \mathbf{y})|) d\mathbf{y} \\
&\quad + \int_{2r \leq |\mathbf{x} - \mathbf{y}| < 2} |\mathbf{K}(\mathbf{y})| |h(\mathbf{x} - \mathbf{y}) - h(\mathbf{x}' - \mathbf{y})| d\mathbf{y} \\
&\quad + \int_{|\mathbf{x} - \mathbf{y}| \geq 2} |\mathbf{K}(\mathbf{y})| |h(\mathbf{x} - \mathbf{y}) - h(\mathbf{x}' - \mathbf{y})| d\mathbf{y} \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}$$

Considering the fact that if $|\mathbf{x} - \mathbf{y}| < 2r$, then $|\mathbf{x}' - \mathbf{y}| < 3r$, we obtain

$$\begin{aligned}
I_1 &\leq \frac{1}{\pi} \sup_{\mathbf{x} \in \mathbb{R}^2} \int_{|\mathbf{x} - \mathbf{y}| < 3r} \frac{1}{|\mathbf{y}|} |h(\mathbf{x} - \mathbf{y})| d\mathbf{y} \\
&\leq \frac{1}{\pi} \|\chi_{\log}^- h\|_{L^\infty} \sup_{\mathbf{x} \in \mathbb{R}^2} \int_{|\mathbf{x} - \mathbf{y}| < 3r} \frac{1 - \log |\mathbf{x} - \mathbf{y}|}{|\mathbf{y}|} d\mathbf{y} \\
&\leq \frac{1}{\pi} \|\chi_{\log}^- h\|_{L^\infty} \sup_{\mathbf{x} \in \mathbb{R}^2} \left[\int_{|\mathbf{y}| < |\mathbf{x} - \mathbf{y}| < 3r} \frac{1 - \log |\mathbf{y}|}{|\mathbf{y}|} d\mathbf{y} \right. \\
&\quad \left. + \int_{|\mathbf{x} - \mathbf{y}| \leq |\mathbf{y}| \cap |\mathbf{x} - \mathbf{y}| < 3r} \frac{1 - \log |\mathbf{x} - \mathbf{y}|}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} \right] \\
&\leq c_1 \varphi(r) \|\chi_{\log}^- h\|_{L^\infty}.
\end{aligned}$$

In order to estimate I_2 and I_3 , we apply the mean value theorem to h . Then, we obtain

$$|h(\mathbf{x} - \mathbf{y}) - h(\mathbf{x}' - \mathbf{y})| \leq r \int_0^1 |(\nabla h)(\mathbf{x} - \mathbf{y} + \tau(\mathbf{x}' - \mathbf{x}))| d\tau.$$

Moreover, it follows from $|\mathbf{x} - \mathbf{y}| \geq 2r$ that

$$|\mathbf{x} - \mathbf{y} + \tau(\mathbf{x}' - \mathbf{x})| \geq ||\mathbf{x} - \mathbf{y}| - \tau r| \geq \frac{1}{2}|\mathbf{x} - \mathbf{y}| + (1 - \tau)r \geq \frac{1}{2}|\mathbf{x} - \mathbf{y}|. \quad (4.2)$$

Thus, setting $D = \{\mathbf{y} \in \mathbb{R}^2 \mid 2r \leq |\mathbf{x} - \mathbf{y}| < 2\}$, we find

$$\begin{aligned}
I_2 &\leq \frac{r}{2\pi} \|\chi_1 \nabla h\|_{L^\infty} \int_D \frac{1}{|\mathbf{y}|} \int_0^1 \frac{1}{|\mathbf{x} - \mathbf{y} + \tau(\mathbf{x}' - \mathbf{x})|} d\tau d\mathbf{y} \\
&\leq \frac{r}{\pi} \|\chi_1 \nabla h\|_{L^\infty} \int_D \frac{1}{|\mathbf{y}| |\mathbf{x} - \mathbf{y}|} d\mathbf{y} \\
&\leq \frac{r}{\pi} \|\chi_1 \nabla h\|_{L^\infty} \left[\frac{1}{2r} \int_{D \cap |\mathbf{y}| < 2r} \frac{1}{|\mathbf{y}|} d\mathbf{y} + \int_{D \cap 2r \leq |\mathbf{y}| < |\mathbf{x} - \mathbf{y}|} \frac{1}{|\mathbf{y}|^2} d\mathbf{y} \right. \\
&\quad \left. + \int_{D \cap |\mathbf{x} - \mathbf{y}| \leq |\mathbf{y}|} \frac{1}{|\mathbf{x} - \mathbf{y}|^2} d\mathbf{y} \right] \\
&\leq 2r \|\chi_1 \nabla h\|_{L^\infty} \left[\frac{1}{2r} \int_0^{2r} dk + 2 \int_{2r}^2 \frac{1}{k} dk \right] \\
&\leq c_2 \varphi(r) \|\chi_1 \nabla h\|_{L^\infty}.
\end{aligned}$$

Finally, it follows from (4.2) that

$$\begin{aligned}
I_3 &\leq \frac{r}{2\pi} \int_{2 \leq |\mathbf{x} - \mathbf{y}|} \frac{1}{|\mathbf{y}|} \int_0^1 |(\nabla h)(\mathbf{x} - \mathbf{y} + \tau(\mathbf{x}' - \mathbf{x}))| d\tau d\mathbf{y} \\
&\leq \frac{r}{2\pi} \left[\|\chi_1 \nabla h\|_{L^\infty} \int_{2 \leq |\mathbf{x} - \mathbf{y}| \cap |\mathbf{y}| < 1} \frac{2}{|\mathbf{y}| |\mathbf{x} - \mathbf{y}|} d\mathbf{y} \right. \\
&\quad \left. + \int_{2 \leq |\mathbf{x} - \mathbf{y}| \cap |\mathbf{y}| \geq 1} \int_0^1 |(\nabla h)(\mathbf{x} - \mathbf{y} + \tau(\mathbf{x}' - \mathbf{x}))| d\tau d\mathbf{y} \right] \\
&\leq r \left[\|\chi_1 \nabla h\|_{L^\infty} + \frac{1}{2\pi} \|\nabla h\|_{L^1} \right].
\end{aligned}$$

Combining the above three estimates, we achieve the conclusion. $\square$

Remark 4.11. As a consequence of above lemmas, we find $\mathbf{K}_h \in C_0(\mathbb{R}^2)$. This plays an essential role to show the existence of a unique global solution of (3.4) with initial vorticity in $\mathcal{M}(\mathbb{R}^2)$. Indeed, owing to the following estimate,

$$\|\mathbf{u}_h\|_{L^\infty} \leq \sup_{\mathbf{x} \in \mathbb{R}^2} \left| \int_{\mathbb{R}^2} \mathbf{K}_h(\mathbf{x} - \mathbf{y}) q(\mathbf{y}) d\mathbf{y} \right| \leq \|\mathbf{K}_h\|_{L^\infty} \|q\|_{\mathcal{M}}, \quad (4.3)$$

and Lemma 4.10, we find that $\mathbf{u}_h$ is uniformly bounded and quasi-Lipschitz continuous, which allows us to apply the classical method to prove Theorem 4.1.

Finally, we investigate the relation between $\mathbf{K}^\varepsilon$ and $\mathbf{K}$. Before that, we show the following lemma.

Lemma 4.12. *Suppose that $h \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2 \setminus B_\delta)$, $\delta > 0$ satisfies $\chi_\alpha^- h \in L^\infty(\mathbb{R}^2)$, $\alpha \in [0, 1]$, and $\chi_{\log}^+ h \in L^1(\mathbb{R}^2)$. Then, the scalar function*

$$G_h(\mathbf{x}) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} (\log |\mathbf{x} - \mathbf{y}|) h(\mathbf{y}) d\mathbf{y},$$

is well-defined for any $\mathbf{x} \in \mathbb{R}^2$ and satisfies the Poisson equation, $-\Delta G_h = h$, in the weak sense that $\langle \nabla G_h, \nabla \phi \rangle = -\langle h, \phi \rangle$ for any $\phi \in C_0^\infty(\mathbb{R}^2)$. Moreover, we have

$$\lim_{|\mathbf{x}| \rightarrow \infty} \frac{G_h(\mathbf{x})}{\log |\mathbf{x}|} = -\frac{1}{2\pi} \int_{\mathbb{R}^2} h(\mathbf{y}) d\mathbf{y}. \quad (4.4)$$

Epecially, if h is a radial function, G_h is also radial, namely $G_h(\mathbf{x}) = G_{h,r}(|\mathbf{x}|)$.

Regarding the derivative of $G_{h,r}$, suppose that the radial function $h \in L^1(\mathbb{R}^2)$ satisfies $\chi_\alpha^- h \in L^\infty(\mathbb{R}^2)$, $\alpha \in [0, 1]$ and $\chi_1^+ h \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Then, we have $G'_{h,r}(0) = 0$ and

$$\lim_{r \rightarrow \infty} r G'_{h,r}(r) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} h(\mathbf{y}) d\mathbf{y}. \quad (4.5)$$

Proof. We first show that G_h is well-defined. For any fixed $\mathbf{x} \in \mathbb{R}^2$, we divide the integral into two parts,

$$|G_h(\mathbf{x})| \leq \frac{1}{2\pi} \left[\int_{|\mathbf{x}-\mathbf{y}| < 1} |\log |\mathbf{x} - \mathbf{y}|| |h(\mathbf{y})| d\mathbf{y} + \int_{|\mathbf{x}-\mathbf{y}| \geq 1} |\log |\mathbf{x} - \mathbf{y}|| |h(\mathbf{y})| d\mathbf{y} \right].$$

Each integral is estimated as follows.

$$\begin{aligned} \int_{|\mathbf{x}-\mathbf{y}| < 1} |\log |\mathbf{x} - \mathbf{y}|| |h(\mathbf{y})| d\mathbf{y} &\leq \|\chi_\alpha^- h\|_{L^\infty} \left[\int_{|\mathbf{y}| < |\mathbf{x}-\mathbf{y}| < 1 \cap |\mathbf{y}| < \delta} \frac{-\log |\mathbf{y}|}{|\mathbf{y}|^\alpha} d\mathbf{y} \right. \\ &\quad \left. + \int_{|\mathbf{x}-\mathbf{y}| \leq |\mathbf{y}| < \delta} \frac{-\log |\mathbf{x} - \mathbf{y}|}{|\mathbf{x} - \mathbf{y}|^\alpha} d\mathbf{y} \right] \\ &\quad + \|h\|_{L^\infty(\mathbb{R}^2 \setminus B_\delta)} \int_{|\mathbf{x}-\mathbf{y}| < 1 \cap |\mathbf{y}| \geq \delta} (-\log |\mathbf{x} - \mathbf{y}|) d\mathbf{y} \\ &\leq 4\pi \|\chi_\alpha^- h\|_{L^\infty} \int_0^\delta k^{1-\alpha} (-\log k) dk \\ &\quad + 2\pi \|h\|_{L^\infty(\mathbb{R}^2 \setminus B_\delta)} \int_0^1 k (-\log k) dk \\ &\leq c \left(\|\chi_\alpha^- h\|_{L^\infty} + \|h\|_{L^\infty(\mathbb{R}^2 \setminus B_\delta)} \right), \end{aligned}$$

and

$$\begin{aligned}
\int_{|\mathbf{x}-\mathbf{y}|\geq 1} |\log |\mathbf{x}-\mathbf{y}|| |h(\mathbf{y})| d\mathbf{y} &\leq \int_{|\mathbf{x}-\mathbf{y}|\geq 1} \log (|\mathbf{x}|+|\mathbf{y}|) |h(\mathbf{y})| d\mathbf{y} \\
&\leq \chi_{\{|\mathbf{x}|\geq 1/2\}} \int_{|\mathbf{y}|<|\mathbf{x}|} \log (2|\mathbf{x}|) |h(\mathbf{y})| d\mathbf{y} \\
&\quad + \int_{|\mathbf{y}|\geq |\mathbf{x}| \cap |\mathbf{y}|\geq 1/2} \log (2|\mathbf{y}|) |h(\mathbf{y})| d\mathbf{y} \\
&\leq \chi_{\log}^+(\mathbf{x}) \|h\|_{L^1} + c \left(\|h\|_{L^1} + \|\chi_{\log}^+ h\|_{L^1} \right).
\end{aligned}$$

Thus, we find

$$|G_h(\mathbf{x})| \leq \chi_{\log}^+(\mathbf{x}) \|h\|_{L^1} + c \left(\|h\|_{L^1} + \|\chi_{\log}^+ h\|_{L^1} + \|\chi_{\log}^- h\|_{L^\infty} + \|h\|_{L^\infty(\mathbb{R}^2 \setminus B_\delta)} \right). \quad (4.6)$$

In order to see an asymptotic behavior of G_h , we decompose the integral as follows.

$$\frac{G_h(\mathbf{x})}{\log |\mathbf{x}|} = -\frac{1}{2\pi} \left[\int_{B_R} \frac{\log |\mathbf{x}-\mathbf{y}|}{\log |\mathbf{x}|} h(\mathbf{y}) d\mathbf{y} + \int_{\mathbb{R}^2 \setminus B_R} \frac{\log |\mathbf{x}-\mathbf{y}|}{\log |\mathbf{x}|} h(\mathbf{y}) d\mathbf{y} \right].$$

As for the first term, since we have the pointwise convergence, $\log |\mathbf{x}-\mathbf{y}|/\log |\mathbf{x}| \rightarrow 1$ as $|\mathbf{x}| \rightarrow \infty$, and

$$\frac{\log |\mathbf{x}-\mathbf{y}|}{\log |\mathbf{x}|} |h(\mathbf{y})| \leq c |h(\mathbf{y})|,$$

for $\mathbf{y} \in B_R$ and sufficiently large $|\mathbf{x}|$, the dominated convergence theorem gives

$$\lim_{|\mathbf{x}| \rightarrow \infty} \int_{B_R} \frac{\log |\mathbf{x}-\mathbf{y}|}{\log |\mathbf{x}|} h(\mathbf{y}) d\mathbf{y} = \int_{B_R} h(\mathbf{y}) d\mathbf{y}.$$

Based on the calculations in (4.6), we find the following estimate for sufficiently large $|\mathbf{x}|$.

$$\begin{aligned}
\left| \int_{\mathbb{R}^2 \setminus B_R} \frac{\log |\mathbf{x}-\mathbf{y}|}{\log |\mathbf{x}|} h(\mathbf{y}) d\mathbf{y} \right| &\leq \|h\|_{L^1(\mathbb{R}^2 \setminus B_R)} \\
&\quad + \frac{c}{\log |\mathbf{x}|} \left(\|h\|_{L^1} + \|\chi_{\log}^+ h\|_{L^1} + \|h\|_{L^\infty(\mathbb{R}^2 \setminus B_\delta)} \right).
\end{aligned}$$

Thus, taking the $|\mathbf{x}| \rightarrow \infty$ limit and then the $R \rightarrow \infty$ limit, we obtain the desired result (4.4). According to the classical theory, we readily confirm that G_h is a weak

solution of the Poisson equation, for example, see [51]. Next, we rewrite G_h in the polar coordinates,

$$G_{h,r}(r, \theta) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\log \sqrt{(r \cos \theta - y_1)^2 + (r \sin \theta - y_2)^2} \right) h(\mathbf{y}) d\mathbf{y}.$$

Then, for a radial function h , we have

$$\begin{aligned} G_{h,r}(r, \theta) &= -\frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \left(\log \sqrt{(r \cos \theta - \tilde{r} \cos \tilde{\theta})^2 + (r \sin \theta - \tilde{r} \sin \tilde{\theta})^2} \right) \tilde{r} h_r(\tilde{r}) d\tilde{\theta} d\tilde{r} \\ &= -\frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \left(\log \sqrt{r^2 + \tilde{r}^2 - 2r\tilde{r} \cos(\theta - \tilde{\theta})} \right) \tilde{r} h_r(\tilde{r}) d\tilde{\theta} d\tilde{r} \\ &= -\frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \left(\log \sqrt{r^2 + \tilde{r}^2 - 2r\tilde{r} \cos \tilde{\theta}} \right) \tilde{r} h_r(\tilde{r}) d\tilde{\theta} d\tilde{r}. \end{aligned}$$

Thus, $G_{h,r}$ is independent of θ and represented by

$$G_{h,r}(r) = G_{h,r}(r, 0) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\log \sqrt{(r - y_1)^2 + y_2^2} \right) h(\mathbf{y}) d\mathbf{y}.$$

Its derivative is given by

$$G'_{h,r}(r) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{r - y_1}{(r - y_1)^2 + y_2^2} h(\mathbf{y}) d\mathbf{y},$$

and we easily find that $G'_{h,r}$ is uniformly bounded and $G'_{h,r}(0) = 0$ based on the calculation in Lemma 4.7. Finally, we see the asymptotic behavior of $G'_{h,r}$. Note that

$$rG'_{h,r}(r) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} h(\mathbf{y}) d\mathbf{y} + \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{-y_1(r - y_1) + y_2^2}{(r - y_1)^2 + y_2^2} h(\mathbf{y}) d\mathbf{y}.$$

Then, setting $\mathbf{x}_r = (r, 0)$, we obtain

$$\left| \int_{\mathbb{R}^2} \frac{-y_1(r - y_1) + y_2^2}{(r - y_1)^2 + y_2^2} h(\mathbf{y}) d\mathbf{y} \right| \leq \int_{\mathbb{R}^2} \frac{|\mathbf{y}|}{|\mathbf{x}_r - \mathbf{y}|} |h(\mathbf{y})| d\mathbf{y} \rightarrow 0,$$

as $r \rightarrow \infty$ in the same way as the proof in Lemma 4.9. Hence, we obtain (4.5) and complete the proof. $\square$

Remark 4.13. Under the condition of Theorem 4.3, it follows from Lemma 4.12 that $G^\varepsilon = G * h^\varepsilon$ is radial, namely $G^\varepsilon(\mathbf{x}) = G_r^\varepsilon(|\mathbf{x}|)$ and

$$\lim_{r \rightarrow \infty} r \frac{dG_r^1}{dr}(r) = -\frac{1}{2\pi} \int_0^\infty k h_r(k) dk = -\frac{1}{2\pi}. \quad (4.7)$$

Note that the relation,

$$G_r^\varepsilon(|\mathbf{x}|) = G_r^1\left(\frac{|\mathbf{x}|}{\varepsilon}\right) - \frac{1}{2\pi} \log \varepsilon. \quad (4.8)$$

Then, we find

$$\mathbf{K}^\varepsilon(\mathbf{x}) = \nabla^\perp G_r^\varepsilon(|\mathbf{x}|) = \mathbf{K}(\mathbf{x}) \mathcal{G}\left(\frac{|\mathbf{x}|}{\varepsilon}\right), \quad \mathcal{G}(r) = -2\pi r \frac{dG_r^1}{dr}(r).$$

Moreover, owing to (4.7) and $\frac{dG_r^1}{dr}(0) = 0$, we have

$$\mathcal{G}(0) = 0, \quad \lim_{r \rightarrow \infty} \mathcal{G}(r) = 1.$$

Hence, if $\mathbf{x} \neq 0$, then $\mathbf{K}^\varepsilon(\mathbf{x})$ converges pointwise to $\mathbf{K}(\mathbf{x})$.

Proposition 4.14. *Let $h \in L^1(\mathbb{R}^2)$ be a radial function that satisfies $\chi_\alpha^- h \in L^\infty(\mathbb{R}^2)$, $\alpha \in [0, 1)$, $\chi_1^+ h \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $\int_0^\infty k h_r(k) dk = 1$. Then, we have*

$$\int_{\mathbb{R}^2} |\mathbf{K}^\varepsilon(\mathbf{x}) - \mathbf{K}(\mathbf{x})| d\mathbf{x} \leq \frac{\varepsilon}{2\pi} \|\chi_1 h\|_{L^1}$$

Proof. We first note that $h \in L^1(\mathbb{R}^2)$ and $\chi_1^+ h \in L^1(\mathbb{R}^2)$ yields $\chi_1 h \in L^1(\mathbb{R}^2)$. We have

$$\begin{aligned} \int_{\mathbb{R}^2} |\mathbf{K}^\varepsilon(\mathbf{x}) - \mathbf{K}(\mathbf{x})| d\mathbf{x} &= \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \mathbf{K}(\mathbf{x} - \mathbf{y}) \frac{1}{2\pi\varepsilon^2} h\left(\frac{\mathbf{y}}{\varepsilon}\right) d\mathbf{y} - \mathbf{K}(\mathbf{x}) \right| d\mathbf{x} \\ &= \varepsilon \int_{\mathbb{R}^2} \left| \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathbf{K}(\mathbf{x} - \mathbf{y}) h(\mathbf{y}) d\mathbf{y} - \mathbf{K}(\mathbf{x}) \right| d\mathbf{x} \\ &= \frac{\varepsilon}{2\pi} \int_{\mathbb{R}^2} |\nabla^\perp (G_h(\mathbf{x}) + \log |\mathbf{x}|)| d\mathbf{x}. \end{aligned}$$

Owing to Lemma 4.12, G_h is a radial function, namely $G_h(\mathbf{x}) = G_{h,r}(|\mathbf{x}|)$. Then, we find

$$\nabla^\perp (G_h(\mathbf{x}) + \log |\mathbf{x}|) = \frac{\mathbf{x}^\perp}{|\mathbf{x}|} \left(G'_{h,r}(|\mathbf{x}|) + \frac{1}{|\mathbf{x}|} \right),$$

and thus

$$\begin{aligned} \int_{\mathbb{R}^2} |\mathbf{K}^\varepsilon(\mathbf{x}) - \mathbf{K}(\mathbf{x})| d\mathbf{x} &= \frac{\varepsilon}{2\pi} \int_{\mathbb{R}^2} \left| G'_{h,r}(|\mathbf{x}|) + \frac{1}{|\mathbf{x}|} \right| d\mathbf{x} \\ &= \varepsilon \int_0^\infty |r G'_{h,r}(r) + 1| dr. \end{aligned}$$

Since G_h is a solution to the Poisson equation,

$$-\Delta G_h(\mathbf{x}) = -\frac{1}{r} \frac{d}{dr} (r G'_{h,r}(r)) = h_r(r),$$

and (4.5) gives $r G'_{h,r}(r) \rightarrow -1$ as $r \rightarrow \infty$, we obtain

$$1 + r G'_{h,r}(r) = \int_r^\infty k h_r(k) dk.$$

Hence, it follows that

$$\begin{aligned} \int_0^\infty |r G'_{h,r}(r) + 1| dr &\leq \int_0^\infty \int_r^\infty k |h_r(k)| dk dr \\ &= \left[r \int_r^\infty k |h_r(k)| dk \right]_0^\infty + \int_0^\infty r^2 |h_r(r)| dr \\ &= \int_0^\infty r^2 |h_r(r)| dr \\ &= \frac{1}{2\pi} \|\chi_1 h\|_{L^1}. \end{aligned}$$

Here, we have used the fact that $h \in L^1(\mathbb{R}^2)$ and $\chi_1 h \in L^1(\mathbb{R}^2)$ yield

$$\left[r \int_r^\infty k |h_r(k)| dk \right]_0^\infty = 0.$$

The proof of the proposition is now complete. $\square$

4.3 Global solvability in the space of Radon measure

We prove Theorem 4.1 by a classical iterative method shown in Appendix B. In this method, a uniform boundedness and a quasi-Lipschitz continuity of the velocity field play an essential role. Our proof is based on that in [56]. We begin with introducing a sequence of approximate solutions for $n \in \mathbb{N}$, which are given by

$$\begin{aligned} q^0(\mathbf{x}, t) &= q_0(\mathbf{x}), \\ \mathbf{u}^n(\mathbf{x}, t) &= (\mathbf{K}_h * q^{n-1})(\mathbf{x}, t), \\ \partial_t \boldsymbol{\eta}^n(\mathbf{x}, t) &= \mathbf{u}^n(\boldsymbol{\eta}^n(\mathbf{x}, t), t), \\ \boldsymbol{\eta}^n(\mathbf{x}, 0) &= \mathbf{x}, \\ q^n(\mathbf{x}, t) &= q_0(\boldsymbol{\eta}^n(\mathbf{x}, -t)). \end{aligned}$$

As the first step, we prove that $\boldsymbol{\eta}^n \in C^1([0, \infty); \mathcal{G})$ for $n \in \mathbb{N}$ by induction. For $n = 1$, it follows from (4.3) and Lemma 4.10 that $\|\mathbf{u}^1\|_{L^\infty} \leq \|\mathbf{K}_h\|_{L^\infty} \|q_0\|_{\mathcal{M}}$ and

$$\begin{aligned} |\mathbf{u}^1(\mathbf{x}) - \mathbf{u}^1(\mathbf{x}')| &\leq \sup_{\mathbf{y} \in \mathbb{R}^2} |\mathbf{K}_h(\mathbf{x} - \mathbf{y}) - \mathbf{K}_h(\mathbf{x}' - \mathbf{y})| \|q_0\|_{\mathcal{M}} \\ &\leq c\varphi(|\mathbf{x} - \mathbf{x}'|) \|q_0\|_{\mathcal{M}}. \end{aligned}$$

Thus, $\mathbf{u}^1$ is uniformly bounded on $\mathbb{R}^2 \times [0, \infty)$, quasi-Lipschitz in space and obviously continuous in time. Using Lemma B.1, we obtain $\boldsymbol{\eta}^1 \in C^1([0, \infty); \mathcal{G})$. For $\boldsymbol{\eta}^{n-1} \in C^1([0, \infty); \mathcal{G})$, considering $\|q^{n-1}(t)\|_{\mathcal{M}} = \|q_0\|_{\mathcal{M}}$, we readily find that $\mathbf{u}^n$ is uniformly bounded and quasi-Lipschitz in space as well. Regarding the continuity in time, we have

$$\begin{aligned} |\mathbf{u}^n(\mathbf{x}, t) - \mathbf{u}^n(\mathbf{x}, t')| &\leq \sup_{\mathbf{y} \in \mathbb{R}^2} |\mathbf{K}_h(\mathbf{x} - \boldsymbol{\eta}^{n-1}(\mathbf{y}, t)) - \mathbf{K}_h(\mathbf{x} - \boldsymbol{\eta}^{n-1}(\mathbf{y}, t'))| \|q_0\|_{\mathcal{M}} \\ &\leq c \sup_{\mathbf{y} \in \mathbb{R}^2} \varphi(|\boldsymbol{\eta}^{n-1}(\mathbf{y}, t) - \boldsymbol{\eta}^{n-1}(\mathbf{y}, t')|) \|q_0\|_{\mathcal{M}} \\ &\leq c \sup_{\mathbf{y} \in \mathbb{R}^2} \sup_{s \in [t, t']} \varphi(|\partial_t \boldsymbol{\eta}^{n-1}(\mathbf{y}, s)| |t - t'|) \|q_0\|_{\mathcal{M}} \\ &\leq c \sup_{\mathbf{y} \in \mathbb{R}^2} \sup_{s \in [t, t']} \varphi(|\mathbf{u}^{n-1}(\mathbf{y}, s)| |t - t'|) \|q_0\|_{\mathcal{M}}. \end{aligned}$$

Since $\mathbf{u}^n$ is uniformly bounded, a uniform continuity in time holds. Hence, it follows from Lemma B.1 that $\boldsymbol{\eta}^n \in C^1([0, \infty); \mathcal{G})$.

Next, we show the existence of a limit flow $\bar{\boldsymbol{\eta}} \in C^1([0, \infty); \mathcal{G})$. The sequence $\{\boldsymbol{\eta}^n\}_{n \in \mathbb{N}}$ is Cauchy in $C([0, T]; \mathcal{G})$ for some $T > 0$. Indeed, for $n \geq N$, we have

$$\begin{aligned} &|\mathbf{u}^n(\boldsymbol{\eta}^n(\mathbf{x})) - \mathbf{u}^{n-1}(\boldsymbol{\eta}^{n-1}(\mathbf{x}))| \\ &\leq \int_{\mathbb{R}^2} |\mathbf{K}_h(\boldsymbol{\eta}^n(\mathbf{x}) - \boldsymbol{\eta}^{n-1}(\mathbf{y})) - \mathbf{K}_h(\boldsymbol{\eta}^{n-1}(\mathbf{x}) - \boldsymbol{\eta}^{n-1}(\mathbf{y}))| |q_0(\mathbf{y})| d\mathbf{y} \\ &\quad + \int_{\mathbb{R}^2} |\mathbf{K}_h(\boldsymbol{\eta}^{n-1}(\mathbf{x}) - \boldsymbol{\eta}^{n-1}(\mathbf{y})) - \mathbf{K}_h(\boldsymbol{\eta}^{n-1}(\mathbf{x}) - \boldsymbol{\eta}^{n-2}(\mathbf{y}))| |q_0(\mathbf{y})| d\mathbf{y} \\ &\leq c\|q_0\|_{\mathcal{M}} \varphi(|\boldsymbol{\eta}^n(\mathbf{x}) - \boldsymbol{\eta}^{n-1}(\mathbf{x})|) + c \int_{\mathbb{R}^2} \varphi(|\boldsymbol{\eta}^{n-1}(\mathbf{y}) - \boldsymbol{\eta}^{n-2}(\mathbf{y})|) |q_0(\mathbf{y})| d\mathbf{y} \\ &\leq c\|q_0\|_{\mathcal{M}} \varphi\left(\sup_{n \geq N-1} \sup_{\mathbf{x} \in \mathbb{R}^2} |\boldsymbol{\eta}^n(\mathbf{x}) - \boldsymbol{\eta}^{n-1}(\mathbf{x})|\right), \end{aligned}$$

in which we drop the explicit time dependence. Defining

$$\rho^N(t) \equiv \sup_{n \geq N} \sup_{\mathbf{x} \in \mathbb{R}^2} |\boldsymbol{\eta}^n(\mathbf{x}, t) - \boldsymbol{\eta}^{n-1}(\mathbf{x}, t)|,$$

we obtain

$$\begin{aligned}\rho^N(t) &\leq \sup_{n \geq N} \sup_{\mathbf{x} \in \mathbb{R}^2} \int_0^t |\mathbf{u}^n(\boldsymbol{\eta}^n(\mathbf{x}, s), s) - \mathbf{u}^{n-1}(\boldsymbol{\eta}^{n-1}(\mathbf{x}, s), s)| ds \\ &\leq c \|q_0\|_{\mathcal{M}} \int_0^t \varphi(\rho^{N-1}(s)) ds.\end{aligned}$$

As shown in Lemma B.1, this implies $\rho^N \rightarrow 0$ uniformly on $[0, T]$ as $N \rightarrow +\infty$ for sufficiently small T . Since T depends only on $\|q_0\|_{\mathcal{M}}$, the convergence holds for arbitrary times and thus the limit flow $\bar{\boldsymbol{\eta}} \in C([0, \infty); \mathcal{G})$ exists.

In turn, we see the convergence of the vorticity and the velocity. Let us define $\bar{q}$ and $\bar{\mathbf{u}}$ by

$$\bar{q}(\mathbf{x}, t) = q_0(\bar{\boldsymbol{\eta}}(\mathbf{x}, -t)), \quad \bar{\mathbf{u}}(\mathbf{x}, t) = \int_{\mathbb{R}^2} \mathbf{K}_h(\mathbf{x} - \mathbf{y}) \bar{q}(\mathbf{y}, t) d\mathbf{y}. \quad (4.9)$$

Then, we find that $q^n \rightharpoonup \bar{q}$ weakly in $\mathcal{M}(\mathbb{R}^2)$ and $\mathbf{u}^n \rightarrow \bar{\mathbf{u}}$ in $C(\mathbb{R}^2)$. Note that both converge uniformly on any finite time interval. Indeed, for any $\psi \in C_0^1(\mathbb{R}^2)$, it follows that

$$\begin{aligned}\left| \int_{\mathbb{R}^2} (q^n(\mathbf{x}, t) - \bar{q}(\mathbf{x}, t)) \psi(\mathbf{x}) d\mathbf{x} \right| &\leq \int_{\mathbb{R}^2} |q_0(\mathbf{x})| |\psi(\boldsymbol{\eta}^n(\mathbf{x}, t)) - \psi(\bar{\boldsymbol{\eta}}(\mathbf{x}, t))| d\mathbf{x} \\ &\leq \|q_0\|_{\mathcal{M}} \|\nabla \psi\|_{L^\infty} \sup_{\mathbf{x} \in \mathbb{R}^2} |\boldsymbol{\eta}^n(\mathbf{x}, t) - \bar{\boldsymbol{\eta}}(\mathbf{x}, t)|,\end{aligned}$$

and

$$|\mathbf{u}^n(\mathbf{x}, t) - \bar{\mathbf{u}}(\mathbf{x}, t)| \leq c \|q_0\|_{\mathcal{M}} \varphi \left(\sup_{\mathbf{y} \in \mathbb{R}^2} |\boldsymbol{\eta}^n(\mathbf{y}, t) - \bar{\boldsymbol{\eta}}(\mathbf{y}, t)| \right).$$

Hence, the convergence of $\boldsymbol{\eta}^n$ and the density argument yield the above assertions.

Finally, we show that $(\bar{\boldsymbol{\eta}}, \bar{q}, \bar{\mathbf{u}})$ is the solution to (3.2). Consider the integral form

of (3.2),

$$\begin{aligned}
& \left| \bar{\eta}(\mathbf{x}, t) - \mathbf{x} - \int_0^t \bar{\mathbf{u}}(\bar{\eta}(\mathbf{x}, s), s) ds \right| \\
&= \left| \bar{\eta}(\mathbf{x}, t) - \mathbf{x} - \int_0^t \bar{\mathbf{u}}(\bar{\eta}(\mathbf{x}, s), s) ds - \left(\eta^n(\mathbf{x}, t) - \mathbf{x} - \int_0^t \mathbf{u}^n(\eta^n(\mathbf{x}, s), s) ds \right) \right| \\
&\leq |\eta^n(\mathbf{x}, t) - \bar{\eta}(\mathbf{x}, t)| + \int_0^t |\mathbf{u}^n(\eta^n(\mathbf{x}, s), s) - \bar{\mathbf{u}}(\eta^n(\mathbf{x}, s), s)| ds \\
&\quad + \int_0^t |\bar{\mathbf{u}}(\eta^n(\mathbf{x}, s), s) - \bar{\mathbf{u}}(\bar{\eta}(\mathbf{x}, s), s)| ds \\
&\leq \sup_{\mathbf{x} \in \mathbb{R}^2} |\eta^n(\mathbf{x}, t) - \bar{\eta}(\mathbf{x}, t)| + c \|q_0\|_{\mathcal{M}} \int_0^t \varphi \left(\sup_{\mathbf{y} \in \mathbb{R}^2} |\eta^{n-1}(\mathbf{y}, s) - \bar{\eta}(\mathbf{y}, s)| \right) ds \\
&\quad + c \|q_0\|_{\mathcal{M}} \int_0^t \varphi \left(\sup_{\mathbf{x} \in \mathbb{R}^2} |\eta^n(\mathbf{x}, s) - \bar{\eta}(\mathbf{x}, s)| \right) ds.
\end{aligned}$$

Then, the right side converges zero uniformly in space as $n \rightarrow +\infty$. Since $\bar{\mathbf{u}}(\bar{\eta}(\mathbf{x}, t), t)$ is continuous in time, $\bar{\eta}$ satisfies (3.2) and $\partial_t \bar{\eta}$ is continuous in time. Considering the time reversibility of (3.2), we find that the solutions are extended to negative times. Moreover, we can see that the solutions $\bar{q}$ and $\bar{\mathbf{u}}$ satisfy

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} (\partial_t \psi + \bar{\mathbf{u}} \cdot \nabla \psi) \bar{q} d\mathbf{x} dt = 0,$$

for any $\psi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^2)$. The uniqueness of the solutions can be proven based on the same estimates in the existence part.

4.4 Convergence to the Euler equations

4.4.1 Proof of Theorem 4.3

In this section, we prove Theorem 4.3. The proof proceeds in the same way as [56]. We estimate the difference of the flow maps.

$$\begin{aligned}
|\boldsymbol{\eta}^\varepsilon(\mathbf{x}, t) - \boldsymbol{\eta}(\mathbf{x}, t)| &\leq \int_0^t |\mathbf{u}^\varepsilon(\boldsymbol{\eta}^\varepsilon(\mathbf{x}, s), s) - \mathbf{u}(\boldsymbol{\eta}(\mathbf{x}, s), s)| ds \\
&\leq \int_0^t \int_{\mathbb{R}^2} |\mathbf{K}^\varepsilon(\boldsymbol{\eta}^\varepsilon(\mathbf{x}, s) - \boldsymbol{\eta}^\varepsilon(\mathbf{y}, s)) - \mathbf{K}(\boldsymbol{\eta}^\varepsilon(\mathbf{x}, s) - \boldsymbol{\eta}^\varepsilon(\mathbf{y}, s))| |\omega_0(\mathbf{y})| d\mathbf{y} ds \\
&\quad + \int_0^t \int_{\mathbb{R}^2} |\mathbf{K}(\boldsymbol{\eta}^\varepsilon(\mathbf{x}, s) - \boldsymbol{\eta}^\varepsilon(\mathbf{y}, s)) - \mathbf{K}(\boldsymbol{\eta}(\mathbf{x}, s) - \boldsymbol{\eta}^\varepsilon(\mathbf{y}, s))| |\omega_0(\mathbf{y})| d\mathbf{y} ds \\
&\quad + \int_0^t \int_{\mathbb{R}^2} |\mathbf{K}(\boldsymbol{\eta}(\mathbf{x}, s) - \boldsymbol{\eta}^\varepsilon(\mathbf{y}, s)) - \mathbf{K}(\boldsymbol{\eta}(\mathbf{x}, s) - \boldsymbol{\eta}(\mathbf{y}, s))| |\omega_0(\mathbf{y})| d\mathbf{y} ds \\
&\equiv \int_0^t J_1 + J_2 + J_3 ds.
\end{aligned}$$

As for J_1 , it follows from Lemma 4.14 that

$$J_1 \leq \|\omega_0\|_{L^\infty} \int_{\mathbb{R}^2} |\mathbf{K}^\varepsilon(\mathbf{y}) - \mathbf{K}(\mathbf{y})| d\mathbf{y} \leq \frac{\varepsilon}{2\pi} \|\omega_0\|_{L^\infty} \|\chi_1 h\|_{L^1}.$$

The other two terms are estimated by using Lemma A.1 and Lemma A.2 respectively. We find

$$\begin{aligned}
J_2 &= \int_{\mathbb{R}^2} |\mathbf{K}(\boldsymbol{\eta}^\varepsilon(\mathbf{x}, s) - \mathbf{y}) - \mathbf{K}(\boldsymbol{\eta}(\mathbf{x}, s) - \mathbf{y})| |q(\mathbf{y}, s)| d\mathbf{y} \\
&\leq c_2 \varphi(|\boldsymbol{\eta}^\varepsilon(\mathbf{x}, s) - \boldsymbol{\eta}(\mathbf{x}, s)|) (\|\omega_0\|_{L^1} + \|\omega_0\|_{L^\infty}),
\end{aligned}$$

and

$$\begin{aligned}
J_3 &= \int_{\mathbb{R}^2} |\mathbf{K}(\boldsymbol{\eta}(\mathbf{x}, s) - \boldsymbol{\eta}^\varepsilon(\boldsymbol{\eta}^{-1}(\mathbf{y}, s), s)) - \mathbf{K}(\boldsymbol{\eta}(\mathbf{x}, s) - \mathbf{y})| |\omega(\mathbf{y}, s)| d\mathbf{y} \\
&\leq c_3 \sup_{\mathbf{x} \in \mathbb{R}^2} \varphi(|\boldsymbol{\eta}^\varepsilon(\mathbf{x}, s) - \boldsymbol{\eta}(\mathbf{x}, s)|) (\|\omega_0\|_{L^1} + \|\omega_0\|_{L^\infty}).
\end{aligned}$$

Combining above estimates and setting $\rho(t) = \sup_{\mathbf{x} \in \mathbb{R}^2} |\boldsymbol{\eta}^\varepsilon(\mathbf{x}, t) - \boldsymbol{\eta}(\mathbf{x}, t)|$, we obtain

$$\rho(t) \leq c \int_0^t [\varepsilon + \varphi(\rho(s))] ds.$$

Thus, Lemma B.3 yields $\rho(t) \leq c(T) \varepsilon^{e^{-cT}}$ for $t \in [0, T]$. The proof is now complete.

4.4.2 Proof of Corollary 4.6

In order to prove Corollary 4.6, we show the following lemma [56].

Lemma 4.15. *Let $\{q^n\}_{n \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^2)$ be a sequence of measures. Suppose that $\{q^n\}_{n \in \mathbb{N}}$ decays uniformly at infinity and converges weakly to zero in $\mathcal{M}(\mathbb{R}^2)$. Then, for any $\psi \in C_0(\mathbb{R}^2)$, we have*

$$\lim_{n \rightarrow \infty} \sup_{\mathbf{x} \in \mathbb{R}^2} \int_{\mathbb{R}^2} \psi(\mathbf{x} - \mathbf{y}) q^n(\mathbf{y}) d\mathbf{y} = 0.$$

Proof. We first remark that $\{q^n\}_{n \in \mathbb{N}}$ is uniformly bounded in $\mathcal{M}(\mathbb{R}^2)$ and set $M = \sup_{n \in \mathbb{N}} \|q^n\|_{\mathcal{M}}$. Since $\{q^n\}_{n \in \mathbb{N}}$ decays uniformly, for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^2 \setminus B_R} |q^n(\mathbf{y})| d\mathbf{y} < \frac{\varepsilon}{6\|\psi\|_{L^\infty}}.$$

Let $R > 0$ be sufficiently large enough so that $|\psi(\mathbf{x})| < 5\varepsilon/(6M)$ for $|\mathbf{x}| > R$. Since ψ is uniformly continuous on compact sets, there exists $\delta > 0$ such that $|\psi(\mathbf{x}) - \psi(\mathbf{x}')| < \varepsilon/(3M)$ for any $\mathbf{x}, \mathbf{x}' \in B_{3R}$ with $|\mathbf{x} - \mathbf{x}'| < \delta$. Here, we cover the ball B_{2R} with a finite number of balls whose radii are δ , and denote the centers of balls by $\mathbf{x}_i, i \in I$. Then, taking sufficiently large $N \in \mathbb{N}$, we have

$$\max_{i \in I} \left| \int_{\mathbb{R}^2} \psi(\mathbf{x}_i - \mathbf{y}) q^n(\mathbf{y}) d\mathbf{y} \right| < \frac{\varepsilon}{3},$$

for any $n \geq N$. Hence, for any $|\mathbf{x}| < 2R$, there exists a index $i \in I$ such that $|\mathbf{x} - \mathbf{x}_i| < \delta$ and

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \psi(\mathbf{x} - \mathbf{y}) q^n(\mathbf{y}) d\mathbf{y} \right| &\leq \int_{\mathbb{R}^2} |\psi(\mathbf{x} - \mathbf{y}) - \psi(\mathbf{x}_i - \mathbf{y})| |q^n(\mathbf{y})| d\mathbf{y} + \left| \int_{\mathbb{R}^2} \psi(\mathbf{x}_i - \mathbf{y}) q^n(\mathbf{y}) d\mathbf{y} \right| \\ &\leq \|q^n\|_{\mathcal{M}} \sup_{\mathbf{y} \in B_R} |\psi(\mathbf{x} - \mathbf{y}) - \psi(\mathbf{x}_i - \mathbf{y})| + 2\|\psi\|_{L^\infty} \int_{\mathbb{R}^2 \setminus B_R} |q^n(\mathbf{y})| d\mathbf{y} \\ &\quad + \max_{i \in I} \left| \int_{\mathbb{R}^2} \psi(\mathbf{x}_i - \mathbf{y}) q^n(\mathbf{y}) d\mathbf{y} \right| \\ &< \frac{\varepsilon}{3M} M + 2\|\psi\|_{L^\infty} \frac{\varepsilon}{6\|\psi\|_{L^\infty}} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

On the other hand, if $|\mathbf{x}| \geq 2R$, then

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \psi(\mathbf{x} - \mathbf{y}) q^n(\mathbf{y}) d\mathbf{y} \right| &\leq \|\psi\|_{L^\infty(\mathbb{R}^2 \setminus B_R)} \int_{B_R} |q^n(\mathbf{y})| d\mathbf{y} + \|\psi\|_{L^\infty} \int_{\mathbb{R}^2 \setminus B_R} |q^n(\mathbf{y})| d\mathbf{y} \\ &< \frac{5\varepsilon}{6M} M + \|\psi\|_{L^\infty} \frac{\varepsilon}{6\|\psi\|_{L^\infty}} \\ &= \varepsilon. \end{aligned}$$

This completes the proof. $\square$

Finally, we show that the Euler-Poincaré flow map is continuous with the initial vorticity in $\mathcal{M}(\mathbb{R}^2)$, whose proof is also based on [56]. The proof of Theorem 4.3 and Lemma 4.16 imply Corollary 4.6.

Lemma 4.16. *Suppose that $q_0 \in L^1(\mathbb{R}^2)$ is approximated by a sequence of measures in $\mathcal{M}(\mathbb{R}^2)$ with $q_0^n \rightharpoonup q_0$ weakly in $\mathcal{M}(\mathbb{R}^2)$ and $\|q_0^n\|_{\mathcal{M}} \rightarrow \|q_0\|_{L^1}$. Then, for any $T > 0$ and $\varepsilon > 0$, we have*

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \sup_{\mathbf{x} \in \mathbb{R}^2} |\boldsymbol{\eta}^{n, \varepsilon}(\mathbf{x}, t) - \boldsymbol{\eta}^\varepsilon(\mathbf{x}, t)| = 0,$$

where $\boldsymbol{\eta}^{n, \varepsilon}$ denotes the flow map in the Euler-Poincaré flows with the parameter ε and the initial vorticity q_0^n .

Proof. Based on the calculation in the proof of Theorem 4.3, we have

$$\begin{aligned} &|\boldsymbol{\eta}^{n, \varepsilon}(\mathbf{x}, t) - \boldsymbol{\eta}^\varepsilon(\mathbf{x}, t)| \\ &\leq \int_0^t \left| \int_{\mathbb{R}^2} \mathbf{K}^\varepsilon(\boldsymbol{\eta}^\varepsilon(\mathbf{x}, s) - \boldsymbol{\eta}^\varepsilon(\mathbf{y}, s)) (q_0^n(\mathbf{y}) - q_0(\mathbf{y})) d\mathbf{y} \right| ds \\ &\quad + \int_0^t \int_{\mathbb{R}^2} |\mathbf{K}^\varepsilon(\boldsymbol{\eta}^\varepsilon(\mathbf{x}, s) - \boldsymbol{\eta}^\varepsilon(\mathbf{y}, s)) - \mathbf{K}^\varepsilon(\boldsymbol{\eta}^\varepsilon(\mathbf{x}, s) - \boldsymbol{\eta}^{n, \varepsilon}(\mathbf{y}, s))| |q_0^n(\mathbf{y})| d\mathbf{y} ds \\ &\quad + \int_0^t \int_{\mathbb{R}^2} |\mathbf{K}^\varepsilon(\boldsymbol{\eta}^\varepsilon(\mathbf{x}, s) - \boldsymbol{\eta}^{n, \varepsilon}(\mathbf{y}, s)) - \mathbf{K}^\varepsilon(\boldsymbol{\eta}^{n, \varepsilon}(\mathbf{x}, s) - \boldsymbol{\eta}^{n, \varepsilon}(\mathbf{y}, s))| |q_0^n(\mathbf{y})| d\mathbf{y} ds \\ &\equiv \int_0^t \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 ds. \end{aligned}$$

Since the convergence $\|q_0^n\|_{\mathcal{M}} \rightarrow \|q_0\|_{L^1}$ implies the uniform decay of the sequence $\{q_0^n - q_0\}_{n \in \mathbb{N}}$, we find from Lemma 4.15 that

$$\sup_{\mathbf{x} \in \mathbb{R}^2} \tilde{J}_1 = \sup_{\mathbf{x} \in \mathbb{R}^2} \left| \int_{\mathbb{R}^2} \mathbf{K}^\varepsilon(\mathbf{x} - \mathbf{y}) (q_0^n(\boldsymbol{\eta}^\varepsilon(\mathbf{y}, -s)) - q_0(\boldsymbol{\eta}^\varepsilon(\mathbf{y}, -s))) d\mathbf{y} \right|,$$

converges to zero in the $n \rightarrow \infty$ limit. Regarding $\tilde{J}_2$ and $\tilde{J}_3$, it follows from Lemma 4.10 that

$$\begin{aligned}\tilde{J}_2 &\leq \|q_0^n\|_{\mathcal{M}} \sup_{\mathbf{x} \in \mathbb{R}^2} \sup_{\mathbf{y} \in \mathbb{R}^2} |\mathbf{K}^\varepsilon(\mathbf{x} - \boldsymbol{\eta}^\varepsilon(\mathbf{y}, s)) - \mathbf{K}^\varepsilon(\mathbf{x} - \boldsymbol{\eta}^{n,\varepsilon}(\mathbf{y}, s))| \\ &\leq \frac{c}{\varepsilon} \|q_0^n\|_{\mathcal{M}} \sup_{\mathbf{y} \in \mathbb{R}^2} \varphi\left(\frac{|\boldsymbol{\eta}^\varepsilon(\mathbf{y}, s) - \boldsymbol{\eta}^{n,\varepsilon}(\mathbf{y}, s)|}{\varepsilon}\right),\end{aligned}$$

and

$$\tilde{J}_3 \leq \frac{c}{\varepsilon} \|q_0^n\|_{\mathcal{M}} \sup_{\mathbf{x} \in \mathbb{R}^2} \varphi\left(\frac{|\boldsymbol{\eta}^\varepsilon(\mathbf{x}, s) - \boldsymbol{\eta}^{n,\varepsilon}(\mathbf{x}, s)|}{\varepsilon}\right).$$

Summing up these estimates, we obtain

$$\tilde{\rho}(t) \leq c \int_0^t \left[\delta(n) + \frac{1}{\varepsilon} \varphi\left(\frac{\tilde{\rho}(s)}{\varepsilon}\right) \right] ds.$$

where $\tilde{\rho}(t) = \sup_{\mathbf{x} \in \mathbb{R}^2} |\boldsymbol{\eta}^{n,\varepsilon}(\mathbf{x}, t) - \boldsymbol{\eta}^\varepsilon(\mathbf{x}, t)|$ and $\delta(n) \rightarrow 0$ as $n \rightarrow \infty$. Hence, Lemma B.3 yields the desired result. $\square$

5 Point vortex problem

5.1 The Euler-Poincaré point vortex system

Let us recall the parametrized Euler-Poincaré equations in the vorticity form,

$$\partial_t q + (\mathbf{u}^\varepsilon \cdot \nabla) q = 0, \quad \mathbf{u}^\varepsilon = \mathbf{K}^\varepsilon * q, \quad \mathbf{K}^\varepsilon = \mathbf{K} * h^\varepsilon, \quad (5.1)$$

and the Lagrangian flow map induced by the regularized velocity,

$$\partial_t \boldsymbol{\eta}^\varepsilon(\mathbf{x}, t) = \mathbf{u}^\varepsilon(\boldsymbol{\eta}^\varepsilon(\mathbf{x}, t), t), \quad \boldsymbol{\eta}^\varepsilon(\mathbf{x}, 0) = \mathbf{x}. \quad (5.2)$$

From now on, we consider the parametrized smoothing function h^ε given by

$$h^\varepsilon(\mathbf{x}) = \frac{1}{\varepsilon^2} h\left(\frac{|\mathbf{x}|}{\varepsilon}\right),$$

and $h(r)$ is a radial function for $r > 0$, which satisfies the assumptions in Theorem 4.1 and

$$\int_{\mathbb{R}^2} h(|\mathbf{x}|) d\mathbf{x} = 2\pi \int_0^\infty r h(r) dr = 1. \quad (5.3)$$

Suppose that the initial vorticity field in the Euler-Poincaré flows is represented by a set of δ -distributions,

$$q_0(\mathbf{x}) = \sum_{n=1}^N \Gamma_n \delta(\mathbf{x} - \mathbf{x}_n^0), \quad (5.4)$$

where $\mathbf{x}_n^0 = (x_n^0, y_n^0) \in \mathbb{R}^2$ for $n = 1, \dots, N$ are the pointwise support of the δ -distributions and $\Gamma_n \in \mathbb{R}$ denotes the strength of the point vortex located at $\mathbf{x}_n^0$. According to Theorem 4.1, there exists a unique global weak solution to (5.1) with the initial data (5.4) and we have the following theorem.

Theorem 5.1. *The solution to (5.1) with the initial data 5.4 is expressed by*

$$q(\mathbf{x}, t) = \sum_{n=1}^N \Gamma_n \delta(\mathbf{x} - \boldsymbol{\eta}^\varepsilon(\mathbf{x}_n^0, t)).$$

Moreover, the point vortices in the Euler-Poincaré flow never collapse.

Proof. Note that we know from Theorem 4.1 that global existence of a unique solution to (5.2). Then, it follows that

$$q(\mathbf{x}, t) = q_0(\boldsymbol{\eta}^\varepsilon(\mathbf{x}, -t)) = \sum_{n=1}^N \Gamma_n \delta(\boldsymbol{\eta}^\varepsilon(\mathbf{x}, -t) - \mathbf{x}_n^0).$$

If $\boldsymbol{\eta}^\varepsilon(\mathbf{x}, -t) = \mathbf{x}_n^0$ then $\mathbf{x} = \boldsymbol{\eta}^\varepsilon(\mathbf{x}_n^0, t)$ else $\mathbf{x} \neq \boldsymbol{\eta}^\varepsilon(\mathbf{x}_n^0, t)$. This implies

$$\delta(\boldsymbol{\eta}^\varepsilon(\mathbf{x}, -t) - \mathbf{x}_n^0) = \delta(\mathbf{x} - \boldsymbol{\eta}^\varepsilon(\mathbf{x}_n^0, t)).$$

Moreover, it follows from the uniqueness of the flow map that $\boldsymbol{\eta}^\varepsilon(\mathbf{x}_m^0, t) \neq \boldsymbol{\eta}^\varepsilon(\mathbf{x}_n^0, t)$ for $m \neq n$ and an arbitrary $t \in \mathbb{R}$. Thus, there is no collapse. $\square$

For simplicity, we introduce the notation of ε -point vortices $\mathbf{x}_n^\varepsilon(t) = \boldsymbol{\eta}^\varepsilon(\mathbf{x}_n^0, t)$. Then, owing to Theorem 5.1, the equations (5.2) with the initial vorticity (5.4) are rewritten by

$$\begin{aligned} \frac{d}{dt} \mathbf{x}_n^\varepsilon(t) &= \mathbf{u}(\mathbf{x}_n^\varepsilon(t), t) = \int_{\mathbb{R}^2} \mathbf{K}^\varepsilon(\mathbf{x}_n^\varepsilon(t) - \mathbf{y}) q(\mathbf{y}, t) d\mathbf{y} \\ &= \sum_{m=1}^N \Gamma_m \int_{\mathbb{R}^2} \mathbf{K}^\varepsilon(\mathbf{x}_n^\varepsilon(t) - \mathbf{y}) \delta(\mathbf{y} - \mathbf{x}_m^\varepsilon(t)) d\mathbf{y} \\ &= \sum_{m=1}^N \Gamma_m \mathbf{K}^\varepsilon(\mathbf{x}_n^\varepsilon(t) - \mathbf{x}_m^\varepsilon(t)). \end{aligned}$$

Hence, considering Remark 4.13 and using the notations there, we obtain the Euler-Poincaré point vortex (PV) system,

$$\frac{d}{dt} \mathbf{x}_n^\varepsilon = -\frac{1}{2\pi} \sum_{m \neq n}^N \Gamma_m \frac{(\mathbf{x}_n^\varepsilon - \mathbf{x}_m^\varepsilon)^\perp}{(l_{nm}^\varepsilon)^2} \mathcal{G} \left(\frac{l_{nm}^\varepsilon}{\varepsilon} \right), \quad n = 1, \dots, N, \quad (5.5)$$

where $l_{nm}^\varepsilon(t) = |\mathbf{x}_n^\varepsilon(t) - \mathbf{x}_m^\varepsilon(t)|$ and $\mathbf{x}_n^\varepsilon(0) = \mathbf{x}_n^0$. Note that the Euler-Poincaré PV system (5.5) has a global solution that yields a global weak solution of the 2D Euler-Poincaré equations. In our strategy to construct the enstrophy dissipating solution, we try to observe the enstrophy dissipation in incompressible and inviscid non-smooth flows by taking the $\varepsilon \rightarrow 0$ limit of a global solution of (5.5). Before that, let us investigate the detail of the Euler-Poincaré PV system.

The Euler-Poincaré PV system (5.5) is a Hamiltonian dynamical system, that is to say, (5.5) is equivalent to

$$\Gamma_n \frac{dx_n^\varepsilon}{dt} = \frac{\partial \mathcal{H}^\varepsilon}{\partial y_n^\varepsilon}, \quad \Gamma_n \frac{dy_n^\varepsilon}{dt} = -\frac{\partial \mathcal{H}^\varepsilon}{\partial x_n^\varepsilon}, \quad n = 1, \dots, N,$$

with its Hamiltonian,

$$\mathcal{H}^\varepsilon = -\frac{1}{2\pi} \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m \left[\log l_{nm}^\varepsilon + H_0 \left(\frac{l_{nm}^\varepsilon}{\varepsilon} \right) \right], \quad (5.6)$$

where $H_0(r) = -\log r - 2\pi G_r^1(r)$ and $G_r^\varepsilon(|\mathbf{x}|) = G^\varepsilon(\mathbf{x}) = (G * h^\varepsilon)(\mathbf{x})$, see (4.8) in Remark 4.13. The system (5.5) has four conserved quantities $(\mathcal{H}^\varepsilon, Q^\varepsilon, P^\varepsilon, I^\varepsilon)$ with

$$Q^\varepsilon + iP^\varepsilon = \sum_{n=1}^N z_n^\varepsilon, \quad I^\varepsilon = \sum_{n=1}^N \Gamma_n |z_n^\varepsilon|^2,$$

in which $z_n^\varepsilon(t) = x_n^\varepsilon(t) + iy_n^\varepsilon(t) \in \mathbb{C}$. Defining the Poisson bracket between two functions f and g by

$$\{f, g\} = \sum_{n=1}^N \frac{1}{\Gamma_n} \left(\frac{\partial f}{\partial x_n} \frac{\partial g}{\partial y_n} - \frac{\partial f}{\partial y_n} \frac{\partial g}{\partial x_n} \right),$$

we find

$$\{\mathcal{H}^\varepsilon, I^\varepsilon\} = 0, \quad \{\mathcal{H}^\varepsilon, (P^\varepsilon)^2 + (Q^\varepsilon)^2\} = 0, \quad \{(P^\varepsilon)^2 + (Q^\varepsilon)^2, I^\varepsilon\} = 0.$$

Thus, the Euler-Poincaré PV system (5.5) with $N = 3$ is integrable. We remark that it is also integrable For $N = 4$ when $\Gamma = \sum_{n=1}^N \Gamma_n = 0$, since, in addition to above quantities, we have

$$\{Q^\varepsilon, P^\varepsilon\} = \Gamma, \quad \{Q^\varepsilon, I^\varepsilon\} = 2P^\varepsilon, \quad \{P^\varepsilon, I^\varepsilon\} = -2Q^\varepsilon.$$

In the case $N = 4$ with $\Gamma \neq 0$ or $N \geq 5$, the system is no longer integrable and the dynamics of ε -point vortices could be chaotic. This is one of the reasons why we consider three point vortex problem in the next part to construct the enstrophy dissipating weak solution.

It is useful to consider the evolution of the distance l_{nm}^ε , which is governed by

$$\frac{d}{dt}(l_{nm}^\varepsilon)^2 = \frac{2}{\pi} \sum_{l \neq n \neq m}^N \Gamma_l A_{nml} \left[\frac{1}{(l_{ml}^\varepsilon)^2} \mathcal{G}\left(\frac{l_{ml}^\varepsilon}{\varepsilon}\right) - \frac{1}{(l_{ln}^\varepsilon)^2} \mathcal{G}\left(\frac{l_{ln}^\varepsilon}{\varepsilon}\right) \right]$$

for $N \geq 3$, where A_{nml} denotes the signed area of the triangle formed by the three ε -point vortices. Its sign is positive if the indices n, m, l at the vertices of the triangle appear counterclockwise and negative if they do clockwise. Its area $|A_{nml}|$ is expressed by

$$|A_{nml}| = \frac{1}{4} \left[2 \left((l_{nm}^\varepsilon l_{ml}^\varepsilon)^2 + (l_{ml}^\varepsilon l_{ln}^\varepsilon)^2 + (l_{ln}^\varepsilon l_{nm}^\varepsilon)^2 \right) - (l_{nm}^\varepsilon)^4 - (l_{ml}^\varepsilon)^4 - (l_{ln}^\varepsilon)^4 \right]^{\frac{1}{2}}.$$

There are two coordinate independent conserved quantities, the Hamiltonian $\mathcal{H}^\varepsilon$ and

$$M^\varepsilon = \sum_{n \neq m}^N \Gamma_n \Gamma_m (l_{nm}^\varepsilon)^2 = 2(\Gamma I^\varepsilon - (Q^\varepsilon)^2 - (P^\varepsilon)^2).$$

5.2 Variations of the enstrophy and energy

We are concerned with the variations of the enstrophy $\mathcal{Z}^\varepsilon(t)$ and the energy $E^\varepsilon(t)$ that vary along with the Euler-Poincaré PV system. They can be derived based on the Novikov' method as [55, 58]. Since the vorticity field generated by N ε -point vortices at $\mathbf{x}_n^\varepsilon(t)$ for $n = 1, \dots, N$ is represented by

$$q(\mathbf{x}, t) = \sum_{n=1}^N \Gamma_n \delta(\mathbf{x} - \mathbf{x}_n^\varepsilon(t)),$$

its Fourier transform is given by

$$\hat{q}(\mathbf{k}, t) = \frac{1}{2\pi} \sum_{n=1}^N \Gamma_n e^{-i\mathbf{k} \cdot \mathbf{x}_n^\varepsilon(t)},$$

which yields

$$|\widehat{q}(\mathbf{k}, t)|^2 = \frac{1}{4\pi^2} \left[\sum_{n=1}^N \Gamma_n^2 + 2 \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m \cos(\mathbf{k} \cdot (\mathbf{x}_n^\varepsilon(t) - \mathbf{x}_m^\varepsilon(t))) \right]$$

Recalling the relation $\omega^\varepsilon = h^\varepsilon * q$, we have $\widehat{\omega}^\varepsilon = 2\pi \widehat{h}^\varepsilon \widehat{q}$ and thus

$$|\widehat{\omega}^\varepsilon(\mathbf{k}, t)|^2 = \frac{1}{4\pi^2} \left| 2\pi \widehat{h}^\varepsilon(\mathbf{k}) \right|^2 \left[\sum_{n=1}^N \Gamma_n^2 + 2 \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m \cos(\mathbf{k} \cdot (\mathbf{x}_n^\varepsilon(t) - \mathbf{x}_m^\varepsilon(t))) \right].$$

Then, we obtain the total enstrophy for the regularized vorticity,

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^2} |\omega^\varepsilon(\mathbf{x}, t)|^2 d\mathbf{x} &= \frac{1}{2} \int_{\mathbb{R}^2} |\widehat{\omega}^\varepsilon(\mathbf{k}, t)|^2 d\mathbf{k} \\ &= \frac{1}{2} \int_0^\infty \int_{-\pi}^\pi r |\widehat{\omega}^\varepsilon(r, \theta, t)|^2 d\theta dr \\ &= \int_0^\infty \pi r \langle |\widehat{\omega}^\varepsilon(r, t)|^2 \rangle dr \end{aligned}$$

in which $r = |\mathbf{k}|$ and $\langle f \rangle = \frac{1}{2\pi} \int_{-\pi}^\pi f(\theta) d\theta$. Here, we define the enstrophy density spectrum $\mathcal{Z}_N^\varepsilon$ by

$$\mathcal{Z}_N^\varepsilon(r, t) = \pi r \langle |\widehat{\omega}^\varepsilon(r, t)|^2 \rangle.$$

and note that $\widehat{h}^\varepsilon$ is radial. Then, it follows that

$$\begin{aligned} \mathcal{Z}_N^\varepsilon(r, t) &= \frac{r}{4\pi} \left| 2\pi \widehat{h}^\varepsilon(r) \right|^2 \left[\sum_{n=1}^N \Gamma_n^2 + 2 \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m \frac{1}{2\pi} \int_{-\pi}^\pi \cos(r l_{nm}^\varepsilon \cos \theta) d\theta \right] \\ &= \frac{r}{4\pi} \left| 2\pi \widehat{h}^\varepsilon(r) \right|^2 \left[\sum_{n=1}^N \Gamma_n^2 + 2 \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m J_0(r l_{nm}^\varepsilon) \right], \end{aligned}$$

in which $J_0(r)$ is a Bessel function of the first kind. Thus, the total enstrophy is given by

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^2} |\omega^\varepsilon(\mathbf{x}, t)|^2 d\mathbf{x} &= \int_0^\infty \mathcal{Z}_N^\varepsilon(r, t) dr \\ &= \frac{1}{4\pi} \int_0^\infty r \left| 2\pi \widehat{h}^\varepsilon(r) \right|^2 dr \sum_{n=1}^N \Gamma_n^2 + \frac{1}{2\pi} \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m \int_0^\infty r \left| 2\pi \widehat{h}^\varepsilon(r) \right|^2 J_0(r l_{nm}^\varepsilon) dr \\ &= \frac{1}{4\pi \varepsilon^2} \int_0^\infty r \left| 2\pi \widehat{h}(r) \right|^2 dr \sum_{n=1}^N \Gamma_n^2 + \frac{1}{2\pi \varepsilon^2} \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m \int_0^\infty r \left| 2\pi \widehat{h}(r) \right|^2 J_0\left(r \frac{l_{nm}^\varepsilon}{\varepsilon}\right) dr. \end{aligned}$$

Here, we have used the relation $\widehat{h}^\varepsilon(\mathbf{k}) = \widehat{h}(\varepsilon\mathbf{k})$. Since the first term in the right-hand side is constant in time, we define the variation of the enstrophy by the second term, namely

$$\mathcal{Z}^\varepsilon(t) = \frac{1}{2\pi\varepsilon^2} \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m \int_0^\infty r \left| 2\pi\widehat{h}(r) \right|^2 J_0 \left(r \frac{l_{nm}^\varepsilon(t)}{\varepsilon} \right) dr. \quad (5.7)$$

Next, we define the variation of the energy of the Euler-Poincaré PV system in the sense of Novikov. The total energy for the regularized velocity is given by

$$\frac{1}{2} \int_{\mathbb{R}^2} |\mathbf{u}^\varepsilon(\mathbf{x}, t)|^2 d\mathbf{x} = \int_0^\infty \pi r \langle |\widehat{\mathbf{u}}^\varepsilon(r, t)|^2 \rangle dr = \int_0^\infty E_N^\varepsilon(r, t) dr,$$

where E_N^ε is the energy density spectrum. Since we have $|\widehat{\omega}^\varepsilon(\mathbf{k}, t)|^2 = |\mathbf{k}|^2 |\widehat{\mathbf{u}}^\varepsilon(\mathbf{k}, t)|^2$, it follows that

$$\begin{aligned} E_N^\varepsilon(r, t) &\equiv \pi r \langle |\widehat{\mathbf{u}}^\varepsilon(r, t)|^2 \rangle = \frac{\pi}{r} \langle |\widehat{\omega}^\varepsilon(r, t)|^2 \rangle \\ &= \frac{1}{4\pi r} \left| 2\pi\widehat{h}^\varepsilon(r) \right|^2 \left[\sum_{n=1}^N \Gamma_n^2 + 2 \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m J_0(r l_{nm}^\varepsilon) \right]. \end{aligned}$$

Hence, the total energy cutting off a larger scale than $1 \ll L < \infty$ is expressed by

$$\begin{aligned} \int_{L^{-1}}^\infty E_N^\varepsilon(r, t) dr &= \frac{1}{4\pi} \int_{\varepsilon L^{-1}}^\infty \frac{1}{r} \left| 2\pi\widehat{h}(r) \right|^2 dr \sum_{n=1}^N \Gamma_n^2 \\ &\quad + \frac{1}{2\pi} \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m \int_{\varepsilon L^{-1}}^\infty \frac{1}{r} \left| 2\pi\widehat{h}(r) \right|^2 J_0 \left(r \frac{l_{nm}^\varepsilon}{\varepsilon} \right) dr. \end{aligned} \quad (5.8)$$

Regarding the second term, we find

$$\begin{aligned} \int_{\varepsilon L^{-1}}^\infty \frac{1}{r} \left| 2\pi\widehat{h}(r) \right|^2 J_0 \left(r \frac{l_{nm}^\varepsilon}{\varepsilon} \right) dr &= \int_{\varepsilon L^{-1}}^\infty \frac{1}{r} J_0 \left(r \frac{l_{nm}^\varepsilon}{\varepsilon} \right) dr \\ &\quad + \int_{\varepsilon L^{-1}}^\infty \frac{1}{r} \left(\left| 2\pi\widehat{h}(r) \right|^2 - 1 \right) J_0 \left(r \frac{l_{nm}^\varepsilon}{\varepsilon} \right) dr, \end{aligned}$$

and the following approximate formula for a sufficiently large L .

$$\begin{aligned} \int_{\varepsilon L^{-1}}^\infty \frac{1}{r} J_0 \left(r \frac{l_{nm}^\varepsilon}{\varepsilon} \right) dr &\sim \int_0^\infty \frac{r}{r^2 + (\varepsilon L^{-1})^2} J_0 \left(r \frac{l_{nm}^\varepsilon}{\varepsilon} \right) dr \\ &= K_0 \left(\frac{l_{nm}^\varepsilon}{L} \right) \\ &\sim -\log l_{nm}^\varepsilon + \log \frac{Le^\beta}{2}, \end{aligned}$$

in which β is the Euler's constant. Thus, taking the $L \rightarrow \infty$ limit in the non-constant part of the total energy (5.8), we obtain the variations of the energy.

$$E^\varepsilon(t) = -\frac{1}{2\pi} \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m \left[\log l_{nm}^\varepsilon(t) + \int_0^\infty \frac{1}{r} \left(1 - |2\pi \hat{h}(r)|^2 \right) J_0 \left(r \frac{l_{nm}^\varepsilon(t)}{\varepsilon} \right) dr \right]. \quad (5.9)$$

Note that the integrand in (5.9) has no singularity since it follows from (5.3) that $2\pi \hat{h}(0) = 1$.

Part III

Enstrophy dissipation and triple collapse of point vortices

6 Review of the point vortex system

6.1 The point vortex system and its self-similar solutions

For later use, let us recall the point vortex (PV) problem, which is expressed by

$$\frac{d}{dt} \mathbf{x}_n = -\frac{1}{2\pi} \sum_{m \neq n}^N \Gamma_m \frac{(\mathbf{x}_n - \mathbf{x}_m)^\perp}{l_{nm}^2}, \quad \mathbf{x}_n(0) = \mathbf{x}_n^0, \quad n = 1, \dots, N, \quad (6.1)$$

where $l_{nm}(t) = |\mathbf{x}_n(t) - \mathbf{x}_m(t)|$ is the distance between m th and n th point vortices. The PV system can be derived formally from the 2D Euler equations when the initial vorticity is given by point vortices. Hence, solutions of the PV system are not those of the 2D Euler equations in general, and this fact is a notable difference from the case of the Euler-Poincaré equations. The PV system is a Hamiltonian dynamical system with its Hamiltonian,

$$\mathcal{H}_{pv} = -\frac{1}{2\pi} \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m \log l_{nm}.$$

Defining the quantities (P, Q, I) in the same manner as $(P^\varepsilon, Q^\varepsilon, I^\varepsilon)$ in the Euler-Poincaré PV system, we similarly find that $(\mathcal{H}_{pv}, P, Q, I)$ are conserved along with

the evolution of (6.1). We now consider self-similar solutions in the PV system based on the argument in [38, 54]. The definition of self-similar evolutions of point vortices are given as follows.

Definition 6.1. *Let $z_n(t) = x_n(t) + iy_n(t)$ be a complex notation for the solution of (6.1). The evolution of point vortices is called self-similarly if we have $z_n(t) = \lambda_n f(t)$ for $n = 1, \dots, N$, where $\lambda_n \in \mathbb{C}$ are scale factors and $f(t) = r(t)e^{i\theta(t)}$ is a function of time.*

Consider the point vortex system with the complex variables,

$$\frac{d}{dt}z_n = \frac{i}{2\pi} \sum_{m \neq n}^N \frac{\Gamma_m}{z_n^* - z_m^*}, \quad n = 1, \dots, N,$$

where z_n^* denotes the complex conjugate of z_n . Assume that the point vortices evolve self-similarly. Then, we have

$$\lambda_n \frac{df}{dt} f^* = \frac{i}{2\pi} \sum_{m \neq n}^N \frac{\Gamma_m}{\lambda_n^* - \lambda_m^*},$$

which is equivalent to the following system.

$$\frac{df}{dt} f^* = C = A + iB, \tag{6.2}$$

$$\lambda_n C = \frac{i}{2\pi} \sum_{m \neq n}^N \frac{\Gamma_m}{\lambda_n^* - \lambda_m^*}. \tag{6.3}$$

In order to solve (6.2), we substitute $f(t) = r(t)e^{i\theta(t)}$ into (6.2). Then, we obtain

$$\frac{1}{2} \frac{d}{dt} r^2 = A, \quad \frac{d}{dt} \theta = \frac{B}{r^2},$$

and thus the solutions are

$$r(t) = \sqrt{2At + r^2(0)}, \quad \theta(t) = \frac{B}{2A} \log(2At + r^2(0)) + \theta(0),$$

for $A \neq 0$. If $A = 0$, we have $r(t) = r(0)$ and $\theta(t) = \frac{B}{r^2(0)}t + \theta(0)$, which corresponds to a relative equilibrium rotating with the constant speed. Hence, the point vortices collapse self-similarly at the origin at the time

$$t_c = -\frac{r^2(0)}{2A}.$$

We note that the collapse occurs at the center of the vorticity z_c , which is an invariant quantity, since we have

$$\Gamma z_c = \sum_{n=1}^N \Gamma_n z_n = \frac{i}{2\pi C} \sum_{n=1}^N \sum_{m \neq n}^N \frac{\Gamma_n \Gamma_m}{\lambda_n^* - \lambda_m^*} = 0.$$

The algebraic system (6.3) is complicated for general N and difficult to solve for the scaling constants λ_n . However, some examples of self-similar solutions, especially self-similar collapsing solutions, are demonstrated in [41, 42] for specific numbers of N . Moreover, in three point vortex problem, we can obtain self-similar collapsing solutions with mathematical rigor [38, 54].

6.2 Self-similar collapsing solutions in the three point vortex system

In the three PV system, the Hamiltonian $\mathcal{H}_{pv}$ is expressed by

$$\mathcal{H}_{pv} = -\frac{1}{2\pi} (\Gamma_2 \Gamma_3 \log l_{23} + \Gamma_3 \Gamma_1 \log l_{31} + \Gamma_1 \Gamma_2 \log l_{12}). \quad (6.4)$$

The system describing the evolution of the three distances are given by

$$\begin{aligned} \frac{d}{dt} l_{12}^2 &= \frac{2}{\pi} \Gamma_3 A_{pv} \left(\frac{1}{l_{23}^2} - \frac{1}{l_{31}^2} \right), \\ \frac{d}{dt} l_{23}^2 &= \frac{2}{\pi} \Gamma_1 A_{pv} \left(\frac{1}{l_{31}^2} - \frac{1}{l_{12}^2} \right), \\ \frac{d}{dt} l_{31}^2 &= \frac{2}{\pi} \Gamma_2 A_{pv} \left(\frac{1}{l_{12}^2} - \frac{1}{l_{23}^2} \right), \end{aligned} \quad (6.5)$$

where A_{pv} denotes the signed area of the triangle formed by the three point vortices. Its sign is positive if the indices 1, 2, 3 at the vertices of the triangle appear counterclockwise and negative if they do clockwise. Here, $|A_{pv}|$ is expressed by

$$|A_{pv}| = \frac{1}{4} [2 (l_{12}^2 l_{23}^2 + l_{23}^2 l_{31}^2 + l_{31}^2 l_{12}^2) - l_{12}^4 - l_{23}^4 - l_{31}^4]^{\frac{1}{2}},$$

and it follows that the distances need to satisfy

$$l_{23}^4 + l_{31}^4 + l_{12}^4 \leq 2 (l_{23}^2 l_{31}^2 + l_{31}^2 l_{12}^2 + l_{12}^2 l_{23}^2). \quad (6.6)$$

We remark that the only relative equilibria of (6.5) are collinear states or equilateral triangles. Here, we focus on the evolution under the following conditions,

$$\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3} = 0, \quad (6.7)$$

$$M_{pv} \equiv \Gamma_1 \Gamma_2 l_{12}^2 + \Gamma_2 \Gamma_3 l_{23}^2 + \Gamma_3 \Gamma_1 l_{31}^2 = 0. \quad (6.8)$$

Note that M_{pv} is a conserved quantity in the PV system. In view of (6.7), we assume $\Gamma_1 \geq \Gamma_2 > 0 > \Gamma_3$ without loss of generality. It is well known that (6.7) and (6.8) are necessary and sufficient conditions for three point vortices to collapse self-similarly in finite time. Based on [2, 54], we now review this result. We first set

$$\sqrt{\kappa} = \frac{l_{31}(t)}{l_{23}(t)}, \quad r_1 = \frac{l_{23}(t)}{l_{12}(t)}, \quad r_2 = \frac{l_{31}(t)}{l_{12}(t)},$$

then, from (6.7) and (6.8), we obtain

$$r_1 = \sqrt{\frac{\Gamma_1 + \Gamma_2}{\kappa \Gamma_1 + \Gamma_2}}, \quad r_2 = \sqrt{\frac{\kappa(\Gamma_1 + \Gamma_2)}{\kappa \Gamma_1 + \Gamma_2}}. \quad (6.9)$$

It is not difficult to see that κ is independent of t . Indeed, it follows from (6.4) that

$$\frac{1}{\theta_{pv}} = l_{23}^{2/\Gamma_1} l_{31}^{2/\Gamma_2} l_{12}^{2/\Gamma_3}, \quad \theta_{pv} = \exp\left(\frac{4\pi \mathcal{H}_{pv}}{\Gamma_1 \Gamma_2 \Gamma_3}\right).$$

and thus we find from (6.7) and (6.9) that

$$\frac{1}{\theta_{pv}} = \left(\frac{\Gamma_1 + \Gamma_2}{\kappa \Gamma_1 + \Gamma_2}\right)^{1/\Gamma_1} \left(\frac{\kappa(\Gamma_1 + \Gamma_2)}{\kappa \Gamma_1 + \Gamma_2}\right)^{1/\Gamma_2}.$$

Differentiating this relation with respect to t , we obtain $\kappa'(1 - 1/\kappa) = 0$, which yields $\kappa' = 0$ or $\kappa = 1$. The former case shows that κ is invariant, and r_1 and r_2 are thus constants owing to (6.9). The latter case corresponds to an equilateral triangle that is a relative equilibrium. In addition, we have the following lemma.

Lemma 6.2. *The ratio κ satisfies $k_- \leq \kappa \leq k_+$, in which k_- and k_+ are given by*

$$k_- = \left(\frac{\Gamma_1 + \Gamma_2 - \sqrt{\mathcal{R}}}{\Gamma_2}\right)^2, \quad k_+ = \left(\frac{\Gamma_1 + \Gamma_2 + \sqrt{\mathcal{R}}}{\Gamma_2}\right)^2, \quad (6.10)$$

and $\mathcal{R} = \Gamma_1^2 + \Gamma_1 \Gamma_2 + \Gamma_2^2$. Moreover, $\kappa = k_{\pm}$ holds if and only if the three point vortices form collinear configurations.

Proof. It follows from (6.6) that

$$1 + \kappa^2 + \frac{1}{r_1^4} \leq 2 \left(\kappa + \frac{1}{r_1^2} \kappa + \frac{1}{r_1^2} \right).$$

Using (6.9), we obtain

$$\Gamma_2^2 \kappa^2 - 2\kappa(2\Gamma_1^2 + 3\Gamma_1\Gamma_2 + 2\Gamma_2^2) + \Gamma_1^2 = \Gamma_2^2 \kappa^2 - 2\kappa(2\mathcal{R} + \Gamma_1\Gamma_2) + \Gamma_1^2 \leq 0,$$

and thus

$$\frac{2\mathcal{R} + \Gamma_1\Gamma_2 - 2\sqrt{\mathcal{R}(\mathcal{R} + \Gamma_1\Gamma_2)}}{\Gamma_2^2} \leq \kappa \leq \frac{2\mathcal{R} + \Gamma_1\Gamma_2 + 2\sqrt{\mathcal{R}(\mathcal{R} + \Gamma_1\Gamma_2)}}{\Gamma_2^2},$$

which is equivalent to $k_- \leq \kappa \leq k_+$. We have $\kappa = k_{\pm}$ if and only if the equality in (6.6) holds, which means that three point vortices are on a straight line. $\square$

It is obvious that $0 < k_- < 1 < k_+$. Next, we show the following lemma describing the range of $\mathcal{H}_{pv}$

Lemma 6.3. *Suppose (6.7) and (6.8). If the initial configuration is not relative equilibria and satisfies $r_1 > 1 > r_2$, then we have*

$$\frac{\Gamma_1^2 \Gamma_2^2}{4\pi(\Gamma_1 + \Gamma_2)} \log \left(\psi \left(\frac{\Gamma_1}{\Gamma_2} k_- \right) \left[\psi \left(\frac{\Gamma_1}{\Gamma_2} \right) \right]^{-1} \right) < \mathcal{H}_{pv} < 0,$$

where $\psi(x)$ is defined by

$$\psi(x) = \left(\frac{1}{1+x} \right)^{1/\Gamma_1} \left(\frac{x}{1+x} \right)^{1/\Gamma_2}.$$

For $r_2 > 1 > r_1$, we have the same result with k_- being replaced by k_+ .

Proof. Using (6.7), we can rewrite $\mathcal{H}_{pv}$ as follows.

$$\mathcal{H}_{pv} = \frac{\Gamma_1 \Gamma_2}{4\pi} \left(\frac{\Gamma_2}{\Gamma_1 + \Gamma_2} \log r_1^2 + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} \log r_2^2 \right).$$

Note that (6.8) is equivalent to

$$\frac{\Gamma_2}{\Gamma_1 + \Gamma_2} r_1^2 + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} r_2^2 = 1.$$

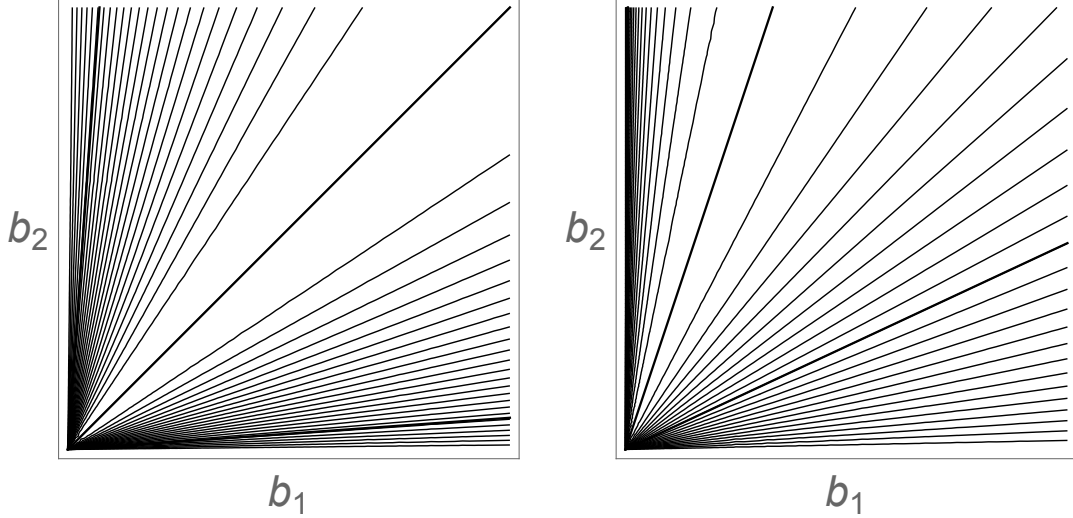


Figure 1: Equi-curves of the Hamiltonian $\mathcal{H}_{pv}$ in the (b_1, b_2) -plane, where $b_1 = l_{23}^2/\Gamma_1$ and $b_2 = l_{31}^2/\Gamma_2$, for (a) $(\Gamma_1, \Gamma_2) = (1, 1)$ and (b) $(\Gamma_1, \Gamma_2) = (3, 1)$. The light curves are equi-curves. The three dark lines from the left correspond to $\kappa = k_+$, $\kappa = 1$ and $\kappa = k_-$, respectively.

Then, it follows from the convexity of $\log x$ that $\mathcal{H}_{pv} \leq 0$. In particular, $\mathcal{H}_{pv} = 0$ if and only if the point vortices form an equilateral triangle, namely $r_1 = r_2 = 1$. On the other hand, it follows from (6.9) that

$$\begin{aligned} \mathcal{H}_{pv} &= \frac{\Gamma_1 \Gamma_2}{4\pi} \left[\frac{\Gamma_2}{\Gamma_1 + \Gamma_2} \log r_1^2 + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} \log r_2^2 \right] \\ &= \frac{\Gamma_1^2 \Gamma_2^2}{4\pi(\Gamma_1 + \Gamma_2)} \log \left[\left(\frac{\Gamma_1 + \Gamma_2}{\kappa \Gamma_1 + \Gamma_2} \right)^{1/\Gamma_1} \left(\frac{\kappa(\Gamma_1 + \Gamma_2)}{\kappa \Gamma_1 + \Gamma_2} \right)^{1/\Gamma_2} \right] \\ &= \frac{\Gamma_1^2 \Gamma_2^2}{4\pi(\Gamma_1 + \Gamma_2)} \log \left(\psi \left(\frac{\Gamma_1}{\Gamma_2} \kappa \right) \left[\psi \left(\frac{\Gamma_1}{\Gamma_2} \right) \right]^{-1} \right). \end{aligned}$$

Since we have $k_- < \kappa < 1$ for $r_1 > 1 > r_2$ and $1 < \kappa < k_+$ for $r_2 > 1 > r_1$, we achieve the conclusion, owing to (C.1) in Appendix C. $\square$

Remark 6.4. The relative equilibria are characterized in terms of the value of $\mathcal{H}_{pv}$. Indeed, we have $\mathcal{H}_{pv} = 0$ if and only if three point vortices form an equilateral triangle.

On the other hand, the Hamiltonian satisfies

$$\mathcal{H}_{pv} = \frac{\Gamma_1^2 \Gamma_2^2}{4\pi(\Gamma_1 + \Gamma_2)} \log \left(\psi \left(\frac{\Gamma_1}{\Gamma_2} k_{\pm} \right) \left[\psi \left(\frac{\Gamma_1}{\Gamma_2} \right) \right]^{-1} \right),$$

if and only if $\kappa = k_{\pm}$ respectively, namely three point vortices are in a collinear state.

We plot equi-curves of the Hamiltonian $\mathcal{H}_{pv}$ in Figure 1. Here, we set $b_1 = l_{23}^2/\Gamma_1$ and $b_2 = l_{31}^2/\Gamma_1$. Then, conditions (6.7) and (6.8) allow us to rewrite (6.4) as follows.

$$\frac{1}{\theta_{pv}} = \left(\frac{b_1}{b_1 + b_2} \right)^{1/\Gamma_1} \left(\frac{b_2}{b_1 + b_2} \right)^{1/\Gamma_2}, \quad (6.11)$$

in which θ_{pv} is a constant defined by

$$\theta_{pv} = \Gamma_1^{1/\Gamma_1} \Gamma_2^{1/\Gamma_2} (-\Gamma_3)^{1/\Gamma_3} \exp \left(\frac{4\pi \mathcal{H}_{pv}}{\Gamma_1 \Gamma_2 \Gamma_3} \right).$$

Then, equi-curves of $\mathcal{H}_{pv}$ is equivalent to (6.11) and thus they can be plotted in the (b_1, b_2) -plane. As we can see in Figure 1, equi-curves are straight lines and all of them are connected to the origin. These facts imply that three point vortices evolve self-similarly and collapse at the finite time. According to [2, 54], we can derive the explicit formulae for the collapse process. Indeed, owing to the temporal invariance of r_1 and r_2 , we can rewrite (6.5) as follows.

$$\frac{d}{dt} l_{12}^2 = \frac{\sigma S}{2\pi} \Gamma_3 \frac{r_2^2 - r_1^2}{r_1^2 r_2^2}, \quad \frac{d}{dt} l_{23}^2 = \frac{\sigma S}{2\pi} \Gamma_1 \frac{1 - r_2^2}{r_2^2}, \quad \frac{d}{dt} l_{31}^2 = \frac{\sigma S}{2\pi} \Gamma_2 \frac{r_1^2 - 1}{r_1^2},$$

where $S = [2(r_1^2 + r_2^2) - (r_1^2 - r_2^2)^2 - 1]^{1/2}$ and σ is the sign of A_{pv} . Then, using (6.7) and (6.8), we obtain

$$l_{12}(t) = \sqrt{\sigma \omega (t_c - t)}, \quad l_{23}(t) = r_1 \sqrt{\sigma \omega (t_c - t)}, \quad l_{31}(t) = r_2 \sqrt{\sigma \omega (t_c - t)},$$

which are equivalent to

$$l_{nm}(t) = l_{nm}(0) \sqrt{1 - \frac{t}{t_c}},$$

where ω and t_c are given by

$$\omega = \frac{S}{2\pi} \Gamma_2 \frac{r_1^2 - 1}{r_1^2 r_2^2}, \quad t_c = \frac{l_{12}^2(0)}{\sigma \omega}.$$

Thus, for $r_1 > 1 > r_2$, three point vortices collapse self-similarly as $t \rightarrow t_c - 0$ when $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ of the triangle appear counterclockwise, namely $\sigma > 0$. On the other hand, if $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ appear clockwise i.e. $\sigma < 0$, three point vortices expand self-similarly for $t > t_c$. In other words, they collapse self-similarly as $t \rightarrow t_c + 0$. As for $r_2 > 1 > r_1$, we similarly find the self-similar collapsing and expanding solutions. We remark that these two self-similar solutions are independent each other and cannot be connected at $t = t_c$ as a solution of the PV system.

7 The α -point vortex system

7.1 The N vortex problem and the enstrophy dissipation

We focus on the Euler- α model. Considering the regularized kernel $\mathbf{K}^\alpha$ in (3.6), we obtain the α -point vortex (α PV) system instead of (5.5).

$$\frac{d}{dt} \mathbf{x}_n^\alpha = -\frac{1}{2\pi} \sum_{m \neq n}^N \Gamma_m \frac{(\mathbf{x}_n^\alpha - \mathbf{x}_m^\alpha)^\perp}{(l_{nm}^\alpha)^2} B_K \left(\frac{l_{nm}^\alpha}{\alpha} \right), \quad n = 1, \dots, N, \quad (7.1)$$

where $l_{nm}^\alpha(t) = |\mathbf{x}_n^\varepsilon(t) - \mathbf{x}_m^\varepsilon(t)|$ and $\mathbf{x}_n^\alpha(0) = \mathbf{x}_n^0$. In this system, the Hamiltonian (5.6) is expressed by,

$$\mathcal{H}^\alpha = -\frac{1}{2\pi} \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m \left[\log l_{nm}^\alpha + K_0 \left(\frac{l_{nm}^\alpha}{\alpha} \right) \right].$$

According to [58], the variations of the enstrophy and the energy in the α PV system have a explicit formulae, which is a remarkable feature of the α PV system. Indeed, since smoothing function h is given by

$$h(r) = \frac{1}{2\pi} K_0(r),$$

and it is a fundamental solution to the Helmholtz operator $1 - \Delta$, we find

$$2\pi \hat{h}(\mathbf{k}) = \frac{1}{1 + |\mathbf{k}|^2}.$$

Then, the integral in (5.7) is calculated by the formula for Bessel functions [65].

$$\int_0^\infty \frac{r}{(1 + r^2)^2} J_0 \left(r \frac{l_{nm}^\alpha}{\alpha} \right) dr = \frac{1}{2} \frac{l_{nm}^\alpha}{\alpha} K_1 \left(\frac{l_{nm}^\alpha}{\alpha} \right). \quad (7.2)$$

and thus the variation of the enstrophy is given by

$$\mathcal{Z}^\alpha(t) = \frac{1}{4\pi\alpha^2} \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m \frac{l_{nm}^\alpha(t)}{\alpha} K_1 \left(\frac{l_{nm}^\alpha(t)}{\alpha} \right).$$

As for the variation of energy, we have

$$\begin{aligned} \int_0^\infty \frac{1}{r} \left(1 - |2\pi\widehat{h}(r)|^2 \right) J_0 \left(r \frac{l_{nm}^\varepsilon(t)}{\varepsilon} \right) dr &= \int_0^\infty \frac{r}{1+r^2} J_0 \left(r \frac{l_{nm}^\varepsilon(t)}{\varepsilon} \right) dr \\ &+ \int_0^\infty \frac{r}{(1+r^2)^2} J_0 \left(r \frac{l_{nm}^\varepsilon(t)}{\varepsilon} \right) dr. \end{aligned}$$

Using (7.2) and the following formula [65],

$$\int_0^\infty \frac{r}{1+r^2} J_0 \left(r \frac{l_{nm}^\alpha}{\alpha} \right) dr = K_0 \left(\frac{l_{nm}^\alpha}{\alpha} \right),$$

we obtain the explicit formula for the variation of the energy,

$$E^\alpha(t) = -\frac{1}{2\pi} \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m \left[\log l_{mn}^\alpha + K_0 \left(\frac{l_{mn}^\alpha}{\alpha} \right) + \frac{l_{mn}^\alpha}{2\alpha} K_1 \left(\frac{l_{mn}^\alpha}{\alpha} \right) \right].$$

We remark that $\mathcal{H}^\alpha$, $\mathcal{Z}^\alpha$ and E^α have the following relation.

$$\mathcal{H}^\alpha = E^\alpha + \alpha^2 \mathcal{Z}^\alpha. \quad (7.3)$$

Let us discuss the enstrophy dissipation with the N -vortex problem in the α PV system. Since the Hamiltonian system (7.1) is not integrable for $N \geq 4$, it is not easy to analyze the dynamics of solutions to the N -vortex problem and show the enstrophy dissipation with mathematical rigor. However, a necessary condition for the enstrophy dissipation in the $\alpha \rightarrow 0$ limit was obtained in [29].

Theorem 7.1. *Suppose that there exist functions $C_{nm}(t) > 0$ for $m \neq n$, which are defined on the interval I , such that*

$$\lim_{\alpha \rightarrow 0} \sup_{t \in I} |l_{mn}^\alpha(t) - C_{mn}(t)| = 0,$$

for any $m \neq n$. Then, we have

$$\lim_{\alpha \rightarrow 0} \sup_{t \in I} |E^\alpha(t) - \mathcal{H}_{pv}| = 0, \quad \lim_{\alpha \rightarrow 0} \sup_{t \in I} |\mathcal{Z}^\alpha(t)| = 0,$$

where $\mathcal{H}_{pv}$ denotes the Hamiltonian of the PV system.

Proof. It follows from (7.3) that

$$|E^\alpha - \mathcal{H}_{pv}| \leq |\mathcal{H}^\alpha - \mathcal{H}_{pv}| + \alpha^2 |\mathcal{Z}^\alpha|. \quad (7.4)$$

Since $\mathcal{H}_{pv}$ and $\mathcal{H}^\alpha$ are invariant in time, we find

$$\mathcal{H}_{pv} = -\frac{1}{2\pi} \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m \log |x_n^0 - x_m^0|,$$

and

$$\begin{aligned} \mathcal{H}^\alpha &= -\frac{1}{2\pi} \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m \left[\log l_{nm}^\alpha(t) + K_0 \left(\frac{l_{nm}^\alpha(t)}{\alpha} \right) \right] \\ &= -\frac{1}{2\pi} \sum_{n=1}^N \sum_{m=n+1}^N \Gamma_n \Gamma_m \left[\log |x_n^0 - x_m^0| + K_0 \left(\frac{|x_n^0 - x_m^0|}{\alpha} \right) \right]. \end{aligned}$$

Hence, we obtain

$$|\mathcal{H}^\alpha - \mathcal{H}_{pv}| \leq \frac{1}{2\pi} \sum_{n=1}^N \sum_{m=n+1}^N |\Gamma_n \Gamma_m| K_0 \left(\frac{|x_n^0 - x_m^0|}{\alpha} \right).$$

On the other hand, we also have

$$\alpha^2 |\mathcal{Z}^\alpha| \leq \frac{1}{4\pi} \sum_{n=1}^N \sum_{m=n+1}^N |\Gamma_n \Gamma_m| \frac{l_{nm}^\alpha(t)}{\alpha} K_1 \left(\frac{l_{nm}^\alpha(t)}{\alpha} \right).$$

Since the assumption implies $l_{mn}^\alpha/\alpha \rightarrow \infty$ as $\alpha \rightarrow 0$ and K_0, K_1 decay exponentially, we conclude that (7.4) converges to zero. The convergence of the variation of the enstrophy follows from the same argument. $\square$

This theorem indicates that the evolution of α -point vortices is equivalent to that of point vortices with the Hamiltonian $\mathcal{H}_{pv}$ in the $\alpha \rightarrow 0$ limit as long as their distances l_{nm}^α converge to non-vanishing functions uniformly in time. Furthermore, there is no enstrophy variation in the same time interval I . In other words, taking its contraposition, we have the following corollary on the necessary condition for the enstrophy variation.

Corollary 7.2. *Suppose that $\mathcal{Z}^\alpha(t_c) \neq 0$ at time t_c in the $\alpha \rightarrow 0$ limit. Then, there exists $m \neq n$ such that $l_{nm}^\alpha(t_c) \rightarrow 0$ as $\alpha \rightarrow 0$.*

We recall that the α -point vortices never collapse in a finite time as shown in Theorem 5.1, that is, $l_{nm}^\alpha(t) \neq 0$ for $\alpha \neq 0$ and all $t \in \mathbb{R}$. However, the distance l_{nm}^α could vanish at a certain time in the $\alpha \rightarrow 0$ limit and, according to Corollary 7.2, the enstrophy variation might occur at that time. It is straightforward to check that the enstrophy variation never occurs for the evolution of two α -point vortices. Indeed, since we find $dl_{12}^\alpha/dt = 0$, it follows that $l_{12}^\alpha(t) = |\mathbf{x}_1^0 - \mathbf{x}_2^0| = l_{12}(0)$ for all $t \in \mathbb{R}$ and thus $\lim_{\alpha \rightarrow 0} l_{nm}^\alpha(t) = l_{12}(0) \neq 0$. Hence, in order to observe the enstrophy dissipation in the $\alpha \rightarrow 0$ limit of the evolution of α -point vortices, we need to consider the α PV system (7.1) with $N \geq 3$. In the following section, we focus on the three vortex problem, since (7.1) is integrable with $N = 3$ and it is suggested with a help of numerical computations that an enstrophy dissipation could occur under a certain condition when the three point vortices collapse self-similarly [58]. The purpose of our study is proving the existence of enstrophy dissipating solutions with mathematical rigor and making it clear what kind of mechanism gives rise to the enstrophy dissipation in the $\alpha \rightarrow 0$ limit.

7.2 The three vortex problem and general properties

We consider the three α -point vortex problem. The three α PV system (7.1) with $N = 3$ is a Hamiltonian system with its Hamiltonian

$$\begin{aligned} \mathcal{H}^\alpha = & -\frac{1}{2\pi} (\Gamma_2\Gamma_3 \log l_{23}^\alpha + \Gamma_3\Gamma_1 \log l_{31}^\alpha + \Gamma_1\Gamma_2 \log l_{12}^\alpha) \\ & -\frac{1}{2\pi} \left[\Gamma_2\Gamma_3 K_0 \left(\frac{l_{23}^\alpha}{\alpha} \right) + \Gamma_3\Gamma_1 K_0 \left(\frac{l_{31}^\alpha}{\alpha} \right) + \Gamma_1\Gamma_2 K_0 \left(\frac{l_{12}^\alpha}{\alpha} \right) \right]. \end{aligned}$$

This system is integrable for any strengths and initial configurations. The system (7.1) is reduced to the following system for the distance l_{nm}^α :

$$\frac{d}{dt} (l_{nm}^\alpha)^2 = \frac{2}{\pi} \Gamma_k A^\alpha \left[\frac{1}{(l_{mk}^\alpha)^2} B_K \left(\frac{l_{mk}^\alpha}{\alpha} \right) - \frac{1}{(l_{kn}^\alpha)^2} B_K \left(\frac{l_{kn}^\alpha}{\alpha} \right) \right], \quad (7.5)$$

and $l_{nm}^\alpha(0) = |\mathbf{x}_n^0 - \mathbf{x}_m^0|$, where A^α denotes the signed area of the triangle formed by the three α -point vortices, $\mathbf{x}_1^\alpha$, $\mathbf{x}_2^\alpha$ and $\mathbf{x}_3^\alpha$, and its sign is determined by the order of indices. Note that three α -point vortices must become a collinear configuration to change the sign of A^α . Remember that we have two invariants in terms of l_{nm}^α , $\mathcal{H}^\alpha$ and

$$M^\alpha = \Gamma_2\Gamma_3 (l_{23}^\alpha)^2 + \Gamma_3\Gamma_1 (l_{31}^\alpha)^2 + \Gamma_1\Gamma_2 (l_{12}^\alpha)^2.$$

We are concerned with the variations of the enstrophy $\mathcal{Z}^\alpha(t)$ and the energy $E^\alpha(t)$ which are given by

$$\mathcal{Z}^\alpha(t) = \frac{1}{4\pi\alpha^2} \left[\Gamma_2\Gamma_3 \frac{l_{23}^\alpha}{\alpha} K_1 \left(\frac{l_{23}^\alpha}{\alpha} \right) + \Gamma_3\Gamma_1 \frac{l_{31}^\alpha}{\alpha} K_1 \left(\frac{l_{31}^\alpha}{\alpha} \right) + \Gamma_1\Gamma_2 \frac{l_{12}^\alpha}{\alpha} K_1 \left(\frac{l_{12}^\alpha}{\alpha} \right) \right],$$

and $E^\alpha(t) = \mathcal{H}^\alpha - \alpha^2 \mathcal{Z}^\alpha(t)$. Note that $\mathcal{Z}^\alpha(t)$ is a part of the enstrophy, defined from the α -vorticity, that varies along with the evolution of the three α point vortices. Here, in order to take the $\alpha \rightarrow 0$ limit, we introduce the following scaled variables,

$$\mathbf{X}_n(t) = \frac{1}{\alpha} \mathbf{x}_n^\alpha(\alpha^2 t + t^*), \quad L_{nm}(t) = \frac{1}{\alpha} l_{nm}^\alpha(\alpha^2 t + t^*). \quad (7.6)$$

for $n = 1, 2, 3$ and $m \neq n$, where $t^* \in \mathbb{R}$ is a constant determined later. Then, the governing equation for $\mathbf{X}_n(t)$ is described by

$$\frac{d}{dt} \mathbf{X}_n = \frac{1}{2\pi} \sum_{m \neq n}^3 \Gamma_m \frac{(\mathbf{X}_n - \mathbf{X}_m)^\perp}{L_{nm}^2} B_K(L_{nm}), \quad \mathbf{X}_n(0) = \frac{\mathbf{x}_n^\alpha(t^*)}{\alpha}. \quad (7.7)$$

Since (7.7) is equivalent to (7.1) with $N = 3$ and $\alpha = 1$, the equation (7.7) is also a Hamiltonian dynamical system with its Hamiltonian,

$$\mathcal{H} = \mathcal{H}^1 = -\frac{1}{2\pi} \left[\Gamma_2\Gamma_3 h_0(L_{23}^2) + \Gamma_3\Gamma_1 h_0(L_{31}^2) + \Gamma_1\Gamma_2 h_0(L_{12}^2) \right]. \quad (7.8)$$

where $h_0(x) = \log \sqrt{x} + K_0(\sqrt{x})$. The evolution of (L_{12}, L_{23}, L_{31}) is explicitly described by

$$\begin{aligned} \frac{d}{dt} L_{12}^2 &= \frac{2}{\pi} \Gamma_3 A \left[\frac{1}{L_{23}^2} B_K(L_{23}) - \frac{1}{L_{31}^2} B_K(L_{31}) \right], \\ \frac{d}{dt} L_{23}^2 &= \frac{2}{\pi} \Gamma_1 A \left[\frac{1}{L_{31}^2} B_K(L_{31}) - \frac{1}{L_{12}^2} B_K(L_{12}) \right], \\ \frac{d}{dt} L_{31}^2 &= \frac{2}{\pi} \Gamma_2 A \left[\frac{1}{L_{12}^2} B_K(L_{12}) - \frac{1}{L_{23}^2} B_K(L_{23}) \right], \end{aligned} \quad (7.9)$$

where $A = A^1$ is the signed area of the triangle formed by the three α -point vortices, and its sign is defined in a same manner to the three PV system. Then, Heron's formula yields

$$A^2 = r(r - L_{12})(r - L_{23})(r - L_{31}), \quad r = \frac{1}{2}(L_{12} + L_{23} + L_{31}). \quad (7.10)$$

Hence, we obtain

$$|A| = \frac{1}{4} \left[2 \left(L_{12}^2 L_{23}^2 + L_{23}^2 L_{31}^2 + L_{31}^2 L_{12}^2 \right) - L_{12}^4 - L_{23}^4 - L_{31}^4 \right]^{\frac{1}{2}},$$

and the triangle inequality;

$$L_{23}^4 + L_{31}^4 + L_{12}^4 \leq 2 \left(L_{23}^2 L_{31}^2 + L_{31}^2 L_{12}^2 + L_{12}^2 L_{23}^2 \right), \quad (7.11)$$

in which the equality holds if and only if the three α -point vortices form a collinear configuration. We note that the solutions of (7.1) and (7.5) are recovered from those of the scaled systems (7.7) and (7.9) via

$$\mathbf{x}_n^\alpha(t) = \alpha \mathbf{X}_n \left(\frac{t - t^*}{\alpha^2} \right), \quad l_{nm}^\alpha(t) = \alpha L_{nm} \left(\frac{t - t^*}{\alpha^2} \right). \quad (7.12)$$

Moreover, in terms of the scaled variables, $E^\alpha(t)$ and $\mathcal{Z}^\alpha(t)$ are simply described by

$$E^\alpha(t) = \mathcal{H}^\alpha + H_K \left(\frac{t - t^*}{\alpha^2} \right), \quad \mathcal{Z}^\alpha(t) = -\frac{1}{\alpha^2} H_K \left(\frac{t - t^*}{\alpha^2} \right), \quad (7.13)$$

when $H_K(\tau)$ is defined by

$$H_K(\tau) = -\frac{1}{4\pi} \left[\Gamma_2 \Gamma_3 L_{23} K_1(L_{23}) + \Gamma_3 \Gamma_1 L_{31} K_1(L_{31}) + \Gamma_1 \Gamma_2 L_{12} K_1(L_{12}) \right].$$

The energy dissipation rate $\mathcal{D}_E^\alpha$ is obtained by differentiating $E^\alpha(t)$,

$$\mathcal{D}_E^\alpha(t) = \frac{1}{\alpha^2} F \left(\frac{t - t^*}{\alpha^2} \right),$$

in which the function $F(\tau)$ is defined by

$$F(\tau) = \frac{1}{8\pi} \left[\Gamma_2 \Gamma_3 \frac{dL_{23}^2}{d\tau} K_0(L_{23}) + \Gamma_3 \Gamma_1 \frac{dL_{31}^2}{d\tau} K_0(L_{31}) + \Gamma_1 \Gamma_2 \frac{dL_{12}^2}{d\tau} K_0(L_{12}) \right]. \quad (7.14)$$

Hence, in order to see the energy and the enstrophy variations together with the evolution of α -point vortices, we need to figure out the behavior of $L_{nm}(t)$ precisely. As we see in Corollary 3.2 of [29], a necessary condition of the enstrophy dissipation at time t^* is $l_{nm}^\alpha(t^*) \rightarrow 0$ as $\alpha \rightarrow 0$. This implies the existence of a singular collapse of the α PV system in the $\alpha \rightarrow 0$ limit, which infers that the limit is related to the

self-similar collapse in the PV system. Thus, we consider the motion of three α -point vortices under the same condition as (6.7);

$$\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3} = 0. \quad (7.15)$$

We also assume that $\Gamma_1 \geq \Gamma_2 > 0 > \Gamma_3$ without loss of generality. Then, it is easy to check that (7.15) yields $\mathcal{H}^\alpha = \mathcal{H}$. We note that

$$M = M^1 = \Gamma_2 \Gamma_3 L_{23}^2 + \Gamma_3 \Gamma_1 L_{31}^2 + \Gamma_1 \Gamma_2 L_{12}^2, \quad (7.16)$$

is invariant and it plays an important role in classifying the dynamics of the three α -point vortices. Let us remark that, for arbitrary solutions (7.7), we have $M^\alpha = \alpha^2 M \rightarrow 0$ as $\alpha \rightarrow 0$ owing to (7.6).

We first investigate relative equilibria in the α PV system. Then, the distance $L_{nm}(t)$ is bounded, which means that the enstrophy dissipation does not occur. Hence, we need to avoid such configurations in order to construct an enstrophy dissipating solution.

Proposition 7.3. *The relative equilibria of (7.9) are equilateral triangles, namely $L_{23} = L_{31} = L_{12}$, or collinear configurations, that is $A = 0$ and $dA/dt = 0$.*

Proof. If the system is in a relative equilibrium, we have $\frac{d}{dt} L_{12}^2 = \frac{d}{dt} L_{23}^2 = \frac{d}{dt} L_{31}^2 = 0$. Then, considering (7.9), we obtain either $A = 0$ or

$$\frac{1}{L_{23}^2} B_K(L_{23}) = \frac{1}{L_{31}^2} B_K(L_{31}) = \frac{1}{L_{12}^2} B_K(L_{12}). \quad (7.17)$$

The former case corresponds to a collinear configuration. Obviously, the α -point vortices need to satisfy $dA/dt = 0$ to form a relative equilibrium. The latter case implies $L_{23} = L_{31} = L_{12}$, namely the α -point vortices form an equilateral triangle. Indeed, since (7.17) is rewritten by

$$h_K(L_{23}^2) = h_K(L_{31}^2) = h_K(L_{12}^2),$$

where $h_K(x) = B_K(\sqrt{x})/x$ is monotone decreasing, as we see (C.6) in Appendix C, it follows that $L_{23} = L_{31} = L_{12}$. $\square$

Remark 7.4. For later use, differentiating (7.10), we derive the evolution equation

of A ;

$$\begin{aligned} \frac{dA}{dt} = & \frac{1}{4\pi} \left[\sum \Gamma_1 \frac{1}{L_{23}} \left\{ \frac{1}{L_{31}^2} B_K(L_{31}) - \frac{1}{L_{12}^2} B_K(L_{12}) \right\} \right] \\ & \times \left[(r - L_{23})(r - L_{31})(r - L_{12}) + r \sum (r - L_{31})(r - L_{12}) \right] \\ & - \frac{r}{2\pi} \sum \Gamma_1 \frac{1}{L_{23}} \left\{ \frac{1}{L_{31}^2} B_K(L_{31}) - \frac{1}{L_{12}^2} B_K(L_{12}) \right\} (r - L_{31})(r - L_{12}), \end{aligned} \quad (7.18)$$

in which $\sum$ denotes the summation over cyclic permutations of indices, for example $\sum L_{12} = L_{12} + L_{23} + L_{31}$. We need to solve $dA/dt = 0$ and $A = 0$ simultaneously to obtain collinear equilibria.

8 Enstrophy dissipation in the three point vortex problem

8.1 Statements of theorems

According to the numerical studies of [58], there exist some singular solutions that dissipate the enstrophy at the event of the triple collapse in the $\alpha \rightarrow 0$ limit for sufficiently small $M \geq 0$. However, it is unclear whether or not the enstrophy variation is always dissipative and the condition $M \geq 0$ is an essential restriction for the dissipation. We prove the existence of the enstrophy dissipating solution without any restriction on M . This indicates that the anomalous enstrophy dissipation is generically observed for a wide range of initial configurations of the three α -point vortices. The first theorem on the enstrophy dissipation is stated as follows.

Theorem 8.1. *Suppose (7.15) and the constant $\mathcal{H}_c$ satisfies*

$$\frac{\Gamma_1^2 \Gamma_2^2}{4\pi(\Gamma_1 + \Gamma_2)} \log \left(\psi \left(\frac{\Gamma_1}{\Gamma_2} k_0 \right) \left[\psi \left(\frac{\Gamma_1}{\Gamma_2} \right) \right]^{-1} \right) < \mathcal{H}_c < 0, \quad (8.1)$$

where k_0 is defined by

$$\psi \left(\frac{\Gamma_1}{\Gamma_2} k_0 \right) = \min \left\{ \psi \left(\frac{\Gamma_1}{\Gamma_2} k_- \right), \psi \left(\frac{\Gamma_1}{\Gamma_2} k_+ \right) \right\}.$$

We also assume that, for any initial configuration with $\mathcal{H}^\alpha = \mathcal{H}_c$, the corresponding orbit does not converge to relative equilibria as either of $t \rightarrow \pm\infty$. Then, there exists

a constant t^* such that $l_{nm}^\alpha(t^*) \rightarrow 0$ as $\alpha \rightarrow 0$ and

$$\lim_{\alpha \rightarrow 0} \mathcal{Z}^\alpha = -z_0 \delta(\cdot - t^*), \quad \lim_{\alpha \rightarrow 0} \mathcal{D}_E^\alpha = 0,$$

in the sense of distributions, where

$$z_0 = \int_{-\infty}^{\infty} H_K(s) ds > 0.$$

Remark 8.2. In this theorem, we assume that the solutions do not converge to relative equilibria as $t \rightarrow \pm\infty$. But, this assumption is not essential. As a matter of fact, it is always satisfied when $M \leq 0$. As for $M > 0$, these solutions exist at most for two values of the Hamiltonian in the range of (8.1) with $M > 0$, which is not generic.

Remark 8.3. Formally, the proof of Theorem 8.1 is straightforward; For any compactly supported smooth function $\phi(\tau)$, it follows from (7.13) that

$$\begin{aligned} \langle \mathcal{Z}^\alpha, \phi \rangle &= - \int_{-\infty}^{\infty} \frac{1}{\alpha^2} H_K \left(\frac{t - t^*}{\alpha^2} \right) \phi(t) dt = - \int_{-\infty}^{\infty} H_K(\tau) \phi(\alpha^2 \tau + t^*) d\tau \\ &\rightarrow -\phi(t^*) \int_{-\infty}^{\infty} H_K(\tau) d\tau = -z_0 \phi(t^*), \end{aligned}$$

and

$$\begin{aligned} \langle \mathcal{D}_E^\alpha, \phi \rangle &= \int_{-\infty}^{\infty} \frac{1}{\alpha^2} F \left(\frac{t - t^*}{\alpha^2} \right) \phi(t) dt = \int_{-\infty}^{\infty} F(\tau) \phi(\alpha^2 \tau + t^*) d\tau \\ &\rightarrow \phi(t^*) \int_{-\infty}^{\infty} F(\tau) d\tau = \phi(t^*) [H_K(\tau)]_{-\infty}^{\infty} = 0, \end{aligned}$$

as $\alpha \rightarrow 0$. Here, we use the relation $F(\tau) = \frac{d}{d\tau} H_K(\tau)$. In order to make above argument mathematically rigorous, we have to show that both $H_K(\tau)$ and $F(\tau)$ are well-defined and integrable for $\tau \in \mathbb{R}$. Owing to the definitions of these functions, it is sufficient to show that $L_{nm}(t) \rightarrow +\infty$ as $t \rightarrow \pm\infty$, since $K_0(x)$ and $K_1(x)$ decay exponentially. More precisely, we show that the ratios L_{23}/L_{12} , L_{31}/L_{12} converge to positive constants in the $t \rightarrow \pm\infty$ limit and thus the equations (7.9) imply that the rate of divergence of $L_{nm}(t)$ is estimated as $\mathcal{O}(t^{1/2})$. Also, we need to choose t^* to satisfy $l_{nm}^\alpha(t^*) \rightarrow 0$ as $\alpha \rightarrow 0$ so that t^* becomes the point support of the limit distribution of $\mathcal{Z}^\alpha$ [29]. There also needs to be shown that z_0 is positive. In Section 8.2, we divide the proof into three cases, $M = 0$, $M < 0$ and $M > 0$, since the dynamics of the three α -point vortices depends on the sign of M .

Regarding the $\alpha \rightarrow 0$ limit of $l_{nm}^\alpha(t)$ in (7.12), we prove that it converges to a singular evolution connecting the self-similar collapsing and expanding solutions obtained in the PV system in Section 6.2, which is stated as follows.

Theorem 8.4. *Under the same assumptions of Theorem 8.1, the three α -point vortices with $\mathcal{H}_c$ converge to the self-similar collapsing solution for $t < t^*$ and the expanding solution for $t > t^*$ with the same value of the Hamiltonian, i.e. $\mathcal{H}_{pv} = \mathcal{H}_c$, in the three PV system, as $\alpha \rightarrow 0$.*

Remark 8.5. As we see in the proof of Section 8.3, the three α -point vortices converge uniformly on any compact interval in $\mathbb{R} \setminus \{t^*\}$, since the limit function cannot be defined at $t = t^*$.

Remark 8.6. Since the α PV system is a Hamiltonian dynamical system, which is conservative and time-reversible, the solutions are symmetric with respect to time reversal. Nevertheless, Theorem 8.1 indicates that the enstrophy variation is invariably dissipative at $t = t^*$, since we have

$$\int_{-\infty}^t \mathcal{L}^\alpha(\tau) d\tau \longrightarrow -z_0 \mathcal{H}(t - t^*),$$

as $\alpha \rightarrow 0$ in the sense of distributions, where $\mathcal{H}$ denotes the Heaviside function. Consequently, this limit self-similar collapsing solution manifests an emergence of the irreversibility of time in this conservative dynamical system.

As we mention in Remark 8.3, the divergence of $L_{nm}(t)$ is essential for the enstrophy dissipation. As a matter of fact, we show that (7.15) is also a necessary condition.

Theorem 8.7. *Suppose $L_{nm}(t) \rightarrow +\infty$ as $t \rightarrow \pm\infty$ for $n \neq m$. Then we have (7.15).*

Proof. Suppose

$$\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3} = h \neq 0.$$

Then, we may assume that $\Gamma_1 \geq \Gamma_2 \geq \Gamma_3$ and $h > 0$ without loss of generality. We first show $0 < \underline{\lim}_{t \rightarrow \pm\infty} L_{23}/L_{12} < \overline{\lim}_{t \rightarrow \pm\infty} L_{23}/L_{12} < +\infty$. Assume that $\overline{\lim}_{t \rightarrow \pm\infty} L_{23}/L_{12} = +\infty$. Then, it follows from $L_{23} \leq L_{31} + L_{12}$ and $L_{31} \leq L_{23} + L_{12}$ that $\lim_{t \rightarrow \pm\infty} L_{31}/L_{23} = 1$. Dividing (7.16) by L_{23}^2 and taking the limit, we obtain $\Gamma_2\Gamma_3 + \Gamma_3\Gamma_1 = 0$. This contradicts (7.15) and we have $\overline{\lim}_{t \rightarrow \pm\infty} L_{23}/L_{12} < +\infty$. Similarly, it follows that $\overline{\lim}_{t \rightarrow \pm\infty} L_{12}/L_{23} < +\infty$ and thus we obtain $0 <$

$\lim_{t \rightarrow \pm\infty} L_{23}/L_{12}$. As for L_{31}/L_{12} , we find the same result in the same way as L_{23}/L_{12} . Next, using (7.8) and (7.16), we find

$$\mathcal{H} = -\frac{\Gamma_1\Gamma_2\Gamma_3}{2\pi} \left[\log H(t) + \frac{1}{\Gamma_1} K_0(L_{23}) + \frac{1}{\Gamma_2} K_0(L_{31}) + \frac{1}{\Gamma_3} K_0(L_{12}) \right],$$

where $H(t)$ is defined by

$$H(t) = (L_{12}(t))^h \left(\frac{L_{23}(t)}{L_{12}(t)} \right)^{1/\Gamma_1} \left(\frac{L_{31}(t)}{L_{12}(t)} \right)^{1/\Gamma_2}.$$

Since $K_0(L_{nm}(t)) \rightarrow 0$ and $H(t) \rightarrow +\infty$ as $t \rightarrow \pm\infty$, we have $\mathcal{H} \rightarrow +\infty$ as $t \rightarrow \pm\infty$. This is a contradiction. $\square$

8.2 Existence of enstrophy dissipating solutions

8.2.1 $M = 0$ case

When we assume that (7.15) and $M = 0$, we employ the same argument as we do in the three PV system. It follows from (7.15) that $M = 0$ is equivalent to

$$\frac{\Gamma_2}{\Gamma_1 + \Gamma_2} L_{23}^2 + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} L_{31}^2 = L_{12}^2. \quad (8.2)$$

This relation indicates that L_{12}^2 internally divides L_{23}^2 and L_{31}^2 into $\Gamma_1 : \Gamma_2$. Let us set

$$\frac{L_{31}}{L_{23}} = \sqrt{k}, \quad \frac{L_{23}}{L_{12}} = \gamma_1, \quad \frac{L_{31}}{L_{12}} = \gamma_2, \quad (8.3)$$

then, it follows from (8.2) that

$$\gamma_1 = \sqrt{\frac{\Gamma_1 + \Gamma_2}{k\Gamma_1 + \Gamma_2}}, \quad \gamma_2 = \sqrt{\frac{k(\Gamma_1 + \Gamma_2)}{k\Gamma_1 + \Gamma_2}}. \quad (8.4)$$

We note that the ratio k depends on t , which is different from the PV system. Since Lemma 6.2 holds in the α PV system [58], we have $k_- \leq k \leq k_+$, where k_- and k_+ are given by (6.10). When $k = k_{\pm}$, the three α -point vortices form collinear configurations. We first show that collinear configurations are not relative equilibria.

Proposition 8.8. *Suppose (7.15) and $M = 0$. Then, we have $dA/dt \neq 0$ when $L_{31}^2/L_{23}^2 = k_{\pm}$.*

Proof. We first consider the case when $L_{31}^2/L_{23}^2 = k_-$, namely $\gamma_1 > 1 > \gamma_2$. Then, it follows from (7.15), $M = 0$ and (7.18) that

$$\begin{aligned}\frac{dA}{dt} &= -\frac{\sqrt{\mathcal{R}}}{4\pi} \left[\frac{\Gamma_1}{\Gamma_1 + \Gamma_2} B_K(\gamma_2 L_{12}) + \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} B_K(\gamma_1 L_{12}) - B_K(L_{12}) \right] \\ &\equiv -\frac{\sqrt{\mathcal{R}}}{4\pi} g_K(L_{12}^2),\end{aligned}$$

in which $g_K(x)$ is given by

$$g_K(x) = \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} B_K\left(\sqrt{\gamma_2^2 x}\right) + \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} B_K\left(\sqrt{\gamma_1^2 x}\right) - B_K(\sqrt{x}).$$

Since $B_K(\sqrt{x})$ is monotone increasing and upward-convex, as we see (C.4) in Appendix C, it follows from (8.2) that

$$B_K\left(\sqrt{L_{12}^2}\right) > \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} B_K\left(\sqrt{\gamma_2^2 L_{12}^2}\right) + \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} B_K\left(\sqrt{\gamma_1^2 L_{12}^2}\right).$$

This yields $dA/dt > 0$. For the other case $L_{31}^2/L_{23}^2 = k_+$, namely $\gamma_2 > 1 > \gamma_1$, then we similarly obtain

$$\frac{dA}{dt} = \frac{\sqrt{\mathcal{R}}}{4\pi} g_K(L_{12}^2) < 0.$$

This concludes the proof. $\square$

When we remember that the evolution of the distances L_{23} , L_{31} and L_{12} described by (7.9) is integrable, it is useful to consider the equi-curves of the Hamiltonian $\mathcal{H}$, since the evolution is represented by a point moving on these curves in a certain two-dimensional phase space. Let us introduce new variables, called the trilinear coordinates (b_1, b_2, b_3) , as follows.

$$b_1 = \frac{L_{23}^2}{\Gamma_1}, \quad b_2 = \frac{L_{31}^2}{\Gamma_2}, \quad b_3 = \frac{L_{12}^2}{\Gamma_3}.$$

Then, the triangle inequality (7.11) is rewritten by

$$(\Gamma_1 b_1)^2 + (\Gamma_2 b_2)^2 + (\Gamma_3 b_3)^2 \leq 2(\Gamma_1 \Gamma_2 b_1 b_2 + \Gamma_2 \Gamma_3 b_2 b_3 + \Gamma_3 \Gamma_1 b_3 b_1). \quad (8.5)$$

For given Γ_1 , Γ_2 and Γ_3 , this inequality gives rise to a region called the *physical region* bounded by a conic section, in which equi-curves of the Hamiltonian are drawn. Especially, any point on the boundary of the physical region corresponds

to a collinear configuration. In the trilinear coordinates, the Hamiltonian $\mathcal{H}$ is expressed by

$$\mathcal{H} = -\frac{1}{2\pi} [\Gamma_2 \Gamma_3 h_0(\Gamma_1 b_1) + \Gamma_3 \Gamma_1 h_0(\Gamma_2 b_2) + \Gamma_1 \Gamma_2 h_0(\Gamma_3 b_3)].$$

By a simple calculation, we find the following equality.

$$\frac{1}{\theta} = b_1^{1/\Gamma_1} b_2^{1/\Gamma_2} (-b_3)^{1/\Gamma_3} \exp \left[\frac{2}{\Gamma_1} K_0(\sqrt{\Gamma_1 b_1}) + \frac{2}{\Gamma_2} K_0(\sqrt{\Gamma_2 b_2}) + \frac{2}{\Gamma_3} K_0(\sqrt{\Gamma_3 b_3}) \right], \quad (8.6)$$

where θ is a constant defined by

$$\theta = \Gamma_1^{1/\Gamma_1} \Gamma_2^{1/\Gamma_2} (-\Gamma_3)^{1/\Gamma_3} \exp \left(\frac{4\pi \mathcal{H}}{\Gamma_1 \Gamma_2 \Gamma_3} \right). \quad (8.7)$$

See [2, 54] for the derivation of this equality. The condition $M = 0$ is equivalent to

$$b_1 + b_2 + b_3 = 0, \quad (8.8)$$

which reduces the trilinear coordinates to the two dimensional (b_1, b_2) -plane. With (7.15) and (8.8), the equality (8.6) is rewritten by $f(b_1, b_2) = 1/\theta$, where $f(b_1, b_2)$ is defined as follows.

$$f(b_1, b_2) = \left(\frac{b_1}{b_1 + b_2} \right)^{1/\Gamma_1} \left(\frac{b_2}{b_1 + b_2} \right)^{1/\Gamma_2} \exp \left[\frac{2}{\Gamma_1} K_0(\sqrt{\Gamma_1 b_1}) + \frac{2}{\Gamma_2} K_0(\sqrt{\Gamma_2 b_2}) - 2 \frac{\Gamma_1 + \Gamma_2}{\Gamma_1 \Gamma_2} K_0 \left(\sqrt{\frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} (b_1 + b_2)} \right) \right]. \quad (8.9)$$

We set $\lambda = b_2/b_1$. Then, $k_- \leq k \leq k_+$ is equivalent to $\lambda_- \leq \lambda \leq \lambda_+$, in which

$$\lambda_- = \frac{\Gamma_1}{\Gamma_2} k_-, \quad \lambda_+ = \frac{\Gamma_1}{\Gamma_2} k_+, \quad (8.10)$$

and the straight lines $b_2 = \lambda_{\pm} b_1$ are the boundary of the physical region in the (b_1, b_2) -plane, which corresponds to collinear configurations.

When $M = 0$, three α -point vortices can form an equilateral triangle. Proposition 8.8 shows that equilateral triangles are relative equilibria. They are characterized in terms of the Hamiltonian as follows.

Lemma 8.9. *Suppose (7.15) and $M = 0$. Then, three α -point vortices form equilateral triangles if and only if $\mathcal{H} = 0$.*

Proof. If $\mathcal{H} = 0$, the constant θ is expressed by

$$\theta = \Gamma_1^{1/\Gamma_1} \Gamma_2^{1/\Gamma_2} (-\Gamma_3)^{1/\Gamma_3}.$$

Then, after a simple calculation with (8.3) and (8.4), the equality (8.6) is rewritten by $f_X(k) = 1$, where

$$f_X(x) = \left[\psi \left(\frac{\Gamma_1}{\Gamma_2} \right) \right]^{-1} \psi \left(\frac{\Gamma_1}{\Gamma_2} x \right) \exp \left(2 \frac{\Gamma_1 + \Gamma_2}{\Gamma_1 \Gamma_2} g_X(x) \right),$$

and

$$g_X(x) = \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} K_0 \left(\sqrt{X} \right) + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} K_0 \left(\sqrt{X} x \right) - K_0 \left(\sqrt{X \frac{x\Gamma_1 + \Gamma_2}{\Gamma_1 + \Gamma_2}} \right),$$

in which $X = \Gamma_1 b_1$. We need to find k such that $f_X(k) = 1$. But, it is easy to confirm that $f_X(1) = 1$ and $k = 1$ are equivalent to the equilateral triangle configuration. To prove the proposition, it is sufficient to show that $k = 1$ is a unique solution to $f_X(k) = 1$. Differentiating $f_X(x)$, we have

$$f'_X(x) = \frac{X}{\Gamma_2} f_X(x) \left[h_K(Xx) - h_K \left(X \frac{x\Gamma_1 + \Gamma_2}{\Gamma_1 + \Gamma_2} \right) \right],$$

and recalling that

$$\frac{x\Gamma_1 + \Gamma_2}{\Gamma_1 + \Gamma_2} > x \quad \text{for } x < 1, \quad x > \frac{x\Gamma_1 + \Gamma_2}{\Gamma_1 + \Gamma_2} \quad \text{for } x > 1,$$

we find that $f'_X(x) > 0$ for $x < 1$, $f'_X(x) < 0$ for $x > 1$ and $f'_X(1) = 0$. This implies that $f_X(k) = 1$ if and only if $k = 1$. On the contrary, for any equilateral triangle, we have $L_{12} = L_{23} = L_{31}$ so that $\mathcal{H} = 0$, owing to (7.15). $\square$

Remark 8.10. Figure 2 shows the physical region and equi-curves of the Hamiltonian for $(\Gamma_1, \Gamma_2) = (1, 1)$ and $(\Gamma_1, \Gamma_2) = (3, 1)$. The three solid lines from the left describe $b_2 = \lambda_+ b_1$, $b_2 = \frac{\Gamma_1}{\Gamma_2} b_1$ and $b_2 = \lambda_- b_1$, respectively. The lines $b_2 = \lambda_{\pm} b_1$ are the boundary of the physical region and, owing to Lemma 8.9, the line $b_2 = \frac{\Gamma_1}{\Gamma_2} b_1$ corresponds to the equi-curve with $\mathcal{H} = 0$. The three lines divide the physical region into two regions and any equi-curve can not intersect the line $b_2 = \frac{\Gamma_1}{\Gamma_2} b_1$.

This observation holds true for any Γ_1 and Γ_2 . That is to say, in terms of γ_1 and γ_2 , configurations of the α -point vortices are divided into three cases, say $\gamma_1 > 1 > \gamma_2$, $\gamma_2 > 1 > \gamma_1$ and $\gamma_1 = \gamma_2 = 1$ (equilateral triangles). Once the relation among the

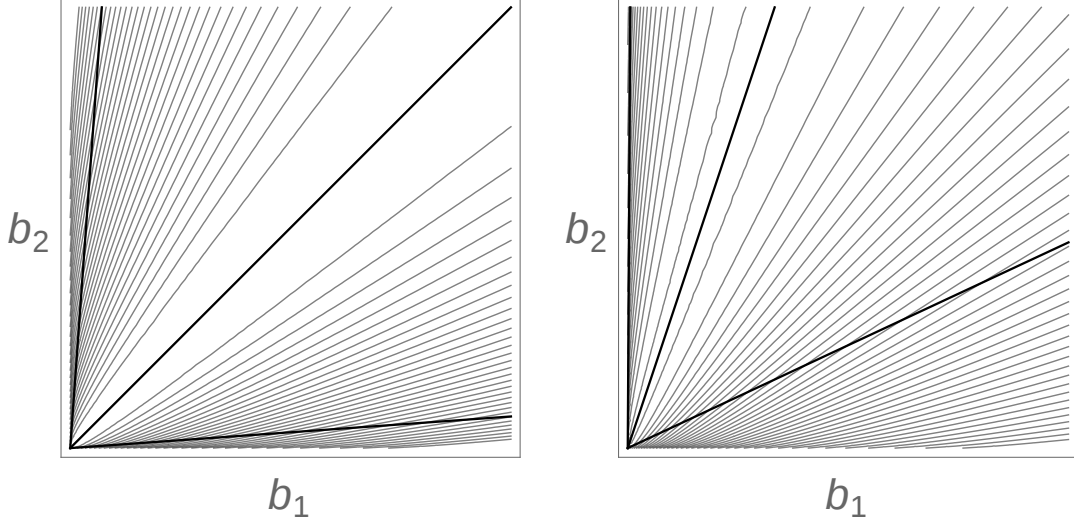


Figure 2: Equi-curves of the Hamiltonian $\mathcal{H}$ and the boundary of the physical region in the (b_1, b_2) -plane for (a) $(\Gamma_1, \Gamma_2) = (1, 1)$ and (b) $(\Gamma_1, \Gamma_2) = (3, 1)$. The light curves are equi-curves. The three dark lines from the left correspond to $b_2 = \lambda_+ b_1$, $b_2 = \frac{\Gamma_1}{\Gamma_2} b_1$ and $b_2 = \lambda_- b_1$, respectively.

ratios is determined by a given initial configuration, it does not change throughout the evolution. Indeed, if the relation changes at some time, the configuration must become an equilateral triangle owing to (8.2). On the other hand, from Lemma 8.9, three α -point vortices form an equilateral triangle if and only if $\mathcal{H} = 0$. This contradicts the invariance of $\mathcal{H}$. Hence, the physical region is divided into two regions. The first region is given by

$$\lambda_- b_1 < b_2 < \frac{\Gamma_1}{\Gamma_2} b_1, \quad (8.11)$$

which corresponds to $\gamma_1 > 1 > \gamma_2$. The second one is

$$\frac{\Gamma_1}{\Gamma_2} b_1 < b_2 < \lambda_+ b_1, \quad (8.12)$$

and we have $\gamma_2 > 1 > \gamma_1$ in this region. In particular, since $b_2 = \lambda_{\pm} b_1$ is the boundary of the physical region, configurations on these lines represent collinear configurations. On the line $b_2 = \frac{\Gamma_1}{\Gamma_2} b_1$, three α -point vortices form equilateral relative equilibria.

Based on this fact, we consider the properties of equi-curves of the Hamiltonian in (8.11) and (8.12) separately. First, we have the range of Hamiltonian as follows.

Lemma 8.11. Suppose (7.15) and $M = 0$. If the initial configuration satisfies $\gamma_1 > 1 > \gamma_2$, then

$$\frac{\Gamma_1^2 \Gamma_2^2}{4\pi(\Gamma_1 + \Gamma_2)} \log \left(\psi \left(\frac{\Gamma_1}{\Gamma_2} k_- \right) \left[\psi \left(\frac{\Gamma_1}{\Gamma_2} \right) \right]^{-1} \right) < \mathcal{H} < 0.$$

For $\gamma_2 > 1 > \gamma_1$, we have the same result with k_- being replaced by k_+ .

Proof. Using (7.15), we rewrite the Hamiltonian as follows.

$$\mathcal{H} = \frac{\Gamma_1 \Gamma_2}{2\pi} \left[\frac{\Gamma_2}{\Gamma_1 + \Gamma_2} h_0(L_{23}^2) + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} h_0(L_{31}^2) - h_0(L_{12}^2) \right]. \quad (8.13)$$

Since $h_0(x)$ is monotone increasing and upward-convex, see (C.7) in Appendix C, it follows from (8.2) and the convexity of $h_0(x)$ that $\mathcal{H} < 0$. On the other hand, owing to the downward-convexity of $K_0(\sqrt{x})$, we find

$$\begin{aligned} \mathcal{H} &= \frac{\Gamma_1 \Gamma_2}{4\pi} \left[\frac{\Gamma_2}{\Gamma_1 + \Gamma_2} \log(\gamma_1 L_{12})^2 + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} \log(\gamma_2 L_{12})^2 - \log L_{12}^2 \right] \\ &\quad + \frac{\Gamma_1 \Gamma_2}{2\pi} \left[\frac{\Gamma_2}{\Gamma_1 + \Gamma_2} K_0(\gamma_1 L_{12}) + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} K_0(\gamma_2 L_{12}) - K_0(L_{12}) \right] \\ &> \frac{\Gamma_1 \Gamma_2}{2\pi} \left(\frac{\Gamma_2}{\Gamma_1 + \Gamma_2} \log \gamma_1^2 + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} \log \gamma_2^2 \right) \\ &= \frac{\Gamma_1^2 \Gamma_2^2}{4\pi(\Gamma_1 + \Gamma_2)} \log \left(\psi \left(\frac{\Gamma_1}{\Gamma_2} k \right) \left[\psi \left(\frac{\Gamma_1}{\Gamma_2} \right) \right]^{-1} \right). \end{aligned}$$

Then, in the same way as Lemma 6.3, we have the conclusion. $\square$

Remark 8.12. Lemma 8.11 shows that $\psi(\lambda_{\pm}) < 1/\theta < \psi(\Gamma_1/\Gamma_2)$ in the regions (8.12) and (8.11), respectively. From Lemma 8.9, we also have $1/\theta = \psi(\Gamma_1/\Gamma_2)$ if and only if $\mathcal{H} = 0$.

Next, we investigate several properties of the equi-curves of $\mathcal{H}$ in the (b_1, b_2) -plane to describe the evolution of the distances. As we easily observe in Figure 2, we have the following properties.

Proposition 8.13. Suppose (7.15) and $M = 0$. Then, the equi-curve in the (b_1, b_2) -plane passes through the origin if and only if $\mathcal{H} = 0$.

Proof. To see how an equi-curve intersects with the boundary of the physical region, we substitute the relation $b_2 = \lambda_{\pm} b_1$ into (8.9). Then, we obtain

$$f_{\lambda_{\pm}}(b_1) \equiv f(b_1, \lambda_{\pm} b_1) = \psi(\lambda_{\pm}) \exp \left(2 \frac{\Gamma_1 + \Gamma_2}{\Gamma_1 \Gamma_2} g_{\lambda_{\pm}}(b_1) \right), \quad (8.14)$$

where $g_{\lambda_{\pm}}(b_1)$ is defined by

$$g_{\lambda_{\pm}}(b_1) = \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} K_0 \left(\sqrt{\Gamma_1 b_1} \right) + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} K_0 \left(\sqrt{\Gamma_2 \lambda_{\pm} b_1} \right) - K_0 \left(\sqrt{\frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} (1 + \lambda_{\pm}) b_1} \right). \quad (8.15)$$

As for $g_{\lambda_{\pm}}(x)$, it follows from $K_0(x) \sim -\beta - \log(x/2)$ as $x \rightarrow 0$, where β is the Euler's constant (see (C.3) in Appendix C), that

$$\begin{aligned} \lim_{b_1 \rightarrow 0} g_{\lambda_{\pm}}(b_1) &= \lim_{b_1 \rightarrow 0} \left[-\frac{\Gamma_2}{\Gamma_1 + \Gamma_2} \left(\beta + \log \frac{1}{2} \sqrt{\Gamma_1 b_1} \right) - \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} \left(\beta + \log \frac{1}{2} \sqrt{\Gamma_2 \lambda_{\pm} b_1} \right) \right. \\ &\quad \left. + \beta + \log \frac{1}{2} \sqrt{\frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} (1 + \lambda_{\pm}) b_1} \right] \\ &= \frac{1}{2} \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} \log \left[\Gamma_1^{-1/\Gamma_1} (\Gamma_2 \lambda_{\pm})^{-1/\Gamma_2} \left(\frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} (1 + \lambda_{\pm}) \right)^{1/\Gamma_1 + 1/\Gamma_2} \right] \\ &= \frac{1}{2} \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} \log \left[\psi \left(\frac{\Gamma_1}{\Gamma_2} \right) [\psi(\lambda_{\pm})]^{-1} \right]. \end{aligned}$$

We assume that the intersection point of the equi-curve and the boundary of physical region is the origin, $b_1 = b_2 = 0$. Then we have

$$f_{\lambda_{\pm}}(0) = \psi(\lambda_{\pm}) \exp \left(2 \frac{\Gamma_1 + \Gamma_2}{\Gamma_1 \Gamma_2} g_{\lambda_{\pm}}(0) \right) = \psi \left(\frac{\Gamma_1}{\Gamma_2} \right).$$

From Remark 8.12, this shows that $f_{\lambda_{\pm}}(0) = 1/\theta$ is equivalent to $\mathcal{H} = 0$. On the contrary, if $\mathcal{H} = 0$ corresponds to an equilateral triangle configuration, then (b_1, b_2) satisfies $b_2 = \frac{\Gamma_1}{\Gamma_2} b_1$ and this equi-curve passes through the origin. $\square$

As we see in Figure 2, every equi-curve with $\mathcal{H} \neq 0$ intersects the boundary of the physical region and approaches a straight line asymptotically. We show that this observation always holds true.

Proposition 8.14. *Suppose (7.15) and $M = 0$. Then, every equi-curve with $\mathcal{H} < 0$ and the boundary of the physical region intersect. As $\mathcal{H}$ decreases, the b_1 -coordinate of the intersection point gets larger. Moreover, for each equi-curve, there exists a constant $\lambda_0 \in (\lambda_-, \lambda_+)$ such that the equi-curve approaches the straight line asymptotically whose slope is λ_0 as $b_1 \rightarrow \infty$.*

Proof. We first show the existence of the equi-curve intersecting with the boundary of the physical region. So, let us see the property of $f_{\lambda_{\pm}}(b_1)$ in (8.14). Let us differentiate $g_{\lambda_{\pm}}(b_1)$ in (8.15), then we have

$$g'_{\lambda_{\pm}}(b_1) = -\frac{1}{2b_1} \left[\frac{\Gamma_2}{\Gamma_1 + \Gamma_2} h_1(\Gamma_1 b_1) + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} h_1(\Gamma_2 \lambda_{\pm} b_1) - h_1 \left(\frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} (1 + \lambda_{\pm}) b_1 \right) \right],$$

where $h_1(x) = \sqrt{x} K_1(\sqrt{x})$ is downward-convex, as we see (C.5) in Appendix C. Thus, we obtain $g'_{\lambda_{\pm}}(x) < 0$ so that $f_{\lambda_{\pm}}(x)$ is monotone decreasing. It follows from $f_{\lambda_{\pm}}(0) = \psi(\Gamma_1/\Gamma_2)$ and $f_{\lambda_{\pm}}(b_1) \rightarrow \psi(\lambda_{\pm})$ as $b_1 \rightarrow +\infty$ that

$$\psi(\lambda_{\pm}) < f_{\lambda_{\pm}}(b_1) < \psi\left(\frac{\Gamma_1}{\Gamma_2}\right),$$

for $b_1 > 0$. Owing to Remark 8.12, for any given $\mathcal{H} < 0$ and θ defined by (8.7), there exists a unique constant $\widehat{b}_1 > 0$ such that $f_{\lambda_{\pm}}(\widehat{b}_1) = 1/\theta$. That is to say, there exists an equi-curve of $\mathcal{H}$ that intersects with the boundary of the physical region at $(\widehat{b}_1, \widehat{b}_2)$, where $\widehat{b}_2 = \lambda_{\pm} \widehat{b}_1$. Since $f_{\lambda_{\pm}}(x)$ is monotone decreasing and $1/\theta$ gets smaller as $\mathcal{H}$ decreases, it follows that as $\mathcal{H}$ decreases, the corresponding solution $\widehat{b}_1$ gets larger.

Next, we show the asymptotic behavior of the equi-curve passing through $(\widehat{b}_1, \widehat{b}_2)$. Differentiating (8.9) with respect to b_1 and b_2 , we obtain

$$\begin{aligned} \frac{\partial}{\partial b_1} f(b_1, b_2) &= f(b_1, b_2) \left[h_K(\Gamma_1 b_1) - h_K \left(\frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} (b_1 + b_2) \right) \right], \\ \frac{\partial}{\partial b_2} f(b_1, b_2) &= f(b_1, b_2) \left[h_K(\Gamma_2 b_2) - h_K \left(\frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} (b_1 + b_2) \right) \right], \end{aligned}$$

respectively. Owing to the monotonicity of $h_K(x)$, it is straightforward to see that $\partial f / \partial b_1 = 0$ and $\partial f / \partial b_2 = 0$ if and only if $\Gamma_1 b_1 = \Gamma_2 b_2 = \Gamma_3 b_3$, which corresponds to $\mathcal{H} = 0$. Thus, in the regions (8.11) and (8.12), we have $\partial f / \partial b_1 \neq 0$ and $\partial f / \partial b_2 \neq 0$. That is to say, for any given $\mathcal{H} < 0$ and θ defined in (8.7), any point (b_1, b_2) with $f(b_1, b_2) = 1/\theta$ is in (8.11) or (8.12). From the implicit function theorem, there exist the neighborhoods U of b_1 and V of b_2 such that there is a unique function $b_2 = \xi(b_1)$

defined for $b_1 \in U$ and $b_2 \in V$ and satisfying $f(b_1, \xi(b_1)) = 1/\theta$. Furthermore, $\xi(b_1)$ is continuously differentiable and we find

$$\frac{d\xi}{db_1} = -\frac{\partial f/\partial b_1}{\partial f/\partial b_2} = \frac{h_K(\Gamma_1 b_1) - h_K\left(\frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2}(b_1 + b_2)\right)}{h_K\left(\frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2}(b_1 + b_2)\right) - h_K(\Gamma_2 b_2)} = \frac{h_K(\gamma_1^2 L_{12}^2) - h_K(L_{12}^2)}{h_K(L_{12}^2) - h_K(\gamma_2^2 L_{12}^2)}.$$

Since we have $\gamma_1 > 1 > \gamma_2$ and $\gamma_2 > 1 > \gamma_1$ in the regions (8.11) and (8.12), respectively, we obtain $d\xi/db_1 > 0$ in both regions. Hence, we conclude that there exists a unique function $\hat{\xi}(b_1)$ that is passing through $(\hat{b}_1, \hat{b}_2)$ and monotone increasing. Owing to the invariance of the Hamiltonian, we obtain

$$\lim_{b_1 \rightarrow +\infty} \frac{\hat{\xi}(b_1)}{b_1} = \lambda_0 > 0.$$

Indeed, if $\hat{\xi}(b_1)/b_1 \rightarrow 0$ or $+\infty$ as $b_1 \rightarrow +\infty$, then we have $f(b_1, \hat{\xi}(b_1)) \rightarrow 0$. This contradicts the fact that $f(b_1, \hat{\xi}(b_1))$ is constant. Hence, we have the claim. By taking the $b_1 \rightarrow +\infty$ limit in (8.9), we obtain $\psi(\lambda_0) = 1/\theta$. Considering Remark 8.12 and (C.1), we find that the constant λ_0 is uniquely determined in $(\Gamma_1/\Gamma_2, \lambda_+)$ and $(\lambda_-, \Gamma_1/\Gamma_2)$ when $(\hat{b}_1, \hat{b}_2)$ satisfies $\hat{b}_2 = \lambda_{\pm} \hat{b}_1$, respectively.

Finally, we show that, for any $\mathcal{H}$, the corresponding equi-curve is uniquely determined and it is $\hat{\xi}(b_1)$. If the point (β_1, β_2) in the physical region corresponding to $\mathcal{H}$ satisfies $\beta_2 \neq \hat{\xi}(\beta_1)$, then, with the same argument as $\hat{\xi}(b_1)$, there exists a unique function $\tilde{\xi}(b_1)$ such that $\beta_2 = \tilde{\xi}(\beta_1)$. Since $\tilde{\xi}(b_1)$ is monotone increasing, it must intersect with the boundary of the physical region. However, we already know that the intersection point is unique for given $\mathcal{H}$, which is $(\hat{b}_1, \hat{b}_2)$. This contradicts that uniqueness of the function $b_2 = \xi(b_1)$ passing through $(\hat{b}_1, \hat{b}_2)$. $\square$

The evolution of the α -point vortices corresponds to a point going along an equi-curve of the Hamiltonian in the physical region. Proposition 8.14 claims that the point on the equi-curve of $\mathcal{H}$ in (8.1) always reaches the boundary of the physical region. That is to say, three α -point vortices become a collinear configuration at a certain time, we set this time as the initial time of the scaled system (7.7). Then, the constant t^* is determined uniquely and we have the following result.

Lemma 8.15. *Suppose $M^\alpha = M = 0$ and $A^\alpha(t^*) = 0$. Then, we have $l_{nm}^\alpha(t^*) \rightarrow 0$ as $\alpha \rightarrow 0$.*

Proof. We first note that $A(0) = 0$ is equivalent to $A^\alpha(t^*) = 0$, which means $l_{23}^\alpha(t^*) = l_{31}^\alpha(t^*) + l_{12}^\alpha(t^*)$ or $l_{31}^\alpha(t^*) = l_{23}^\alpha(t^*) + l_{12}^\alpha(t^*)$. We first consider the case when $l_{23}^\alpha(t^*) =$

$l_{31}^\alpha(t^*) + l_{12}^\alpha(t^*)$. Suppose $l_{nm}^\alpha(t^*) \rightarrow l_{nm}^*$ as $\alpha \rightarrow 0$. Then, we find $l_{23}^* = l_{31}^* + l_{12}^*$ and $\Gamma_2\Gamma_3(l_{23}^*)^2 + \Gamma_3\Gamma_1(l_{31}^*)^2 + \Gamma_1\Gamma_2(l_{12}^*)^2 = 0$. Hence, we have

$$l_{23}^* = \sqrt{\frac{\Gamma_1 + \Gamma_2}{k_- \Gamma_1 + \Gamma_2}} l_{12}^*, \quad l_{31}^* = \sqrt{\frac{k_- (\Gamma_1 + \Gamma_2)}{k_- \Gamma_1 + \Gamma_2}} l_{12}^*.$$

If $l_{12}^* > 0$, then we obtain

$$\lim_{\alpha \rightarrow 0} \mathcal{H}^\alpha = \frac{\Gamma_1^2 \Gamma_2^2}{4\pi(\Gamma_1 + \Gamma_2)} \log \left(\psi \left(\frac{\Gamma_1}{\Gamma_2} k_- \right) \left[\psi \left(\frac{\Gamma_1}{\Gamma_2} \right) \right]^{-1} \right).$$

This contradicts that $\mathcal{H}^\alpha = \mathcal{H}$ satisfies (8.1). Thus, we conclude that $l_{12}^* = 0$ so that $l_{23}^* = l_{31}^* = 0$. As for $l_{31}^\alpha(t^*) = l_{23}^\alpha(t^*) + l_{12}^\alpha(t^*)$, the same argument is valid with k_- being replaced by k_+ . $\square$

After the point in the physical region reaches the boundary of the physical region, according to Proposition 8.8, the point moves back along the same equi-curve. Moreover, it also follows that $L_{nm}(t) \rightarrow \infty$ as $t \rightarrow \pm\infty$. Consequently, since the energy and enstrophy variations are well-defined for $t \in \mathbb{R}$, we have the following theorem, which indicates that the enstrophy dissipates at $t = t^*$ and the energy variation is conserved in the sense of distributions associated with the evolution of the three α -point vortices in the $\alpha \rightarrow 0$ limit.

Theorem 8.16. *Suppose (7.15) and $M = 0$. Then, for each three α -point vortices with $\mathcal{H}$ in (8.1), we have*

$$\lim_{\alpha \rightarrow 0} \mathcal{Z}^\alpha = -z_0 \delta(\cdot - t^*), \quad \lim_{\alpha \rightarrow 0} \mathcal{D}_E^\alpha = 0,$$

in the sense of distributions and z_0 is positive.

Proof. Owing to Lemma 8.15, we can define t^* as the time when the three α -point vortices become a collinear configuration. Then, the convergence is easily shown as Remark 8.3, since $\mathcal{Z}^\alpha$ and $\mathcal{D}_E^\alpha$ are both well-defined for $t \in \mathbb{R}$ and exponentially decay as $t \rightarrow \pm\infty$. Regarding the positivity of z_0 , we consider the function $H_K(\tau)$ which is represented by

$$H_K(\tau) = \frac{\Gamma_1 \Gamma_2}{4\pi} \left[\frac{\Gamma_1}{\Gamma_1 + \Gamma_2} h_1(L_{31}^2) + \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} h_1(L_{23}^2) - h_1(L_{12}^2) \right].$$

Since $h_1(x)$ is downward-convex, we have $H_K(\tau) > 0$ so that $z_0 > 0$. $\square$

8.2.2 $M \neq 0$ case, general results

We first see some common properties hold for $M \neq 0$. It follows from (7.15) and (7.16) that

$$\frac{\Gamma_2}{\Gamma_1 + \Gamma_2} L_{23}^2 + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} L_{31}^2 = L_{12}^2 + \frac{3\widetilde{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2}, \quad (8.16)$$

where $\widetilde{M} = \frac{M}{3\Gamma_1\Gamma_2\Gamma_3}$. We introduce the ratios in the same way as $M = 0$.

$$\frac{L_{31}}{L_{23}} = \sqrt{\rho}, \quad \frac{L_{23}}{L_{12}} = \mu_1, \quad \frac{L_{31}}{L_{12}} = \mu_2.$$

On the other hand, we define the different trilinear coordinates (b_1, b_2, b_3) as

$$b_1 = \frac{L_{23}^2}{\Gamma_1 \widetilde{M}}, \quad b_2 = \frac{L_{31}^2}{\Gamma_2 \widetilde{M}}, \quad b_3 = \frac{L_{12}^2}{\Gamma_3 \widetilde{M}}.$$

Then, (7.16) is rewritten by

$$b_1 + b_2 + b_3 = 3. \quad (8.17)$$

In the same mannar as $M = 0$, we use (8.17) to investigate the evolution of α -point vortices by analyzing equi-curves in the (b_1, b_2) -plane. So, we rewrite the physical region (8.5) in terms of the trilinear coordinates as

$$\left(\frac{\Gamma_1}{\Gamma_2} b_1\right)^2 + \left(\frac{\Gamma_2}{\Gamma_1} b_2\right)^2 - 2\left(2\frac{\Gamma_1}{\Gamma_2} + 2\frac{\Gamma_2}{\Gamma_1} + 3\right)b_1 b_2 + 6\frac{\Gamma_1}{\Gamma_2} b_1 + 6\frac{\Gamma_2}{\Gamma_1} b_2 + 9 \leq 0. \quad (8.18)$$

The boundary of the physical region is a hyperbola whose asymptotes are given by

$$b_2 = \lambda_- b_1 + 3\sqrt{\frac{\Gamma_1\Gamma_2}{\mathcal{R}}}\lambda_-, \quad b_2 = \lambda_+ b_1 - 3\sqrt{\frac{\Gamma_1\Gamma_2}{\mathcal{R}}}\lambda_+, \quad (8.19)$$

where $\lambda_{\pm}$ are given by (8.10). It is important to remark that the physical region exists in the region $b_1, b_2 > 0$ for $M < 0$ and in the region $b_1, b_2 < 0$ for $M > 0$. Especially when $M < 0$, we have $b_3 < 0$, which yields

$$b_1 + b_2 > 3, \quad (8.20)$$

and the line $b_1 + b_2 = 3$ is a tangent to the hyperbola at its vertex point $(\frac{3\Gamma_2}{\Gamma_1 + \Gamma_2}, \frac{3\Gamma_1}{\Gamma_1 + \Gamma_2})$. To see the equi-curves of the Hamiltonian, we also rewrite $\mathcal{H}$ as

$$\mathcal{H} = -\frac{1}{2\pi} \left[\Gamma_2 \Gamma_3 h_0 \left(\widetilde{M} \Gamma_1 b_1 \right) + \Gamma_3 \Gamma_1 h_0 \left(\widetilde{M} \Gamma_2 b_2 \right) + \Gamma_1 \Gamma_2 h_0 \left(\widetilde{M} \Gamma_3 b_3 \right) \right].$$

This is equivalent to

$$\frac{1}{\theta} = b_1^{1/\Gamma_1} b_2^{1/\Gamma_2} (-b_3)^{1/\Gamma_3} \exp \left[\frac{2}{\Gamma_1} K_0 \left(\sqrt{\widetilde{M} \Gamma_1 b_1} \right) + \frac{2}{\Gamma_2} K_0 \left(\sqrt{\widetilde{M} \Gamma_2 b_2} \right) + \frac{2}{\Gamma_3} K_0 \left(\sqrt{\widetilde{M} \Gamma_3 b_3} \right) \right], \quad (8.21)$$

where θ is the constant given in (8.7). Moreover, defining a function $f(b_1, b_2)$ as

$$f(b_1, b_2) = \left(\frac{b_1}{b_1 + b_2 - 3} \right)^{1/\Gamma_1} \left(\frac{b_2}{b_1 + b_2 - 3} \right)^{1/\Gamma_2} \exp \left[\frac{2}{\Gamma_1} K_0 \left(\sqrt{\widetilde{M} \Gamma_1 b_1} \right) + \frac{2}{\Gamma_2} K_0 \left(\sqrt{\widetilde{M} \Gamma_2 b_2} \right) - 2 \frac{\Gamma_1 + \Gamma_2}{\Gamma_1 \Gamma_2} K_0 \left(\sqrt{\frac{\widetilde{M} \Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} (b_1 + b_2 - 3)} \right) \right], \quad (8.22)$$

we find that (8.21) is simply expressed by $f(b_1, b_2) = 1/\theta$.

We now consider the relative equilibria for $M \neq 0$. Owing to (8.16), three α -point vortices can not form an equilateral triangle. Thus, from Proposition 7.3, collinear configurations can be relative equilibria, which are characterized as follows.

Proposition 8.17. *Suppose (7.15) and $M \neq 0$. Then, we have $dA/dt = 0$ if and only if the configuration corresponds to a tangent point of an equi-curve at the boundary of the physical region.*

Proof. We first consider the tangent line of the equi-curve. Differentiating $f(b_1, b_2)$ with respect to b_1 and b_2 , we obtain

$$\begin{aligned} \frac{\partial}{\partial b_1} f(b_1, b_2) &= \widetilde{M} f(b_1, b_2) \left[h_K \left(\widetilde{M} \Gamma_1 b_1 \right) - h_K \left(\frac{\widetilde{M} \Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} (b_1 + b_2 - 3) \right) \right], \\ \frac{\partial}{\partial b_2} f(b_1, b_2) &= \widetilde{M} f(b_1, b_2) \left[h_K \left(\widetilde{M} \Gamma_2 b_2 \right) - h_K \left(\frac{\widetilde{M} \Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} (b_1 + b_2 - 3) \right) \right]. \end{aligned}$$

It is easy to see that $\partial f / \partial b_1 = 0$ and $\partial f / \partial b_2 = 0$ if and only if $b_2 = \frac{\Gamma_1}{\Gamma_2} b_1 + 3$ and $b_2 = \frac{\Gamma_1}{\Gamma_2} (b_1 - 3)$, namely $L_{23} = L_{12}$ and $L_{31} = L_{12}$, respectively. Thus, using the implicit function theorem, we find that for any (b_1, b_2) satisfying $b_2 \neq \frac{\Gamma_1}{\Gamma_2} (b_1 - 3)$, there exist a unique function $b_2 = \xi(b_1)$ defined in neighborhoods of b_1 and b_2 and

satisfying $f(b_1, \xi(b_1)) = 1/\theta$. Furthermore, we have

$$\begin{aligned} \frac{d\xi}{db_1} &= -\frac{\partial f/\partial b_1}{\partial f/\partial b_2} = \frac{h_K\left(\widetilde{M}\Gamma_1 b_1\right) - h_K\left(\frac{\widetilde{M}\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}(b_1+b_2-3)\right)}{h_K\left(\frac{\widetilde{M}\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}(b_1+b_2-3)\right) - h_K\left(\widetilde{M}\Gamma_2 b_2\right)} \\ &= \frac{h_K(\mu_1^2 L_{12}^2) - h_K(L_{12}^2)}{h_K(L_{12}^2) - h_K(\mu_2^2 L_{12}^2)}. \end{aligned} \quad (8.23)$$

Similarly, for any (b_1, b_2) satisfying $b_2 \neq \frac{\Gamma_1}{\Gamma_2}b_1 + 3$, there is a unique function $b_1 = \eta(b_2)$ satisfying $f(\eta(b_2), b_2) = 1/\theta$ in some neighborhoods and

$$\frac{d\eta}{db_2} = \frac{h_K(\mu_2^2 L_{12}^2) - h_K(L_{12}^2)}{h_K(L_{12}^2) - h_K(\mu_1^2 L_{12}^2)}.$$

Next, we see the tangent line of the boundary of the physical region. Let us define $g(b_1, b_2)$ as the left hand side of (8.18), then

$$\begin{aligned} \frac{\partial}{\partial b_1} g(b_1, b_2) &= 2\left(\frac{\Gamma_1}{\Gamma_2}\right)^2 b_1 - 2\left(2\frac{\Gamma_1}{\Gamma_2} + 2\frac{\Gamma_2}{\Gamma_1} + 3\right) b_2 + 6\frac{\Gamma_1}{\Gamma_2} b_1 \\ &= \frac{2}{\Gamma_1^2 \Gamma_2^2 \widetilde{M}} \left(\Gamma_1^3 L_{23}^2 - \Gamma_1(2\Gamma_1^2 + 2\Gamma_2^2 + 3\Gamma_1\Gamma_2)L_{31}^2 + 3\widetilde{M}\Gamma_1^3\Gamma_2 \right). \end{aligned}$$

Moreover, using (8.16), we obtain

$$\frac{\partial g}{\partial b_1} = \frac{2(\Gamma_1 + \Gamma_2)}{\Gamma_1 \Gamma_2^2 \widetilde{M}} (\Gamma_1(L_{23}^2 - L_{12}^2) - (\Gamma_1 + 2\Gamma_2)L_{31}^2). \quad (8.24)$$

Similarly, we find

$$\frac{\partial g}{\partial b_2} = \frac{2(\Gamma_1 + \Gamma_2)}{\Gamma_1^2 \Gamma_2 \widetilde{M}} (\Gamma_2(L_{31}^2 - L_{12}^2) - (2\Gamma_1 + \Gamma_2)L_{23}^2). \quad (8.25)$$

Since points on the boundary represent collinear configurations, they are divided into three cases, $L_{31} + L_{12} = L_{23}$, $L_{23} + L_{12} = L_{31}$ and $L_{23} + L_{31} = L_{12}$. Let us now consider $L_{31} + L_{12} = L_{23}$, namely $\mu_2 + 1 = \mu_1$. Then, (8.24) and (8.25) are rewritten by

$$\frac{\partial g}{\partial b_1} = \frac{4(\Gamma_1 + \Gamma_2)L_{12}^2}{\Gamma_1 \Gamma_2^2 \widetilde{M}} \mu_2 (\Gamma_1 - \mu_2 \Gamma_2), \quad \frac{\partial g}{\partial b_2} = -\frac{4(\Gamma_1 + \Gamma_2)L_{12}^2}{\Gamma_1^2 \Gamma_2 \widetilde{M}} \mu_1 (\Gamma_2 + \mu_1 \Gamma_1).$$

Note that $\partial g/\partial b_2 \neq 0$, then we find from the implicit function theorem that there exists a unique function $\xi_0(b_1)$ such that the boundary of physical region is expressed by $g(b_1, \xi_0(b_1)) = 0$. Moreover, we have

$$\frac{d\xi_0}{db_1} = -\frac{\partial g/\partial b_1}{\partial g/\partial b_2} = \frac{\mu_2\Gamma_1(\Gamma_1 - \mu_2\Gamma_2)}{\mu_1\Gamma_2(\Gamma_2 + \mu_1\Gamma_1)}.$$

We remark that, for $\Gamma_1 L_{12} \neq \Gamma_2 L_{31}$, we similarly find that there exists a unique function $\eta_0(b_2)$ satisfying $g(\eta_0(b_2), b_2) = 0$ and

$$\frac{d\eta_0}{db_2} = \frac{\mu_1\Gamma_2(\Gamma_2 + \mu_1\Gamma_1)}{\mu_2\Gamma_1(\Gamma_1 - \mu_2\Gamma_2)}.$$

On the other hand, when $L_{31} + L_{12} = L_{23}$, it follows from (7.18) and $r = L_{23}$ that

$$\begin{aligned} \frac{dA}{dt} = \frac{L_{12}^2}{4\pi(\Gamma_1 + \Gamma_2)} & \left[\mu_1\Gamma_2(\Gamma_2 + \mu_1\Gamma_1) (h_K(L_{12}^2) - h_K(\mu_1^2 L_{12}^2)) \right. \\ & \left. - \mu_2\Gamma_1(\Gamma_1 - \mu_2\Gamma_2) (h_K(\mu_2^2 L_{12}^2) - h_K(L_{12}^2)) \right]. \end{aligned}$$

Then, considering $d\xi/db_1 = d\xi_0/db_1$ for $L_{31} \neq L_{12}$ and $d\eta/db_2 = d\eta_0/db_2$ for $L_{31} = L_{12}$ and $\Gamma_1 \neq \Gamma_2$ at contact points, we obtain $dA/dt = 0$. As for $L_{31} = L_{12}$ and $\Gamma_1 = \Gamma_2$, we have $\partial f/\partial b_1 \neq 0$, $\partial f/\partial b_2 = 0$, $\partial g/\partial b_1 = 0$ and $\partial g/\partial b_2 \neq 0$. Thus, the equi-curve is not tangent to the boundary of the physical region. Regarding $L_{23} + L_{12} = L_{31}$ and $L_{23} + L_{31} = L_{12}$, we achieve the conclusion in the same way as $L_{31} + L_{12} = L_{23}$. $\square$

8.2.3 $M < 0$ case

We prove the energy conservation and the enstrophy dissipation for $M < 0$, namely $\widetilde{M} > 0$, similarly as in $M = 0$; We first show that collinear configurations can not be relative equilibria in this case. After checking the range of the Hamiltonian, we then investigate the asymptotic behavior of equi-curves of the Hamiltonian.

Proposition 8.18. *Suppose (7.15) and $M < 0$. Then, we have $dA/dt \neq 0$ when three α -point vortices form collinear configurations.*

Proof. By the assumptions, there are two collinear configurations with $\mu_1 + 1 = \mu_2$ and $\mu_2 + 1 = \mu_1$. When $\mu_1 + 1 = \mu_2$, it follows from (7.18) that

$$\frac{dA}{dt} = \frac{\mu_2\Gamma_1 + \mu_1\Gamma_2}{4\pi} [aB_K(\mu_2 L_{12}) + bB_K(\mu_1 L_{12}) - B_K(L_{12})],$$

where a and b are given by

$$a = \frac{\Gamma_1 + \mu_2 \Gamma_2}{\mu_2(\mu_2 \Gamma_1 + \mu_1 \Gamma_2)} \frac{\Gamma_1}{\Gamma_1 + \Gamma_2}, \quad b = \frac{\Gamma_2 - \mu_1 \Gamma_1}{\mu_1(\mu_2 \Gamma_1 + \mu_1 \Gamma_2)} \frac{\Gamma_2}{\Gamma_1 + \Gamma_2}.$$

If $\Gamma_2 > \mu_1 \Gamma_1$, namely $b > 0$, since $B_K(\sqrt{x})$ is upward-convex and $a\mu_2^2 + b\mu_1^2 = 1$, we obtain

$$\begin{aligned} \frac{dA}{dt} &\leq \frac{\mu_2 \Gamma_1 + \mu_1 \Gamma_2}{4\pi} \left[(a+b) B_K \left(\sqrt{L_{12}^2/(a+b)} \right) - B_K(L_{12}) \right] \\ &= \frac{\mu_2 \Gamma_1 + \mu_1 \Gamma_2}{4\pi} L_{12}^2 \left[h_K(L_{12}^2/(a+b)) - h_K(L_{12}^2) \right]. \end{aligned}$$

It is easy to see that $a+b < 1$ for $M < 0$. Since $h_K(x)$ is monotone decreasing, we conclude that $dA/dt < 0$. For the case $\Gamma_2 \leq \mu_1 \Gamma_1$, we use the relation $a\mu_2^2 + b\mu_1^2 = 1$, then

$$\begin{aligned} \frac{dA}{dt} &= \frac{\mu_2 \Gamma_1 + \mu_1 \Gamma_2}{4\pi} [a B_K(\mu_2 L_{12}) - (-b) B_K(\mu_1 L_{12}) - B_K(L_{12})] \\ &= \frac{\mu_2 \Gamma_1 + \mu_1 \Gamma_2}{4\pi} L_{12}^2 [a \mu_2^2 h_K(\mu_2^2 L_{12}^2) - (-b) \mu_1^2 h_K(\mu_1^2 L_{12}^2) - h_K(L_{12}^2)] \\ &= \frac{\mu_2 \Gamma_1 + \mu_1 \Gamma_2}{4\pi} L_{12}^2 [(-b) \mu_1^2 (h_K(\mu_2^2 L_{12}^2) - h_K(\mu_1^2 L_{12}^2)) \\ &\quad + (h_K(\mu_2^2 L_{12}^2) - h_K(L_{12}^2))], \end{aligned}$$

and we thus have $dA/dt < 0$. For $\mu_2 + 1 = \mu_1$, it follows from (7.18) that

$$\frac{dA}{dt} = -\frac{\mu_2 \Gamma_1 + \mu_1 \Gamma_2}{4\pi} \left[\tilde{a} B_K(\mu_2 L_{12}) + \tilde{b} B_K(\mu_1 L_{12}) - B_K(L_{12}) \right],$$

where $\tilde{a}$ and $\tilde{b}$ are given by

$$\tilde{a} = \frac{\Gamma_1 - \mu_2 \Gamma_2}{\mu_2(\mu_2 \Gamma_1 + \mu_1 \Gamma_2)} \frac{\Gamma_1}{\Gamma_1 + \Gamma_2}, \quad \tilde{b} = \frac{\Gamma_2 + \mu_1 \Gamma_1}{\mu_1(\mu_2 \Gamma_1 + \mu_1 \Gamma_2)} \frac{\Gamma_2}{\Gamma_1 + \Gamma_2}.$$

Then, we can show $dA/dt > 0$ in the same way as the first case. $\square$

Before observing the range of the Hamiltonian, we show a significant property about the relation between μ_1 and μ_2 for configurations with $\mathcal{H} \leq 0$.

Lemma 8.19. *Suppose (7.15) and $M < 0$. If $\mathcal{H} \leq 0$, then we have either $\mu_1 > 1 > \mu_2$ or $\mu_2 > 1 > \mu_1$ and these relations can not change throughout the evolution.*

Proof. We note that either μ_1 or μ_2 is greater than 1. Indeed, if $\mu_1, \mu_2 \leq 1$, then it follows from (8.16) that

$$\frac{\Gamma_2}{\Gamma_1 + \Gamma_2}(\mu_1^2 - 1)L_{12}^2 + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2}(\mu_2^2 - 1)L_{12}^2 = \frac{3\widetilde{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2} \leq 0.$$

This contradicts that $\widetilde{M}$ is positive. On the other hand, let us recall that

$$\mathcal{H} = \frac{\Gamma_1\Gamma_2}{2\pi(\Gamma_1 + \Gamma_2)} \left[\Gamma_2 (h_0(\mu_1^2 L_{12}^2) - h_0(L_{12}^2)) + \Gamma_1 (h_0(\mu_2^2 L_{12}^2) - h_0(L_{12}^2)) \right], \quad (8.26)$$

and $h_0(x)$ is monotone increasing. Hence, if $\mu_1, \mu_2 > 1$, then $\mathcal{H} > 0$, which contradicts the assumption $\mathcal{H} \leq 0$.

Moreover, when the relation changes at a certain time, it must be either $\mu_1 > \mu_2 = 1$ or $\mu_2 > \mu_1 = 1$. However, it follows from (8.26) that $\mathcal{H} > 0$ for both cases. This contradicts that $\mathcal{H}$ is invariant in time. $\square$

Next, we see the range of the Hamiltonian $\mathcal{H}$ when $M < 0$. It is different from Lemma 8.11 for $M = 0$, since $\mathcal{H}$ can be positive.

Lemma 8.20. *Suppose (7.15) and $M < 0$. Then, we have*

$$\mathcal{H} < \frac{\Gamma_1\Gamma_2}{2\pi} \left[h_0 \left(\frac{3\widetilde{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2} \right) + \beta - \log 2 \right].$$

Moreover, if the initial configuration satisfies $\mu_1 > 1 > \mu_2$, then

$$\mathcal{H} > \frac{\Gamma_1^2\Gamma_2^2}{4\pi(\Gamma_1 + \Gamma_2)} \log \left(\psi \left(\frac{\Gamma_1}{\Gamma_2} k_- \right) \left[\psi \left(\frac{\Gamma_1}{\Gamma_2} \right) \right]^{-1} \right).$$

For $\mu_2 > 1 > \mu_1$, we have the same result with k_- being replaced by k_+ .

Proof. Owing to (8.16) and the upward-convexity of $h_0(x)$, we find from (8.13) that

$$\begin{aligned} \mathcal{H} &< \frac{\Gamma_1\Gamma_2}{2\pi} \left[h_0 \left(L_{12}^2 + \frac{3\widetilde{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2} \right) - h_0(L_{12}^2) \right] \\ &< \frac{\Gamma_1\Gamma_2}{2\pi} \left[h_0 \left(\frac{3\widetilde{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2} \right) + \beta - \log 2 \right]. \end{aligned}$$

Here, we use

$$0 < h_0 \left(x + \frac{3\widetilde{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2} \right) - h_0(x) < h_0 \left(\frac{3\widetilde{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2} \right) - h_0(0),$$

for $x > 0$, and (C.8) in Appendix C. As for the lower bound of $\mathcal{H}$, since (8.16) with $\widetilde{M} > 0$ yields

$$\frac{\Gamma_2}{\Gamma_1 + \Gamma_2} \mu_1^2 + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} \mu_2^2 > 1,$$

it follows from $\mu_2 = \sqrt{\rho}\mu_1$ that

$$\mu_1 > \sqrt{\frac{\Gamma_1 + \Gamma_2}{\rho\Gamma_1 + \Gamma_2}}, \quad \mu_2 > \sqrt{\frac{\rho(\Gamma_1 + \Gamma_2)}{\rho\Gamma_1 + \Gamma_2}}.$$

Considering the monotonicity of $h_0(x)$, we obtain

$$\mathcal{H} > \frac{\Gamma_1\Gamma_2}{2\pi} \left[\frac{\Gamma_2}{\Gamma_1 + \Gamma_2} h_0 \left(\frac{\Gamma_1 + \Gamma_2}{\rho\Gamma_1 + \Gamma_2} L_{12}^2 \right) + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} h_0 \left(\frac{\rho(\Gamma_1 + \Gamma_2)}{\rho\Gamma_1 + \Gamma_2} L_{12}^2 \right) - h_0(L_{12}^2) \right].$$

In the same way as the proof of Lemma 8.11, we find

$$\mathcal{H} > \frac{\Gamma_1^2\Gamma_2^2}{4\pi(\Gamma_1 + \Gamma_2)} \log \left(\psi \left(\frac{\Gamma_1}{\Gamma_2} \rho \right) \left[\psi \left(\frac{\Gamma_1}{\Gamma_2} \right) \right]^{-1} \right).$$

We note that ρ satisfies $k_- < \rho < \Gamma_1/\Gamma_2$ for $\mu_1 > 1 > \mu_2$ and $\Gamma_1/\Gamma_2 < \rho < k_+$ for $\mu_2 > 1 > \mu_1$. Indeed, if $\rho \leq k_-$ or $\rho \geq k_+$, then the line $b_2 = \rho b_1$ is not in the physical region, since the asymptotes (8.19) are parallel to the lines $b_2 = \lambda_{\pm} b_1$. Hence, the lemma follows immediately from (C.1). $\square$

Remark 8.21. We can show that, for any Hamiltonian value $\mathcal{H}_c$ in the range of (8.1), there exists a configuration corresponding to $\mathcal{H}_c$ with $M < 0$ as follows. We first note that, for any small $\varepsilon > 0$, there exists a constant $\rho_c \in (k_-, k_+)$ such that $H(\rho_c) - \varepsilon < \mathcal{H}_c < H(\rho_c) + \varepsilon$, where

$$H(\rho_c) = \frac{\Gamma_1^2\Gamma_2^2}{4\pi(\Gamma_1 + \Gamma_2)} \log \left(\psi \left(\frac{\Gamma_1}{\Gamma_2} \rho_c \right) \left[\psi \left(\frac{\Gamma_1}{\Gamma_2} \right) \right]^{-1} \right).$$

Next, for any large $L_{23} > 0$, we define $L_{31} = \sqrt{\rho_c}L_{23}$ and L_{12} so that (8.16) holds. Then, regarding the Hamiltonian $\mathcal{H}$ as the function of L_{23} , we obtain $\mathcal{H}(L_{23}) \rightarrow H(\rho_c)$ as $L_{23} \rightarrow +\infty$. Thus, we conclude that there exists a configuration with the Hamiltonian value $\mathcal{H}_c$ by taking a sufficiently large L_{23} and defining $L_{31} = \sqrt{\rho_c}L_{23}$ and L_{12} so that (8.16) holds.

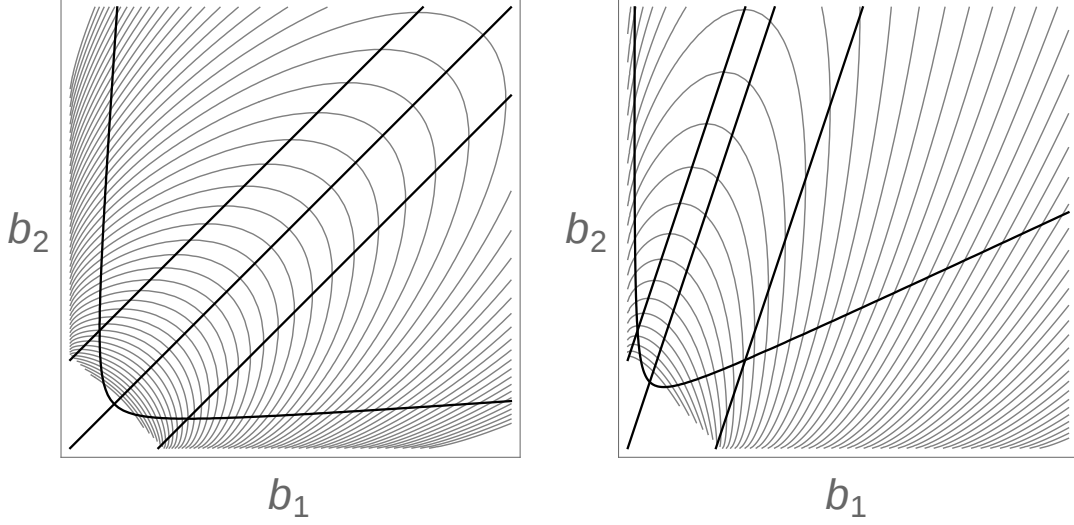


Figure 3: Equi-curves of the Hamiltonian $\mathcal{H}$ and the boundary of the physical region in the (b_1, b_2) -plane when $\widetilde{M} = 0.01$ for (a) $(\Gamma_1, \Gamma_2) = (1, 1)$ and (b) $(\Gamma_1, \Gamma_2) = (3, 1)$. The light curves are equi-curves. The boundary of the physical region is a hyperbola drawn as the dark curve in each figure. The three dark straight lines correspond to $b_2 = \frac{\Gamma_1}{\Gamma_2}b_1 + 3$, $b_2 = \frac{\Gamma_1}{\Gamma_2}(b_1 - 3)$, where the implicit function theorems in Proposition 8.17 don't hold, and $b_2 = \frac{\Gamma_1}{\Gamma_2}b_1$. The three α -point vortices form isosceles triangles on these three lines.

Remark 8.22. From Lemma 8.20, we find that θ defined in (8.7) satisfies $\psi(\lambda_{\pm}) < 1/\theta < 1/\theta_+$, where θ_+ is given by

$$\begin{aligned} \frac{1}{\theta_+} = & \psi\left(\frac{\Gamma_1}{\Gamma_2}\right) \left(\frac{3\widetilde{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2}\right)^{1/\Gamma_1 + 1/\Gamma_2} \\ & \times \exp\left[2\frac{\Gamma_1 + \Gamma_2}{\Gamma_1\Gamma_2} \left((\beta - \log 2) + K_0\left(\sqrt{\frac{3\widetilde{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2}}\right)\right)\right]. \end{aligned}$$

We note that $1/\theta_+ > \psi(\Gamma_1/\Gamma_2)$ and if $\mathcal{H} \leq 0$, then $\psi(\lambda_{\pm}) < 1/\theta \leq \psi(\Gamma_1/\Gamma_2)$.

Figure 3 shows the physical region and equi-curves of the Hamiltonian when $\widetilde{M} = 0.01$ for $(\Gamma_1, \Gamma_2) = (1, 1)$ and $(\Gamma_1, \Gamma_2) = (3, 1)$. We can see that every equi-curve intersects the boundary of the physical region. Although it seems that every equi-curve is bounded in the physical region, as a matter of fact, the equi-curve for

$\mathcal{H} \leq 0$ approaches a straight line asymptotically. That is to say, for any initial configuration with $\mathcal{H} \leq 0$, three α -point vortices form a collinear configuration at some time and diverge to the infinity, which is shown as follows.

Proposition 8.23. *Suppose (7.15), $M < 0$ and $\mathcal{H} \leq 0$. Then, every equi-curve intersects the boundary of the physical region. As $\mathcal{H}$ decreases, the b_1 -coordinate of the intersection point gets larger. Moreover, for each equi-curve, there exists a constant $\lambda_b \in (\lambda_-, \lambda_+)$ such that it asymptotically approaches a straight line whose slope is λ_b as $b_1 \rightarrow \infty$.*

Proof. We first show the existence of the equi-curve intersecting the straight lines $b_2 = \lambda_{\pm} b_1$, which are parallel to the asymptotes of the boundary hyperbola of the physical region, for any $\mathcal{H} < 0$. To see the property of $f(b_1, b_2)$ along these lines, we substitute $b_2 = \lambda_{\pm} b_1$ into (8.22). Then, we find

$$\begin{aligned} f_{\lambda_{\pm}}(b_1) \equiv f(b_1, \lambda_{\pm} b_1) &= \left(\frac{b_1}{(1 + \lambda_{\pm})b_1 - 3} \right)^{1/\Gamma_1} \left(\frac{\lambda_{\pm} b_1}{(1 + \lambda_{\pm})b_1 - 3} \right)^{1/\Gamma_2} \\ &\quad \times \exp \left[\frac{2}{\Gamma_1} K_0 \left(\sqrt{\widetilde{M} \Gamma_1 b_1} \right) + \frac{2}{\Gamma_2} K_0 \left(\sqrt{\widetilde{M} \Gamma_2 \lambda_{\pm} b_1} \right) \right. \\ &\quad \left. - 2 \frac{\Gamma_1 + \Gamma_2}{\Gamma_1 \Gamma_2} K_0 \left(\sqrt{\frac{\widetilde{M} \Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} \{(1 + \lambda_{\pm})b_1 - 3\}} \right) \right], \end{aligned}$$

and $f_{\lambda_{\pm}}(b_1) \rightarrow \psi(\lambda_{\pm})$ as $b_1 \rightarrow +\infty$. Owing to (8.20), we have $b_1 > \frac{3}{1+\lambda_{\pm}}$ and

$$\begin{aligned} f_+ \equiv \lim_{b_1 \rightarrow \frac{3}{1+\lambda_+}} f_{\lambda_{\pm}}(b_1) &= \psi(\lambda_{\pm}) \left(\frac{3\widetilde{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2} \right)^{1/\Gamma_1 + 1/\Gamma_2} \exp \left[2 \frac{\Gamma_1 + \Gamma_2}{\Gamma_1 \Gamma_2} (\beta - \log 2) \right] \\ &\quad \times \exp \left[\frac{2}{\Gamma_1} K_0 \left(\sqrt{\frac{3\widetilde{M}\Gamma_1}{1 + \lambda_{\pm}}} \right) + \frac{2}{\Gamma_2} K_0 \left(\sqrt{\frac{3\widetilde{M}\Gamma_2 \lambda_{\pm}}{1 + \lambda_{\pm}}} \right) \right]. \end{aligned}$$

We note that

$$f_+ > \psi \left(\frac{\Gamma_1}{\Gamma_2} \right), \quad (8.27)$$

since it follows from the positive monotonicity of $h_0(x)$ that

$$\begin{aligned}\log\left(\frac{f_+}{\psi(\Gamma_1/\Gamma_2)}\right) &= \frac{2}{\Gamma_1}h_0\left(\frac{3\widetilde{M}\Gamma_1}{\lambda_{\pm}+1}\right) + \frac{2}{\Gamma_2}h_0\left(\frac{3\widetilde{M}\Gamma_2\lambda_{\pm}}{\lambda_{\pm}+1}\right) + 2\frac{\Gamma_1+\Gamma_2}{\Gamma_1\Gamma_2}(\beta - \log 2) \\ &> \frac{2}{\Gamma_1}h_0(0) + \frac{2}{\Gamma_2}h_0(0) + 2\frac{\Gamma_1+\Gamma_2}{\Gamma_1\Gamma_2}(\beta - \log 2) \\ &= 0.\end{aligned}$$

On the other hand, by differentiating $f_{\lambda_{\pm}}(b_1)$ with respect to b_1 , we obtain

$$\begin{aligned}f'_{\lambda_{\pm}}(b_1) &= \widetilde{M}f_{\lambda_{\pm}}(b_1) \left[h_K\left(\widetilde{M}\Gamma_1 b_1\right) + \lambda_{\pm}h_K\left(\widetilde{M}\Gamma_2\lambda_{\pm}b_1\right) \right. \\ &\quad \left. - (1 + \lambda_{\pm})h_K\left(\widetilde{M}\frac{\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2}\{(1 + \lambda_{\pm})b_1 - 3\}\right) \right] \\ &= \frac{\Gamma_1 + \Gamma_2}{\Gamma_1\Gamma_2} \frac{f_{\lambda_{\pm}}(b_1)}{b_1} \left[\frac{\Gamma_2}{\Gamma_1 + \Gamma_2} B_K\left(\sqrt{\widetilde{M}\Gamma_1 b_1}\right) + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} B_K\left(\sqrt{\widetilde{M}\Gamma_2\lambda_{\pm}b_1}\right) \right. \\ &\quad \left. - B_K\left(\sqrt{\frac{\widetilde{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2}(1 + \lambda_{\pm})b_1}\right) \right] \\ &\quad + \widetilde{M}f_{\lambda_{\pm}}(b_1)(1 + \lambda_{\pm}) \left[h_K\left(\frac{\widetilde{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2}(1 + \lambda_{\pm})b_1\right) \right. \\ &\quad \left. - h_K\left(\frac{\widetilde{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2}\{(1 + \lambda_{\pm})b_1 - 3\}\right) \right].\end{aligned}$$

Owing to the upward convexity of $B_K(\sqrt{x})$ and the negative monotonicity of $h_K(x)$, we find $f'_{\lambda_{\pm}}(b_1) < 0$, namely $f_{\lambda_{\pm}}(b_1)$ is monotone decreasing. Hence, we obtain $\psi(\lambda_{\pm}) < f_{\lambda_{\pm}}(b_1) < f_+$. On the other hand, we also have $\psi(\lambda_{\pm}) < 1/\theta < 1/\theta_+$ in Remark 8.22 and it is easy to confirm that $f_+ < 1/\theta_+$. Let us define the constant H_+ as the value of the Hamiltonian satisfying $1/\theta = f_+$, then (8.27) implies that $H_+ > 0$. Hence, for any given $\mathcal{H} < H_+$ and θ defined by (8.7), there exists a unique constant $\widehat{b}_1$ such that $f_{\lambda_{\pm}}(\widehat{b}_1) = 1/\theta$, namely the equi-curve intersects with $b_2 = \lambda_{\pm}b_1$ at $(\widehat{b}_1, \widehat{b}_2)$, where $\widehat{b}_2 = \lambda_{\pm}\widehat{b}_1$. Since $f_{\lambda_{\pm}}(b_1)$ is monotone decreasing, as $\mathcal{H}$ decreases, $\widehat{b}_1$ gets larger.

Next, we investigate the asymptotic behavior of the equi-curve passing through $(\widehat{b}_1, \widehat{b}_2)$ and we also show that it intersects the boundary of the physical region when $\mathcal{H} \leq 0$. Lemma 8.19 shows that if $\mathcal{H} \leq 0$, then we have either $\mu_1 > 1 > \mu_2$ or

$\mu_2 > 1 > \mu_1$, which are equivalent to the regions

$$b_2 < \frac{\Gamma_1}{\Gamma_2}(b_1 - 3), \quad b_2 > \frac{\Gamma_1}{\Gamma_2}b_1 + 3, \quad (8.28)$$

respectively. That is to say, for any given $\mathcal{H} < 0$ and θ defined by (8.7), the intersection point $(\widehat{b}_1, \widehat{b}_2)$ satisfying $f(\widehat{b}_1, \widehat{b}_2) = 1/\theta$ is in the regions (8.28). Thus, using the implicit function theorem as in the proof of Proposition 8.17, we find that there is a unique function $b_2 = \widehat{\xi}(b_1)$ passing through $(\widehat{b}_1, \widehat{b}_2)$ and satisfying $f(b_1, \widehat{\xi}(b_1)) = 1/\theta$. The derivative of $\widehat{\xi}(b_1)$ is given by (8.23) with ξ being replaced by $\widehat{\xi}$. Note that we have $d\widehat{\xi}/db_1 > 0$ in the regions (8.28). As for the asymptotic behavior of $\widehat{\xi}(b_1)$, owing to the invariance of $f(b_1, b_2)$ in (8.22), we obtain

$$\lim_{b_1 \rightarrow +\infty} \frac{\widehat{\xi}(b_1)}{b_1} = \lambda_b > 0,$$

by the same argument as in the proof of Proposition 8.14. Then, taking the $b_1 \rightarrow +\infty$ limit in (8.21), we obtain $1/\theta = \psi(\lambda_b)$. Thus, when $\mathcal{H} \leq 0$, it follows from Remark 8.22 and (C.1) that λ_b is uniquely determined in $(\Gamma_1/\Gamma_2, \lambda_+)$ and $(\lambda_-, \Gamma_1/\Gamma_2)$ for $(\widehat{b}_1, \widehat{b}_2)$ satisfying $\widehat{b}_2 = \lambda_{\pm}\widehat{b}_1$, respectively. Hence, we conclude that there exists an equi-curve $\xi(b_1)$ with $\mathcal{H} \leq 0$, which intersects the line $b_2 = \lambda_{\pm}b_1$ and approaches the straight line whose slope is λ_b asymptotically. Since the slopes of the asymptotes (8.19) are $\lambda_{\pm}$, $b_2 = \xi(b_1)$ intersects with the boundary of physical region. In addition, as $\mathcal{H}$ decreases, the b_1 -coordinate of the intersection point gets larger.

As for the uniqueness of the equi-curve corresponding to $\mathcal{H}$, we can prove in the same way as Proposition 8.14. $\square$

Remark 8.24. As we see in Figure 3, the physical region is divided into three regions by the two lines $b_2 = \frac{\Gamma_1}{\Gamma_2}b_1 + 3$ and $b_2 = \frac{\Gamma_1}{\Gamma_2}(b_1 - 3)$. That is to say, we can observe that equi-curves between these two line are monotone decreasing and those in other regions are monotone increasing. These facts are shown for any Γ_1 and Γ_2 as follows. Considering (8.23) in Proposition 8.17, we have $d\xi/db_1 > 0$ only if $\mu_1 > 1 > \mu_2$ or $\mu_2 > 1 > \mu_1$, which are equivalent to two regions in (8.28), respectively. On the other hand, we find $d\xi/db_1 < 0$ if and only if $\mu_1, \mu_2 > 1$, which is equivalent to the region,

$$\frac{\Gamma_1}{\Gamma_2}(b_1 - 3) < b_2 < \frac{\Gamma_1}{\Gamma_2}b_1 + 3.$$

and we note that every equi-curve in this region corresponds to $\mathcal{H} > 0$.

We finally show the enstrophy dissipation and the energy conservation in the $\alpha \rightarrow 0$ limit. From the above arguments, three α -point vortices with the Hamiltonian value $\mathcal{H}_c$ in the range of (8.1) become a collinear configuration at certain time, we set this time as the initial time of (7.7) similarly to $M = 0$. Then, t^* is determined uniquely and the following lemma holds.

Lemma 8.25. *Suppose $M^\alpha \rightarrow 0$ as $\alpha \rightarrow 0$ and $A^\alpha(t^*) = 0$. Then, we have $l_{nm}^\alpha(t^*) \rightarrow 0$ as $\alpha \rightarrow 0$.*

Proof. The result follows in the same way as Lemma 8.15. $\square$

Moreover, we have $L_{nm}(t) \rightarrow +\infty$ as $t \rightarrow \pm\infty$, which guarantees that the energy and enstrophy variations are well-defined for $t \in \mathbb{R}$. The following lemma needs to be shown before we finish the proof of the enstrophy dissipation.

Lemma 8.26. *Let $f_\mu(x)$, $x > 0$ be defined by*

$$f_\mu(x) = - \left(h_K(\mu_2^2 x) - h_K(x) \right) \left(K_0(\sqrt{x}) - K_0\left(\sqrt{\mu_1^2 x}\right) \right) \\ + \left(h_K(x) - h_K(\mu_1^2 x) \right) \left(K_0\left(\sqrt{\mu_2^2 x}\right) - K_0(\sqrt{x}) \right).$$

Then, we have $f_\mu(x) > 0$ if $\mu_1 > 1 > \mu_2$, and $f_\mu(x) < 0$ if $\mu_2 > 1 > \mu_1$.

Proof. Suppose $\mu_1 > 1 > \mu_2$. Since $h_K(x)$ and $K_0(\sqrt{x})$ are monotone decreasing, $f_\mu(x) > 0$ is equivalent to

$$f_0(x) \equiv \frac{K_0(\sqrt{x}) - K_0\left(\sqrt{\mu_1^2 x}\right)}{K_0\left(\sqrt{\mu_2^2 x}\right) - K_0(\sqrt{x})} < f_K(x) \equiv \frac{B_K(\sqrt{x}) - \frac{1}{\mu_1^2} B_K\left(\sqrt{\mu_1^2 x}\right)}{\frac{1}{\mu_2^2} B_K\left(\sqrt{\mu_2^2 x}\right) - B_K(\sqrt{x})}.$$

In addition, it follows from (C.3) in Appendix C that $f_\mu(0) = 0$ and

$$f_0(0) = f_K(0) = -\frac{\log \mu_1}{\log \mu_2}. \quad (8.29)$$

We first show that $f_0(x)$ is monotone decreasing. Differentiating it, we find

$$f_0'(x) = \frac{-1}{2x \left(K_0\left(\sqrt{\mu_2^2 x}\right) - K_0(\sqrt{x}) \right)^2} \\ \times \left[\left(B_K\left(\sqrt{\mu_1^2 x}\right) - B_K(\sqrt{x}) \right) \left(K_0\left(\sqrt{\mu_2^2 x}\right) - K_0(\sqrt{x}) \right) \right. \\ \left. - \left(K_0(\sqrt{x}) - K_0\left(\sqrt{\mu_1^2 x}\right) \right) \left(B_K(\sqrt{x}) - B_K\left(\sqrt{\mu_2^2 x}\right) \right) \right].$$

Since $f'_0(x) < 0$ is equivalent to

$$\frac{B_K(\sqrt{\mu_1^2 x}) - B_K(\sqrt{x})}{K_0(\sqrt{x}) - K_0(\sqrt{\mu_1^2 x})} > \frac{B_K(\sqrt{x}) - B_K(\sqrt{\mu_2^2 x})}{K_0(\sqrt{\mu_2^2 x}) - K_0(\sqrt{x})},$$

we show this relation. Owing to the mean value theorem, there exist two points c_1 and c_2 with $\mu_2^2 x < c_2 < x < c_1 < \mu_1^2 x$ such that

$$\begin{aligned} \frac{B_K(\sqrt{\mu_1^2 x}) - B_K(\sqrt{x})}{K_0(\sqrt{x}) - K_0(\sqrt{\mu_1^2 x})} &= \frac{\sqrt{c_1} K_0(\sqrt{c_1})}{K_1(\sqrt{c_1})} \equiv k(c_1), \\ \frac{B_K(\sqrt{x}) - B_K(\sqrt{\mu_2^2 x})}{K_0(\sqrt{\mu_2^2 x}) - K_0(\sqrt{x})} &= \frac{\sqrt{c_2} K_0(\sqrt{c_2})}{K_1(\sqrt{c_2})} \equiv k(c_2). \end{aligned}$$

Thus, it is sufficient to show that $k(x)$ is monotone increasing. We have

$$\begin{aligned} k'(x) &= \frac{1}{2\sqrt{x} (K_1(\sqrt{x}))^2} \left[2K_0(\sqrt{x})K_1(\sqrt{x}) - \sqrt{x} (K_1(\sqrt{x}))^2 + \sqrt{x} (K_0(\sqrt{x}))^2 \right] \\ &\equiv \frac{1}{2\sqrt{x} (K_1(\sqrt{x}))^2} k_1(x). \end{aligned}$$

Since $k_1(x) \rightarrow 0$ as $x \rightarrow +\infty$ and

$$k'_1(x) = -\frac{1}{x} K_0(\sqrt{x})K_1(\sqrt{x}) - \frac{1}{2\sqrt{x}} (K_0(\sqrt{x}))^2 - \frac{1}{2\sqrt{x}} (K_1(\sqrt{x}))^2 < 0,$$

we find that $k_1(x)$ is positive, so is $k'(x)$. Hence, $k(x)$ is monotone increasing so that $f_0(x)$ is monotone decreasing.

Next, we see the function $f_K(x)$. Differentiating it, we have

$$\begin{aligned} f'_K(x) &= \frac{1}{2x (h_K(\mu_2^2 x) - h_K(x))^2} \left[\left(K_0(\sqrt{x}) - K_0(\sqrt{\mu_1^2 x}) \right) (h_K(\mu_2^2 x) - h_K(x)) \right. \\ &\quad \left. - (h_K(x) - h_K(\mu_1^2 x)) \left(K_0(\sqrt{\mu_2^2 x}) - K_0(\sqrt{x}) \right) \right] \\ &= \frac{-1}{2x (h_K(\mu_2^2 x) - h_K(x))^2} f_\mu(x). \end{aligned}$$

Since we have $f_\mu(0) = 0$, for small constant $\varepsilon > 0$, we have either $f_\mu(x) > 0$ or $f_\mu(x) \leq 0$ for $x \in (0, \varepsilon)$. If $f_\mu(x) \leq 0$, then we have $f_0(x) \geq f_K(x)$ and $f'_K(x) \geq 0$. Owing to (8.29), this contradicts the negative monotonicity of $f_0(x)$. Let us assume that there exists a constant $x_0 > 0$ such that $f_\mu(x) > 0$ for $x \in (0, x_0)$ and $f_\mu(x_0) = 0$, namely $f_0(x) < f_K(x)$ for $x \in (0, x_0)$, $f_0(x_0) = f_K(x_0)$ and $f'_K(x_0) = 0$. Then it follows that

$$\frac{f_0(x_0) - f_0(x)}{x_0 - x} > \frac{f_K(x_0) - f_K(x)}{x_0 - x},$$

for $x \in (0, x_0)$. This yields $f'_0(x_0) > f'_K(x_0) = 0$ and contradicts $f'_0(x) < 0$. Hence, we conclude that $f_\mu(x) > 0$ for $x > 0$. As for $\mu_2 > 1 > \mu_1$, we can show $f_\mu(x) < 0$ for $x > 0$ by the same argument as above. $\square$

Theorem 8.27. *Suppose (7.15) and $M < 0$. Then, for each three α -point vortices with $\mathcal{H}$ in (8.1), we have*

$$\lim_{\alpha \rightarrow 0} \mathcal{Z}^\alpha(t) = -z_0 \delta(t - t^*) , \quad \lim_{\alpha \rightarrow 0} \mathcal{D}_E^\alpha(t) = 0,$$

in the sense of distributions and z_0 is positive.

Proof. Considering Lemma 8.25, we can define t^* as the time when the three α -point vortices become a collinear configuration similarly to $M = 0$. Then, the convergence follows immediately as Theorem 8.16. To show the positivity of z_0 , we investigate the property of $F(\tau)$ in (7.14). It follows from (7.9) that

$$\begin{aligned} F(\tau) &= \frac{\Gamma_1 \Gamma_2}{8\pi} \left[\frac{dL_{12}^2}{d\tau} K_0(L_{12}) - \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} \frac{dL_{23}^2}{d\tau} K_0(L_{23}) - \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} \frac{dL_{31}^2}{d\tau} K_0(L_{31}) \right] \\ &= \frac{\Gamma_1 \Gamma_2 \Gamma_3}{4\pi^2} A(\tau) f_\mu(L_{12}^2(\tau)). \end{aligned}$$

Owing to Lemma 8.19, we have either $\mu_1 > 1 > \mu_2$ or $\mu_2 > 1 > \mu_1$ for any initial configuration with $\mathcal{H} \leq 0$. For $\mu_1 > 1 > \mu_2$, it follows from the proof of Proposition 8.18 that $dA/dt > 0$. Thus, setting t^* as the time when the configuration become collinear, we find that $A(\tau) < 0$ for $\tau < t^*$ and $A(\tau) > 0$ for $\tau > t^*$. On the other hand, Lemma 8.26 yields $f_\mu(L_{12}^2) > 0$ for $t \in \mathbb{R}$. Then, we have $F(\tau) > 0$ for $\tau < t^*$ and $F(\tau) < 0$ for $\tau > t^*$. Considering $H_K(\tau) \rightarrow 0$ as $t \rightarrow \pm\infty$, we conclude that $H_K(\tau)$ is positive, so that z_0 is positive. Regarding $\mu_2 > 1 > \mu_1$, we can prove with the same argument as above case. $\square$

8.2.4 $M > 0$ case

We show the enstrophy dissipation and the energy conservation for $M > 0$, namely $\widetilde{M} < 0$. In this case, there is a possibility that some collinear configurations can become relative equilibria. In addition, the equi-curve does not necessarily intersect the boundary of the physical region. That is to say, the three α -point vortices do not become a collinear configuration throughout the evolution. These situations are different from those for $M \leq 0$. In this section, after we see the range of the Hamiltonian, we investigate the property of the equi-curves.

When $M > 0$, the physical region (8.18) is in the region of $b_1, b_2 < 0$. We replace the trilinear coordinates (b_1, b_2, b_3) with a new one (c_1, c_2, c_3) as follows.

$$c_1 = -b_1 = \frac{L_{23}^2}{\Gamma_1 \widehat{M}}, \quad c_2 = -b_2 = \frac{L_{31}^2}{\Gamma_2 \widehat{M}}, \quad c_3 = -b_3 = \frac{L_{12}^2}{\Gamma_3 \widehat{M}},$$

where $\widehat{M} = -\widetilde{M} > 0$. Then, (8.17) is replaced by $c_1 + c_2 + c_3 = -3$. The physical region in the region of $c_1, c_2 > 0$ is rewritten by

$$\left(\frac{\Gamma_1}{\Gamma_2}c_1\right)^2 + \left(\frac{\Gamma_2}{\Gamma_1}c_2\right)^2 - 2\left(2\frac{\Gamma_1}{\Gamma_2} + 2\frac{\Gamma_2}{\Gamma_1} + 3\right)c_1c_2 - 6\frac{\Gamma_1}{\Gamma_2}c_1 - 6\frac{\Gamma_2}{\Gamma_1}c_2 + 9 \leq 0.$$

The boundary of this region is a hyperbola whose asymptotes are given by

$$c_2 = \lambda_- c_1 - 3\sqrt{\frac{\Gamma_1\Gamma_2}{\mathcal{R}}}\lambda_-, \quad c_2 = \lambda_+ c_1 + 3\sqrt{\frac{\Gamma_1\Gamma_2}{\mathcal{R}}}\lambda_+. \quad (8.30)$$

Note that the c_1 -axis and the c_2 -axis are tangent to the boundary at $(3\Gamma_2/\Gamma_1, 0)$ and $(0, 3\Gamma_1/\Gamma_2)$, respectively. In the new coordinates, the Hamiltonian is expressed as follows.

$$\mathcal{H} = -\frac{1}{2\pi} \left[\Gamma_2\Gamma_3 h_0 \left(\widehat{M}\Gamma_1 c_1 \right) + \Gamma_3\Gamma_1 h_0 \left(\widehat{M}\Gamma_2 c_2 \right) + \Gamma_1\Gamma_2 h_0 \left(\widehat{M}\Gamma_3 c_3 \right) \right].$$

Thus, similarly to $M \leq 0$, we obtain $f(c_1, c_2) = 1/\theta$, where $f(c_1, c_2)$ is defined by

$$f(c_1, c_2) = \left(\frac{c_1}{c_1 + c_2 + 3} \right)^{1/\Gamma_1} \left(\frac{c_2}{c_1 + c_2 + 3} \right)^{1/\Gamma_2} \exp \left[\frac{2}{\Gamma_1} K_0 \left(\sqrt{\widehat{M}\Gamma_1 c_1} \right) \right. \\ \left. + \frac{2}{\Gamma_2} K_0 \left(\sqrt{\widehat{M}\Gamma_2 c_2} \right) - 2\frac{\Gamma_1 + \Gamma_2}{\Gamma_1\Gamma_2} K_0 \left(\sqrt{\frac{\widehat{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2}} (c_1 + c_2 + 3) \right) \right].$$

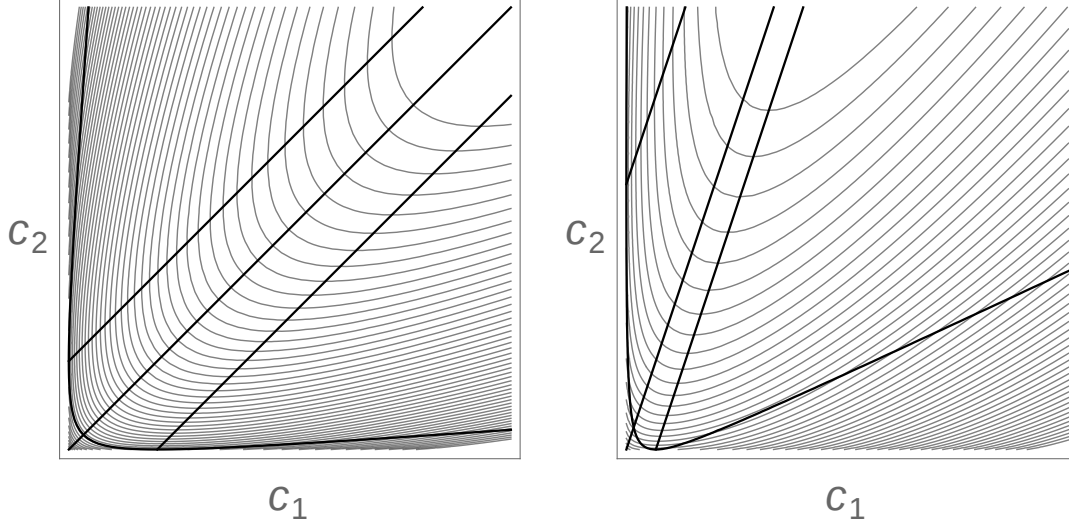


Figure 4: Equi-curves of the Hamiltonian $\mathcal{H}$ and the boundary of the physical region in the (c_1, c_2) -plane when $\widehat{M} = 0.01$ for (a) $(\Gamma_1, \Gamma_2) = (1, 1)$ and (b) $(\Gamma_1, \Gamma_2) = (3, 1)$. The light curves are equi-curves. The dark curve is the boundary of the physical region. The three dark straight lines are $c_2 = \frac{\Gamma_1}{\Gamma_2}(c_1 + 3)$, $c_2 = \frac{\Gamma_1}{\Gamma_2}c_1 - 3$, where the implicit function theorems in Proposition 8.17 don't hold, and $c_2 = \frac{\Gamma_1}{\Gamma_2}c_1$ passing the origin. The three α -point vortices form isosceles triangles on these three lines.

We note that the Hamiltonian $\mathcal{H}$ is always negative when $M > 0$, since it follows from the proof of Lemma 8.20 that

$$\mathcal{H} < \frac{\Gamma_1 \Gamma_2}{2\pi} \left[h_0 \left(L_{12}^2 + \frac{3\widehat{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2} \right) - h_0(L_{12}^2) \right] < 0.$$

This yields

$$0 < \frac{1}{\theta} < \psi \left(\frac{\Gamma_1}{\Gamma_2} \right). \quad (8.31)$$

We remark that there might exist the configuration whose Hamiltonian does not be in the range (8.1). Furthermore, similarly as in Remark 8.21, we can show that for any value of the Hamiltonian $\mathcal{H}_c$ in the range of (8.1), there exists a configuration moving along the equi-curve corresponding to $\mathcal{H}_c$.

Figure 4 shows the physical region and the equi-curves of the Hamiltonian when $\widehat{M} = 0.01$ for (a) $(\Gamma_1, \Gamma_2) = (1, 1)$ and (b) $(\Gamma_1, \Gamma_2) = (3, 1)$, respectively. Then, we

observe that the equi-curves are roughly categorized into the following three parts. (i) Curves that do not intersect with the boundary of the physical region. (ii) Curves that are tangent to the boundary. (iii) Curves that intersect the boundary of the physical region. Moreover, as we show later, every equi-curve is monotone decreasing between the two lines $c_2 = \frac{\Gamma_1}{\Gamma_2}(c_1 + 3)$ and $c_2 = \frac{\Gamma_1}{\Gamma_2}c_1 - 3$, and monotone increasing in the other regions. In addition, each equi-curve corresponds to the different value of the Hamiltonian. To see the distribution of the equi-curves in detail, we show the following lemma.

Lemma 8.28. *For any $p \in \mathbb{R}$, every equi-curve has at most one intersection point with the line $c_2 = \frac{\Gamma_1}{\Gamma_2}c_1 + p$. Moreover, as the value of $\mathcal{H}$ increases, the c_1 -coordinate of the intersection points gets larger.*

Proof. Substituting $c_2 = \frac{\Gamma_1}{\Gamma_2}c_1 + p$ into $f(c_1, c_2)$ and setting $f_p(c_1) = f\left(c_1, \frac{\Gamma_1}{\Gamma_2}c_1 + p\right)$, we obtain

$$\begin{aligned} f_p(c_1) = & \left(\frac{c_1}{\left(1 + \frac{\Gamma_1}{\Gamma_2}\right)c_1 + p + 3} \right)^{1/\Gamma_1} \left(\frac{\frac{\Gamma_1}{\Gamma_2}c_1 + p}{\left(1 + \frac{\Gamma_1}{\Gamma_2}\right)c_1 + p + 3} \right)^{1/\Gamma_2} \\ & \times \exp \left[\frac{2}{\Gamma_1} K_0 \left(\sqrt{\widehat{M} \Gamma_1 c_1} \right) + \frac{2}{\Gamma_2} K_0 \left(\sqrt{\widehat{M} (\Gamma_1 c_1 + \Gamma_2 p)} \right) \right. \\ & \left. - 2 \frac{\Gamma_1 + \Gamma_2}{\Gamma_1 \Gamma_2} K_0 \left(\sqrt{\widehat{M} \left(\Gamma_1 c_1 + \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} (p + 3) \right)} \right) \right], \end{aligned}$$

and $f_p(c_1) \rightarrow \psi(\Gamma_1/\Gamma_2)$ as $c_1 \rightarrow +\infty$. The derivative of $f_p(c_1)$ is given by

$$\begin{aligned} f'_p(c_1) = & \frac{\Gamma_1 + \Gamma_2}{\Gamma_2} \widehat{M} f_p(c_1) \left[\frac{\Gamma_2}{\Gamma_1 + \Gamma_2} h_K \left(\widehat{M} \Gamma_1 c_1 \right) + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} h_K \left(\widehat{M} (\Gamma_1 c_1 + \Gamma_2 p) \right) \right. \\ & \left. - h_K \left(\widehat{M} \left(\Gamma_1 c_1 + \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} (p + 3) \right) \right) \right]. \end{aligned}$$

Owing to the convexity and monotonicity of $h_K(x)$, it follows that

$$\begin{aligned} f'_p(c_1) > & \frac{\Gamma_1 + \Gamma_2}{\Gamma_2} \widehat{M} f_p(c_1) \left[h_K \left(\widehat{M} \left(\Gamma_1 c_1 + \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} p \right) \right) \right. \\ & \left. - h_K \left(\widehat{M} \left(\Gamma_1 c_1 + \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} (p + 3) \right) \right) \right] > 0. \end{aligned}$$

Hence, $f_p(c_1)$ is monotone increasing and we get $f_p(c_1) < \psi(\Gamma_1/\Gamma_2)$. Considering (8.31), we find that, for any given $\mathcal{H} < 0$ and $p \in \mathbb{R}$, there exists at most one solution

of $f_p(c_1) = 1/\theta$, where θ is defined by (8.7). This means that every equi-curve with the Hamiltonian value $\mathcal{H}$ has at most one intersection point with $c_2 = \frac{\Gamma_1}{\Gamma_2}c_1 + p$. Since $f_p(c_1)$ is monotone increasing, as the value of $\mathcal{H}$ increases, namely $1/\theta$ increases, the c_1 -coordinate of the intersection points gets larger. $\square$

Figure 4 illustrates that each equi-curve tends to two different asymptotic lines as $c_1 \rightarrow +\infty$, which is proven as follows.

Proposition 8.29. *Suppose (7.15) and $M > 0$. For every equi-curve in the region $c_2 < \frac{\Gamma_1}{\Gamma_2}c_1 - 3$, there exists a constant λ_{c_-} such that it asymptotically approaches the straight line whose slope is $\lambda_{c_-} < \Gamma_1/\Gamma_2$ as $c_1 \rightarrow +\infty$. For the equi-curve in the region $c_2 > \frac{\Gamma_1}{\Gamma_2}(c_1 + 3)$, we have the same result with $\lambda_{c_+} > \Gamma_1/\Gamma_2$.*

Proof. We first show the monotonicity of the equi-curves. Differentiating $f(c_1, c_2)$ with respect to c_1 and c_2 , we obtain

$$\begin{aligned}\frac{\partial}{\partial c_1} f(c_1, c_2) &= \widehat{M}f(c_1, c_2) \left[h_K \left(\widehat{M}\Gamma_1 c_1 \right) - h_K \left(\frac{\widehat{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2} (c_1 + c_2 + 3) \right) \right], \\ \frac{\partial}{\partial c_2} f(c_1, c_2) &= \widehat{M}f(c_1, c_2) \left[h_K \left(\widehat{M}\Gamma_2 c_2 \right) - h_K \left(\frac{\widehat{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2} (c_1 + c_2 + 3) \right) \right].\end{aligned}$$

Thus, $\partial f/\partial c_1 = 0$ and $\partial f/\partial c_2 = 0$ if and only if $c_2 = \frac{\Gamma_1}{\Gamma_2}c_1 - 3$ and $c_2 = \frac{\Gamma_1}{\Gamma_2}(c_1 + 3)$, namely $L_{23} = L_{12}$ and $L_{31} = L_{12}$, respectively. For any given $\mathcal{H}$ and θ defined by (8.7), we consider a point (c_1, c_2) satisfying $f(c_1, c_2) = 1/\theta$ and $c_2 \neq \frac{\Gamma_1}{\Gamma_2}(c_1 + 3)$. Then, it follows from the implicit function theorem that there exists a unique function $c_2 = \xi(c_1)$ defined in neighborhoods of c_1 and c_2 and satisfying $f(c_1, \xi(c_1)) = 1/\theta$. Regarding the points on the line $c_2 = \frac{\Gamma_1}{\Gamma_2}(c_1 + 3)$, we similarly find a unique function $c_1 = \eta(c_2)$ satisfying $f(\eta(c_2), c_2) = 1/\theta$ in neighborhoods. Note that the derivatives of ξ and η are given by

$$\frac{d\xi}{dc_1} = \frac{h_K(\mu_1^2 L_{12}^2) - h_K(L_{12}^2)}{h_K(L_{12}^2) - h_K(\mu_2^2 L_{12}^2)}, \quad \frac{d\eta}{dc_2} = \frac{h_K(\mu_2^2 L_{12}^2) - h_K(L_{12}^2)}{h_K(L_{12}^2) - h_K(\mu_1^2 L_{12}^2)},$$

and that we have $\eta = \xi^{-1}$ if $c_2 \neq \frac{\Gamma_1}{\Gamma_2}c_1 - 3$ and $c_2 \neq \frac{\Gamma_1}{\Gamma_2}(c_1 + 3)$, owing to the uniqueness of η and ξ . Then, defining $f(c_1, \xi(c_1)) = f(\eta(c_2), c_2)$ on the line $c_2 = \frac{\Gamma_1}{\Gamma_2}(c_1 + 3)$, we can make $\xi(c_1)$ continuous in the region $c_1, c_2 > 0$. Similar to Remark 8.24, we find that $d\xi/dc_1 > 0$ holds only if $\mu_1 > 1 > \mu_2$ or $\mu_2 > 1 > \mu_1$, which correspond to the regions

$$c_2 < \frac{\Gamma_1}{\Gamma_2}c_1 - 3, \quad c_2 > \frac{\Gamma_1}{\Gamma_2}(c_1 + 3), \quad (8.32)$$

respectively. On the other hand, $d\xi/dc_1 < 0$ holds if and only if $\mu_1, \mu_2 < 1$, which corresponds to

$$\frac{\Gamma_1}{\Gamma_2}c_1 - 3 < c_2 < \frac{\Gamma_1}{\Gamma_2}(c_1 + 3). \quad (8.33)$$

Note that this is the situation we observe in Figure 4.

We next see the asymptotic behavior of the equi-curves. Let us consider the equi-curve $\widehat{\xi}(c_1)$ passing through a point $(\widehat{c}_1, \widehat{c}_2)$ in the regions (8.32). As we see in the proof of Propositions 8.14 and 8.23, the invariance of $f(c_1, c_2)$ yields

$$\lim_{c_1 \rightarrow +\infty} \frac{\widehat{\xi}(c_1)}{c_1} = \lambda_c.$$

Thus, taking the $c_1 \rightarrow +\infty$ limit in $f(c_1, \widehat{\xi}(c_1)) = 1/\theta$, we obtain

$$\frac{1}{\theta} = \psi(\lambda_c). \quad (8.34)$$

Since we know from Lemma 8.28 that $d\xi/dc_1 < \Gamma_1/\Gamma_2$ for $c_2 < \frac{\Gamma_1}{\Gamma_2}c_1 - 3$ and $d\xi/dc_1 > \Gamma_1/\Gamma_2$ for $c_2 > \frac{\Gamma_1}{\Gamma_2}(c_1 + 3)$, owing to (8.31) and (C.1), (8.34) has a unique solution $\lambda_{c_-} < \Gamma_1/\Gamma_2$ and $\lambda_{c_+} > \Gamma_1/\Gamma_2$, respectively.

We finally show that, for any Hamiltonian value $\mathcal{H}$, the corresponding equi-curve is unique in the regions (8.32) and (8.33). We consider the equi-curve $\widehat{\xi}(c_1)$ with the Hamiltonian value $\mathcal{H}$. Suppose that there is a point (β_1, β_2) corresponding to $\mathcal{H}$ and satisfies $\beta_1 \neq \widehat{\xi}(\beta_1)$. Then, there exists a equi-curve $\widetilde{\xi}(c_1)$ passing through (β_2, β_1) . Since it follows that $\widehat{\xi}(c_1)/c_1, \widetilde{\xi}(c_1)/c_1 \rightarrow \lambda_{c_{\pm}}$ as $c_1 \rightarrow +\infty$ in the region (8.32), there exists a constant $p \in \mathbb{R}$ such that $c_2 = \frac{\Gamma_1}{\Gamma_2}c_1 + p$ intersects with both $\widehat{\xi}(c_1)$ and $\widetilde{\xi}(c_1)$. In the region (8.33), $\widehat{\xi}(c_1)$ and $\widetilde{\xi}(c_1)$ must intersect the line $c_1 = \frac{\Gamma_1}{\Gamma_2}c_1$. These contradict Lemma 8.28. We complete the proof. $\square$

Remark 8.30. In Proposition 8.29, we see the property of the equi-curves in the region of $c_1, c_2 > 0$ and do not mention the relation between equi-curves and the physical region. We already know that the divergence of $L_{nm}(t)$ is essential to show the enstrophy dissipation. This indicates that the equi-curve $\widehat{\xi}(c_1)$ needs to stay in the physical region as $c_1 \rightarrow +\infty$. That is to say, $\lambda_{c_{\pm}}$ should satisfy $\lambda_{c_-} \in (\lambda_-, \Gamma_1/\Gamma_2)$ and $\lambda_{c_+} \in (\Gamma_1/\Gamma_2, \lambda_+)$, since the asymptotes of hyperbola are given by (8.30). Hence, it is essential for the enstrophy dissipation that $\mathcal{H}$ stays in the range (8.1), since $\lambda_{c_{\pm}}$ are solutions of (8.34).

Remark 8.31. Based on Proposition 8.29, we consider the equi-curves of (i), (ii) and (iii) in the physical region and describe their asymptotic behavior in detail as follows.

- (i) Owing to Proposition 8.29, the equi-curve $\widehat{\xi}(c_1)$ is monotone increasing in (8.32) and decreasing in (8.33). Moreover, it asymptotically approaches two lines whose slopes are $\lambda_{c_{\pm}}$. Such a behavior is not observed when $M \leq 0$.
- (ii) There exist at most two points of tangency, one satisfies $\mu_1 > \mu_2$ and the other does $\mu_2 > \mu_1$. Since, owing to Proposition 8.17, the point of tangency corresponds to a relative collinear equilibrium, the configuration approaches the collinear configuration as $t \rightarrow -\infty$ (or $t \rightarrow +\infty$) while $L_{nm}(t) \rightarrow +\infty$ as $t \rightarrow +\infty$ (or $t \rightarrow -\infty$). We define the values of $\mathcal{H}$ corresponding to such equi-curves as $\mathcal{H}_1$ and $\mathcal{H}_2$.
- (iii) Since the intersection point corresponds to a collinear configuration that is not a equilibrium, we obtain $\widehat{\xi}(c_1)/c_1 \rightarrow \lambda_{c_{\pm}}$ as $t \rightarrow \pm\infty$ for $\mu_1 > 1 > \mu_2$ and $\mu_2 > 1 > \mu_1$, respectively. This behavior is same as that we observe when $M \leq 0$.

We finally show the enstrophy dissipation and the energy conservation for $M > 0$. According to Remark 8.31, the divergence of $L_{nm}(t)$ as $t \rightarrow \pm\infty$ holds for the evolution corresponding to the equi-curves (i) and (iii). For (iii) case, similarly to $M \leq 0$, we define t^* as the time at which three α -point vortices become a collinear configuration. As for (i), the equi-curves must intersect with the line $c_2 = \frac{\Gamma_1}{\Gamma_2}c_1$. That is to say, three α -point vortices form an isosceles triangle with $L_{23} = L_{31}$, we define this time as the initial time of the system (7.7). Then, we have the following Lemma.

Lemma 8.32. *Suppose $M^\alpha \rightarrow 0$ as $\alpha \rightarrow 0$ and $l_{23}^\alpha(t^*) = l_{31}^\alpha(t^*)$. Then, we have $l_{nm}^\alpha(t^*) \rightarrow 0$ as $\alpha \rightarrow 0$.*

Proof. Note that $L_{23}(0) = L_{31}(0)$ is equivalent to $l_{23}^\alpha(t^*) = l_{31}^\alpha(t^*)$. Suppose $l_{nm}^\alpha(t^*) \rightarrow l_{nm}^*$ as $\alpha \rightarrow 0$. Then, we obtain $l_{23}^* = l_{31}^*$ and $\Gamma_2\Gamma_3(l_{23}^*)^2 + \Gamma_3\Gamma_1(l_{31}^*)^2 + \Gamma_1\Gamma_2(l_{12}^*)^2 = 0$ so that $l_{23}^* = l_{31}^* = l_{12}^*$. If $l_{12}^* > 0$, then we have $\mathcal{H}^\alpha \rightarrow 0$ as $\alpha \rightarrow 0$. This contradicts that $\mathcal{H}^\alpha = \mathcal{H}$ satisfies (8.1). Thus, we achieve the conclusion. $\square$

We remark that Lemma 8.32 holds when the condition $l_{23}^\alpha(t^*) = l_{31}^\alpha(t^*)$ is replaced by $l_{23}^\alpha(t^*), l_{31}^\alpha(t^*) \leq l_{12}^\alpha(t^*)$. Finally, we show the enstrophy dissipation for $M > 0$.

Theorem 8.33. *Suppose (7.15) and $M > 0$. Then, for each three α -point vortices with $\mathcal{H}$ satisfying (8.1) and $\mathcal{H} \neq \mathcal{H}_1, \mathcal{H}_2$, we have*

$$\lim_{\alpha \rightarrow 0} \mathcal{Z}^\alpha(t) = -z_0\delta(t - t^*), \quad \lim_{\alpha \rightarrow 0} \mathcal{D}_E^\alpha(t) = 0,$$

in the sense of distributions and z_0 is positive.

Proof. Owing to Lemma 8.32, we can define t^* as the time when the three α -point vortices form an isosceles triangle. Then, similar to Theorem 8.16, we can easily show the convergence. As for the positivity of z_0 , since $h_1(x)$ is downward-convex, it follows from (8.16) and

$$H_K(\tau) > \frac{\Gamma_1 \Gamma_2}{4\pi} \left[h_1 \left(L_{12}^2 + \frac{3\widetilde{M}\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2} \right) - h_1(L_{12}^2) \right] > 0,$$

that $z_0 > 0$. □

8.3 Convergence to self-similar collapsing solutions

As we have discussed, the $\alpha \rightarrow 0$ limit solution of the three α PV system is related to the self-similar collapsing and expanding solutions in the PV system. In this section, we show that the limit converges to a singular solution connecting these singular solutions and the enstrophy dissipation occurs at the critical time.

We consider the three α -point vortices with the value of Hamiltonian $\mathcal{H}_c$ satisfying the assumptions in Theorem 8.1. Then, we have $L_{nm}(t) \rightarrow +\infty$ as $t \rightarrow \pm\infty$. More precisely, we already know that, for $M \leq 0$ and some special cases in $M > 0$, there exists a constant $\nu \in (k_-, 1)$ or $(1, k_+)$ such that

$$\lim_{t \rightarrow \pm\infty} \frac{L_{31}(t)}{L_{23}(t)} = \nu,$$

and, for the other cases in $M > 0$, there exist constants $\nu_- \in (k_-, 1)$ and $\nu_+ \in (1, k_+)$ such that

$$\lim_{t \rightarrow \pm\infty} \frac{L_{31}(t)}{L_{23}(t)} = \nu_{\pm}. \quad (8.35)$$

In what follows, we consider (8.35) with ν_+ without loss of generality. The same argument applies to the case of ν_- . Recalling the invariant quantity M^α and (7.12);

$$l_{nm}^\alpha(t) = \alpha L_{nm} \left(\frac{t - t^*}{\alpha^2} \right),$$

we obtain

$$M = \frac{M^\alpha}{\alpha^2} = \Gamma_2 \Gamma_3 L_{23}^2 \left(\frac{t - t^*}{\alpha^2} \right) + \Gamma_3 \Gamma_1 L_{31}^2 \left(\frac{t - t^*}{\alpha^2} \right) + \Gamma_1 \Gamma_2 L_{12}^2 \left(\frac{t - t^*}{\alpha^2} \right).$$

Here, we set

$$\lim_{t \rightarrow +\infty} \frac{L_{23}(t)}{L_{12}(t)} = \zeta_1, \quad \lim_{t \rightarrow +\infty} \frac{L_{31}(t)}{L_{12}(t)} = \zeta_2, \quad (8.36)$$

Then, it follows from (7.15) and (8.36) that

$$1 = \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} \zeta_1^2 + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} \zeta_2^2, \quad (8.37)$$

for $t > t^*$ in the $\alpha \rightarrow 0$ limit. Thus, using (8.35), we obtain

$$\zeta_1 = \sqrt{\frac{\Gamma_1 + \Gamma_2}{\nu_+ \Gamma_1 + \Gamma_2}}, \quad \zeta_2 = \sqrt{\frac{\nu_+ (\Gamma_1 + \Gamma_2)}{\nu_+ \Gamma_1 + \Gamma_2}}.$$

Regarding the Hamiltonian $\mathcal{H}$, we similarly obtain

$$\mathcal{H} = \frac{\Gamma_1 \Gamma_2}{2\pi} \left(\frac{\Gamma_2}{\Gamma_1 + \Gamma_2} \log \zeta_1 + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} \log \zeta_2 \right),$$

in the $\alpha \rightarrow 0$ limit. In the same argument as the PV system in §2, we find that ν_+ , ζ_1 and ζ_2 are independent of t . On the other hand, owing to (7.5) and (7.12), l_{12}^α satisfies

$$(l_{12}^\alpha(t))^2 = (l_{12}^\alpha(t^*))^2 + \int_{t^*}^t \frac{2}{\pi} \Gamma_3 A^\alpha \left[\frac{1}{\alpha^2 L_{23}^2 \left(\frac{s-t^*}{\alpha^2} \right)} B_K \left(L_{23} \left(\frac{s-t^*}{\alpha^2} \right) \right) - \frac{1}{\alpha^2 L_{31}^2 \left(\frac{s-t^*}{\alpha^2} \right)} B_K \left(L_{31} \left(\frac{s-t^*}{\alpha^2} \right) \right) \right] ds.$$

Taking the $\alpha \rightarrow 0$ limit and setting $l_{nm}^0(t) = \lim_{\alpha \rightarrow 0} l_{nm}^\alpha(t)$, we obtain

$$(l_{12}^0(t))^2 = \frac{1}{2\pi} \sigma_0 R_0 \Gamma_3 \frac{\zeta_2^2 - \zeta_1^2}{\zeta_1^2 \zeta_2^2} (t - t^*).$$

where $R_0 = [2(\zeta_1^2 + \zeta_2^2) - (\zeta_1^2 - \zeta_2^2)^2 - 1]^{1/2}$ and σ_0 denotes the sign of A^α . Here, we use $l_{nm}^\alpha(t^*) \rightarrow 0$ as $\alpha \rightarrow 0$. Regarding l_{23}^α and l_{31}^α , we similarly find

$$(l_{23}^0(t))^2 = \frac{1}{2\pi} \sigma_0 R_0 \Gamma_1 \frac{1 - \zeta_2^2}{\zeta_2^2} (t - t^*), \quad (l_{31}^0(t))^2 = \frac{1}{2\pi} \sigma_0 R_0 \Gamma_2 \frac{\zeta_1^2 - 1}{\zeta_1^2} (t - t^*).$$

Using (7.15) and (8.37), we consequently obtain

$$l_{12}^0(t) = \sqrt{\sigma_0 \omega_0 (t - t^*)}, \quad l_{23}^0(t) = \zeta_1 \sqrt{\sigma_0 \omega_0 (t - t^*)}, \quad l_{31}^0(t) = \zeta_2 \sqrt{\sigma_0 \omega_0 (t - t^*)},$$

where ω_0 is defined by

$$\omega_0 = \frac{1}{2\pi} R_0 \Gamma_2 \frac{\zeta_1^2 - 1}{\zeta_1^2 \zeta_2^2}.$$

Considering the argument of the PV system in §2, we can conclude that the three α -point vortices converge to a self-similar singular solution with the same Hamiltonian value as $\mathcal{H}_c$ in the PV system, which collides as $t \rightarrow t^* + 0$ and expands as $t \rightarrow +\infty$ self-similarly. As for ν_- , we similarly obtain the self-similar singular solution in the PV system defined for $t < t^*$, which also collides as $t \rightarrow t^* - 0$ and expands as $t \rightarrow -\infty$ self-similarly.

9 Concluding Remarks

We have successfully proven that the existence of the enstrophy dissipating solutions and that such a dissipation occurs because of the triple collapse of point vortices. In order to obtain these results, we first showed that the Euler-Poincaré equations have a unique global weak solutions with initial vorticity in the space of Radon measure. Then, we considered point vortices as a measure valued initial vorticity and introduced the Euler-Poincaré PV system. In this research, we focused on the α PV system that is a specific example of the Euler-Poincaré PV system, and gave a mathematical proof of the existence of the evolution of the three α -point vortices that dissipates the enstrophy in the sense of distribution in the $\alpha \rightarrow 0$ limit. In this limit, it converges to a singular orbit that connects the collapsing and the expanding solutions at the collapse time in the three PV system, and the enstrophy dissipation occurs at the collapsing time. Moreover, our theorems indicates that the enstrophy dissipation is observed for a wide range of the initial configurations of the three α -point vortices. Consequently, we conclude that the self-similar triple collapse is one of the generic mechanisms of the enstrophy dissipation in 2D incompressible and inviscid flows.

We discuss some future problems suggested by the present study. In this paper, we restrict our attention to the three α -point vortices whose mutual distances tend to infinity as $t \rightarrow \pm\infty$ and that converge to the singular solution in the PV system. On the other hand, when the vortex strengths do not satisfy (7.15) or the Hamiltonian value is out of (8.1), their distances could be finite throughout the evolution. Then, it is worth investigating the convergence of these evolutions in the $\alpha \rightarrow 0$ limit as a regular dynamical system, although the enstrophy is conserved in this case. As in [29], considering the evolution of the two α -point vortices, we easily confirm that their distance remains finite and it converges to that of the two PV system. Hence, it is likely that the three α -point vortices also converge to the solution of the three PV system in the $\alpha \rightarrow 0$ limit, but it is not trivial.

Regarding the enstrophy dissipation caused by the vortex collapse, it is important to investigate whether the α PV system is the essential regularization to show the

enstrophy dissipation, or not. It is challenging to try to obtain the same results for the Euler-Poincaré PV system. Furthermore, it is also interesting to investigate the enstrophy variation together with the evolution of many point vortices. According to [38, 41, 42], the N point vortices in the PV system can collapse self-similarly in finite time under certain circumstances. Thus, there is a possibility of obtaining another enstrophy dissipating solution by considering the collapse of the N vortex problem in a Euler-Poincaré PV system. As a matter of fact, it is numerically observed in [29] that the enstrophy dissipation occurs in the α PV system via a quadruple self-similar collapse as $\alpha \rightarrow 0$. However, since the Euler-Poincaré PV system (5.5) is not integrable for $N \geq 4$ in general, further analytic investigation is required.

A Quasi-Lipschitz estimates for the singular kernel

We review the quasi-Lipschitz estimates for the singular kernel $\mathbf{K}$. The first lemma is a classical estimate for bounded functions. The proof is straightforward and we find it, for example, in [4, 49, 52].

Lemma A.1. *For $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$,*

$$\int_{\mathbb{R}^2} |\mathbf{K}(\mathbf{x} - \mathbf{y}) - \mathbf{K}(\mathbf{x}' - \mathbf{y})| |\omega(\mathbf{y})| d\mathbf{y} \leq c\varphi(|\mathbf{x} - \mathbf{x}'|) (\|\omega\|_{L^1} + \|\omega\|_{L^\infty}).$$

Proof. We set $r = |\mathbf{x} - \mathbf{x}'|$ and note prove the lemma for sufficiently small $r > 0$.

$$\begin{aligned} & \int_{\mathbb{R}^2} |\mathbf{K}(\mathbf{x} - \mathbf{y}) - \mathbf{K}(\mathbf{x}' - \mathbf{y})| |\omega(\mathbf{y})| d\mathbf{y} \\ & \leq \int_{|\mathbf{x} - \mathbf{y}| < 2r} (|\mathbf{K}(\mathbf{x} - \mathbf{y})| + |\mathbf{K}(\mathbf{x}' - \mathbf{y})|) |\omega(\mathbf{y})| d\mathbf{y} \\ & \quad + \int_{2r \leq |\mathbf{x} - \mathbf{y}| < 2} |\mathbf{K}(\mathbf{x} - \mathbf{y}) - \mathbf{K}(\mathbf{x}' - \mathbf{y})| |\omega(\mathbf{y})| d\mathbf{y} \\ & \quad + \int_{|\mathbf{x} - \mathbf{y}| \geq 2} |\mathbf{K}(\mathbf{x} - \mathbf{y}) - \mathbf{K}(\mathbf{x}' - \mathbf{y})| |\omega(\mathbf{y})| d\mathbf{y} \\ & \equiv L_1 + L_2 + L_3. \end{aligned} \tag{A.1}$$

Since $|\mathbf{x} - \mathbf{y}| < 2r$ yields $|\mathbf{x}' - \mathbf{y}| < 3r$, we have

$$L_1 \leq \frac{1}{\pi} \|\omega\|_{L^\infty} \sup_{\mathbf{x} \in \mathbb{R}^2} \int_{|\mathbf{x} - \mathbf{y}| < 3r} \frac{1}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} \leq c_1 r \|\omega\|_{L^\infty}.$$

As for L_2 and L_3 , we use the mean value theorem. Then,

$$|\mathbf{K}(\mathbf{x} - \mathbf{y}) - \mathbf{K}(\mathbf{x}' - \mathbf{y})| \leq r \int_0^1 |(\nabla \mathbf{K})(\mathbf{x} - \mathbf{y} + \tau(\mathbf{x}' - \mathbf{x}))| d\tau.$$

If $|\mathbf{x} - \mathbf{y}| \geq 2r$, $|\mathbf{x} - \mathbf{y} + \tau(\mathbf{x}' - \mathbf{x})| \geq |\mathbf{x} - \mathbf{y}|/2$. Thus, we find

$$\begin{aligned} L_2 &\leq c_2 r \|\omega\|_{L^\infty} \int_{2r \leq |\mathbf{x} - \mathbf{y}| < 2} \int_0^1 \frac{1}{|\mathbf{x} - \mathbf{y} + \tau(\mathbf{x}' - \mathbf{x})|^2} d\tau d\mathbf{y} \\ &\leq c_2 r \|\omega\|_{L^\infty} \int_{2r \leq |\mathbf{x} - \mathbf{y}| < 2} \frac{1}{|\mathbf{x} - \mathbf{y}|^2} d\mathbf{y} \\ &\leq c_2 \varphi(r) \|\omega\|_{L^\infty}. \end{aligned}$$

Finally, it follows that

$$L_3 \leq c_3 r \int_{2 \leq |\mathbf{x} - \mathbf{y}|} \frac{1}{|\mathbf{x} - \mathbf{y}|^2} |\omega(\mathbf{y})| d\mathbf{y} \leq c_3 r \|\omega\|_{L^1}.$$

Combining three estimates, we obtain the desired result. $\square$

The following lemma is a modified estimate of the singular kernel, which was proven in [4, 56].

Lemma A.2. *Let $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and ϕ be an area preserving measurable transformation on $\mathbb{R}^2$. Then*

$$\int_{\mathbb{R}^2} |\mathbf{K}(\mathbf{x} - \mathbf{y}) - \mathbf{K}(\mathbf{x} - \phi(\mathbf{y}))| |\omega(\mathbf{y})| d\mathbf{y} \leq c \sup_{\mathbf{x} \in \mathbb{R}^2} \varphi(|\mathbf{x} - \phi(\mathbf{x})|) (\|\omega\|_{L^1} + \|\omega\|_{L^\infty}).$$

Proof. Set $r = \sup_{\mathbf{x} \in \mathbb{R}^2} |\mathbf{x} - \phi(\mathbf{x})|$ and suppose that $r > 0$ is sufficiently small. we split the integral in the same way as (A.1). The first term L_1 is estimated as follows.

$$\begin{aligned} L_1 &\leq \frac{1}{2\pi} \|\omega\|_{L^\infty} \int_{|\mathbf{x} - \mathbf{y}| < 2r} \left(\frac{1}{|\mathbf{x} - \mathbf{y}|} + \frac{1}{|\mathbf{x} - \phi(\mathbf{y})|} \right) d\mathbf{y} \\ &\leq \frac{1}{\pi} \|\omega\|_{L^\infty} \int_{|\mathbf{x} - \mathbf{y}| < 2r} \frac{1}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} \\ &\leq c_1 r \|\omega\|_{L^\infty}. \end{aligned}$$

Here, we have used the inequality,

$$\int_{|\mathbf{x}-\mathbf{y}|<2r} \frac{1}{|\mathbf{x}-\phi(\mathbf{y})|} d\mathbf{y} \leq \int_{|\mathbf{x}-\mathbf{y}|<2r} \frac{1}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y},$$

since ϕ is area preserving and the symmetric rearrangement about $\mathbf{x}$ maximizes the integral. Applying the mean value theorem and the fact that

$$\begin{aligned} |\mathbf{x}-\mathbf{y}+\tau(\mathbf{y}-\phi(\mathbf{y}))| &\geq \frac{1}{2}|\mathbf{x}-\mathbf{y}|+r-\tau|\mathbf{y}-\phi(\mathbf{y})| \\ &\geq \frac{1}{2}|\mathbf{x}-\mathbf{y}|+r(1-\tau) \\ &\geq \frac{1}{2}|\mathbf{x}-\mathbf{y}|, \end{aligned}$$

for $|\mathbf{x}-\mathbf{y}| \geq 2r$. Thus, based on the proof of Lemma A.1, we find

$$\begin{aligned} L_2 &\leq c_2 r \|\omega\|_{L^\infty} \int_{2r \leq |\mathbf{x}-\mathbf{y}| < 2} \frac{1}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{y} \\ &\leq c_2 \varphi(r) \|\omega\|_{L^\infty}. \end{aligned}$$

It is also easy to see that $L_3 \leq c_3 r \|\omega\|_{L^1}$. We achieve the conclusion. □

B Fundamental lemmas related to ordinary differential problems

We prove the lemma about the global existence and uniqueness of the solutions to the ordinary differential equation. The proof is based on that in [49].

Lemma B.1. *Consider the initial value problem for $\mathbf{x}(t) \in \mathbb{R}^n$, $t \in [0, \infty)$.*

$$\frac{d}{dt} \mathbf{x}(t) = \mathbf{b}(\mathbf{x}(t), t), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{B.1}$$

where $\mathbf{b} \in C(\mathbb{R}^n \times [0, \infty); \mathbb{R}^n)$ is uniformly bounded and quasi-Lipschitz continuous in space, namely

$$|\mathbf{b}(\mathbf{x}, t) - \mathbf{b}(\mathbf{x}', t)| \leq c\varphi(|\mathbf{x} - \mathbf{x}'|),$$

and c is independent of t . Then, (B.1) has a global unique solution $\mathbf{x}(\cdot) \in C^1[0, \infty)$.

Proof. We use the classical iterative method. We define a sequence of approximate solutions for $0 \leq t \leq T$,

$$\mathbf{x}_n(t) = \int_0^t \mathbf{b}(\mathbf{x}_{n-1}(s), s) ds + \mathbf{x}_0, \quad \mathbf{x}_n(0) = \mathbf{x}_0. \quad (\text{B.2})$$

Then, we have

$$\begin{aligned} |\mathbf{x}_n(t) - \mathbf{x}_{n-1}(t)| &\leq \int_0^t |\mathbf{b}(\mathbf{x}_{n-1}(s), s) - \mathbf{b}(\mathbf{x}_{n-2}(s), s)| ds \\ &\leq c \int_0^t \varphi(|\mathbf{x}_{n-1}(s) - \mathbf{x}_{n-2}(s)|) ds \\ &\leq c \int_0^t L(\delta) |\mathbf{x}_{n-1}(s) - \mathbf{x}_{n-2}(s)| ds + cT\delta, \end{aligned} \quad (\text{B.3})$$

in which we have used the inequality,

$$\varphi(r) \leq L(\delta)r + \delta, \quad L(\delta) = -\log \delta,$$

for $0 < \delta < 1$. By iterating the inequality (B.3), we obtain

$$|\mathbf{x}_n(t) - \mathbf{x}_{n-1}(t)| \leq \frac{(cTL(\delta))^{n-1}}{(n-1)!} \sup_{0 \leq t \leq T} |\mathbf{x}_1(t) - \mathbf{x}_0| + cT\delta \sum_{k=0}^{n-2} \frac{(cTL(\delta))^k}{k!}.$$

Since $\mathbf{b}$ is uniformly bounded, we have

$$\sup_{0 \leq t \leq T} |\mathbf{x}_1(t) - \mathbf{x}_0| \leq T \|\mathbf{b}\|_{L^\infty(\mathbb{R}^n \times [0, \infty))}.$$

Thus, it follows that

$$|\mathbf{x}_n(t) - \mathbf{x}_{n-1}(t)| \leq \frac{L(\delta)^{n-1}}{(n-1)!} c^{n-1} T^n \|\mathbf{b}\|_{L^\infty} + cT\delta e^{cTL(\delta)}.$$

Choosing $\delta = e^{-n}$, we find

$$|\mathbf{x}_n(t) - \mathbf{x}_{n-1}(t)| \leq \frac{n^{n-1}}{(n-1)!} c^{n-1} T^n \|\mathbf{b}\|_{L^\infty} + cT e^{cTn^{-n}}.$$

Suppose that T is sufficiently small such that $1 - cT > 1/2$ and $2cTe^c < 1$, and remember the Stirling formula $n^n \leq n!e^{cn}$. Then,

$$\begin{aligned} |\mathbf{x}_n(t) - \mathbf{x}_{n-1}(t)| &\leq (cTe^c)^{n-1} e^{cT} \|\mathbf{b}\|_{L^\infty} + cT e^{-n/2} \\ &\leq \left(\frac{1}{2}\right)^{n-1} e^{cT} \|\mathbf{b}\|_{L^\infty} + cT e^{-n/2}. \end{aligned}$$

Hence, $\{\mathbf{x}_n(t)\}_{n=1}^\infty$ for $t \in [0, T]$ is a uniformly convergent Cauchy sequence. We denote the limit function by $\mathbf{x}(t)$ and find that it satisfies (B.1) in integral form/ Indeed, we have (B.2) and

$$\begin{aligned} \left| \int_0^t \mathbf{b}(\mathbf{x}_n(s), s) ds - \int_0^t \mathbf{b}(\mathbf{x}(s), s) ds \right| &\leq c \int_0^T \varphi(|\mathbf{x}_n(s) - \mathbf{x}(s)|) ds \\ &\leq cT\varphi\left(\sup_{0 \leq t \leq T} |\mathbf{x}_n(s) - \mathbf{x}(s)|\right). \end{aligned}$$

Taking the $n \rightarrow \infty$ limit, we find that $\mathbf{x}(t)$ satisfies (B.2). Finally, it follows from the continuity of $\mathbf{b}(\mathbf{x}(t), t)$ in time that $\mathbf{x}(t) \in C^1[0, T]$. Since T does not depend on $\mathbf{x}_0$, the result can be extended to arbitrary large times.

Regarding the uniqueness, suppose that $\mathbf{x}(t)$ and $\mathbf{y}$ are solutions to (B.1). Using the same method in the existence part, we find

$$\begin{aligned} |\mathbf{x}(t) - \mathbf{y}(t)| &\leq \int_0^t |\mathbf{b}(\mathbf{x}(s), s) - \mathbf{b}(\mathbf{y}(s), s)| ds \\ &\leq c \int_0^t \varphi(|\mathbf{x}(s) - \mathbf{y}(s)|) ds \\ &\leq \sup_{0 \leq t \leq T} |\mathbf{x}(t) - \mathbf{y}(t)| \left(\frac{1}{2}\right)^{n-1} e^c T + cT e^{-n/2}. \end{aligned}$$

For sufficiently large n , we obtain

$$\sup_{0 \leq t \leq T} |\mathbf{x}(t) - \mathbf{y}(t)| \leq 2cT e^{-n/2},$$

and conclude that $\mathbf{x} = \mathbf{y}$ on $[0, T]$ in the $n \rightarrow \infty$ limit. □

Next, we show the useful integral inequality. This proof is also based on [49].

Lemma B.2. *Let $x \in C([0, T]; \mathbb{R}^+)$, $\mathbb{R}^+ = [0, \infty)$, and $f \in C(\mathbb{R}^+; \mathbb{R}^+)$ be a nondecreasing function, such that*

$$x(t) \leq x(0) + \int_0^t f(x(s)) ds,$$

for $t \in [0, T]$. Suppose that $y = y(t)$ is a solution of the initial value problem,

$$\frac{d}{dt}y(t) = f(y(t)), \quad y(0) = x(0), \tag{B.4}$$

that is continuous with respect to the initial data. Then, we have

$$x(t) \leq y(t), \tag{B.5}$$

for any $t \in [0, T]$.

Proof. Let $y_\delta(t)$ be a solution of (B.4) with the initial data $y(0) = x(0) + \delta$ for $\delta > 0$. Owing to the continuity of the solution, there exists a time t^* defined by

$$t^* = \sup\{t \in [0, T] \mid x(t) < y_\delta(t)\}.$$

We show that $t^* = T$. Indeed, if $t^* < T$, we have

$$0 = y_\delta(t^*) - x(t^*) \geq \delta + \int_0^{t^*} [f(y_\delta(s)) - f(x(s))] ds \geq \delta,$$

since f is nondecreasing. This is a contradiction. Thus, it follows that

$$x(t) \leq y_\delta(t),$$

for $t \in [0, T]$ and, taking the $\delta \rightarrow 0$ limit, we obtain (B.5). $\square$

The following lemma is a concrete example of the application of Lemma B.2. The proof is given in [56].

Lemma B.3. *Suppose that $x \in C^1([0, T]; \mathbb{R}^+)$ satisfies*

$$x(t) \leq \int_0^t [\varepsilon + \varphi(x(s))] ds,$$

for $0 < \delta < 1$. Then, we have

$$x(t) \leq \frac{e^t - 1}{e} \varepsilon e^{-t} + e^t \frac{\varepsilon^{e^{-t}} - \varepsilon}{-\log \varepsilon}. \tag{B.6}$$

Epecially, $x(t) \leq c(T) \varepsilon^{e^{-T}}$ for $t \in [0, T]$.

Proof. Using the tangent approximation of $\varphi(r)$, namely

$$\varphi(r) \leq \varphi'(\delta)(r - \delta) + \varphi(\delta) = (-\log \delta)r + \delta,$$

for any $0 < \delta < 1$, we find

$$x(t) \leq \int_0^t [(-\log \delta)x(s) + \delta + \varepsilon] ds.$$

Consider the ordinary differential equation,

$$\frac{d}{dt}y(t) = (-\log \delta)y(t) + \delta + \varepsilon, \quad y(0) = 0,$$

with $\delta = e^{-1}\varepsilon^{e^{-t}}$. Then, it is easy to confirm that the solution is given by

$$y(t) = \frac{e^t - 1}{e}\varepsilon^{e^{-t}} + e^t \frac{\varepsilon^{e^{-t}} - \varepsilon}{-\log \varepsilon}.$$

Applying Lemma B.2, we obtain (B.6). Note that it follows from (B.6) that

$$x(t) \leq \left(\frac{e^T - 1}{e} + \frac{e^T}{-\log \varepsilon} \right) \varepsilon^{e^{-t}} - \frac{\varepsilon}{-\log \varepsilon} \leq c(T)\varepsilon^{e^{-t}},$$

which is the desired result. □

C Properties of some auxiliary functions

Here, we introduce several functions and their properties that are often used in this paper.

1. The function $\psi(x)$: The function $\psi(x)$ is defined by

$$\psi(x) = \left(\frac{1}{1+x} \right)^{1/\Gamma_1} \left(\frac{x}{1+x} \right)^{1/\Gamma_2},$$

where $\Gamma_1, \Gamma_2 > 0$. Then, we easily find that $\psi(0) = 0$ and $\psi(x) \rightarrow 0$ as $x \rightarrow +\infty$. Moreover, since $\psi'(x) > 0$ for $0 < x < \Gamma_1/\Gamma_2$ and $\psi'(x) < 0$ for $x > \Gamma_1/\Gamma_2$, $\psi(x)$ has a maximal value at $x = \Gamma_1/\Gamma_2$. Hence, we have

$$0 < \psi(x) < \psi\left(\frac{\Gamma_1}{\Gamma_2}\right), \tag{C.1}$$

for $0 < x < \Gamma_1/\Gamma_2$ and $x > \Gamma_1/\Gamma_2$.

2. The function $K_0(x)$ and $K_1(x)$: As for the modified Bessel function of the second kind $K_0(x)$ and $K_1(x)$, we know

$$K'_0(x) = -K_1(x), \quad K'_1(x) = -\frac{1}{x}K_1(x) - K_0(x). \quad (\text{C.2})$$

Thus, both functions are monotone decreasing and downward-convex. Moreover, we have

$$K_0(x) \sim -\left(\beta + \log \frac{1}{2}x\right), \quad K_1(x) \sim \frac{1}{x} + \left(\beta + \log \frac{1}{2}x\right)\frac{1}{2}x - \frac{1}{4}x, \quad (\text{C.3})$$

as $x \rightarrow 0$, where β is the Euler's constant[65].

3. The function $B_K(\sqrt{x})$: The function $B_K(\sqrt{x})$ is monotone increasing and upward-convex. Indeed, owing to $B_K(x) = 1 - xK_1(x)$ and (C.2), we have

$$(B_K(\sqrt{x}))' = \frac{1}{2}K_0(\sqrt{x}) > 0, \quad (B_K(\sqrt{x}))'' = -\frac{1}{4\sqrt{x}}K_1(\sqrt{x}) < 0. \quad (\text{C.4})$$

4. The function $h_1(x)$: The function $h_1(x) = \sqrt{x}K_1(\sqrt{x})$ is monotone decreasing and downward-convex, since

$$h'_1(x) = -\frac{1}{2}K_0(\sqrt{x}) < 0, \quad h''_1(x) = \frac{1}{4\sqrt{x}}K_1(\sqrt{x}) > 0. \quad (\text{C.5})$$

5. The function $h_K(x)$: The function $h_K(x)$ is defined by

$$h_K(x) = \frac{1}{x}B_K(\sqrt{x}).$$

Then, $h_K(x)$ is monotone decreasing and downward-convex. Differentiating it, we have

$$h'_K(x) = -\frac{1}{x^2}\tilde{h}_0(x), \quad \tilde{h}_0(x) = B_K(\sqrt{x}) - \frac{1}{2}xK_0(\sqrt{x}),$$

and $\tilde{h}'_0(x) = \frac{1}{4}\sqrt{x}K_1(\sqrt{x}) > 0$. With $\tilde{h}_0(0) = 0$, we obtain $\tilde{h}_0(x) > 0$, and so $h'_K(x) < 0$. Moreover, its second derivative becomes

$$h''_K(x) = \frac{2}{x^3}\tilde{h}_1(x), \quad \tilde{h}_1(x) = B_K(\sqrt{x}) - \frac{1}{2}xK_0(\sqrt{x}) - \frac{1}{4}x^{3/2}K_1(\sqrt{x}).$$

Since we have $\tilde{h}_1(0) = 0$ and $\tilde{h}'_1(x) = \frac{1}{8}xK_0(\sqrt{x}) > 0$, it follows that $\tilde{h}_1(x) > 0$. Thus, we conclude that

$$h'_K(x) < 0, \quad h''_K(x) > 0. \quad (\text{C.6})$$

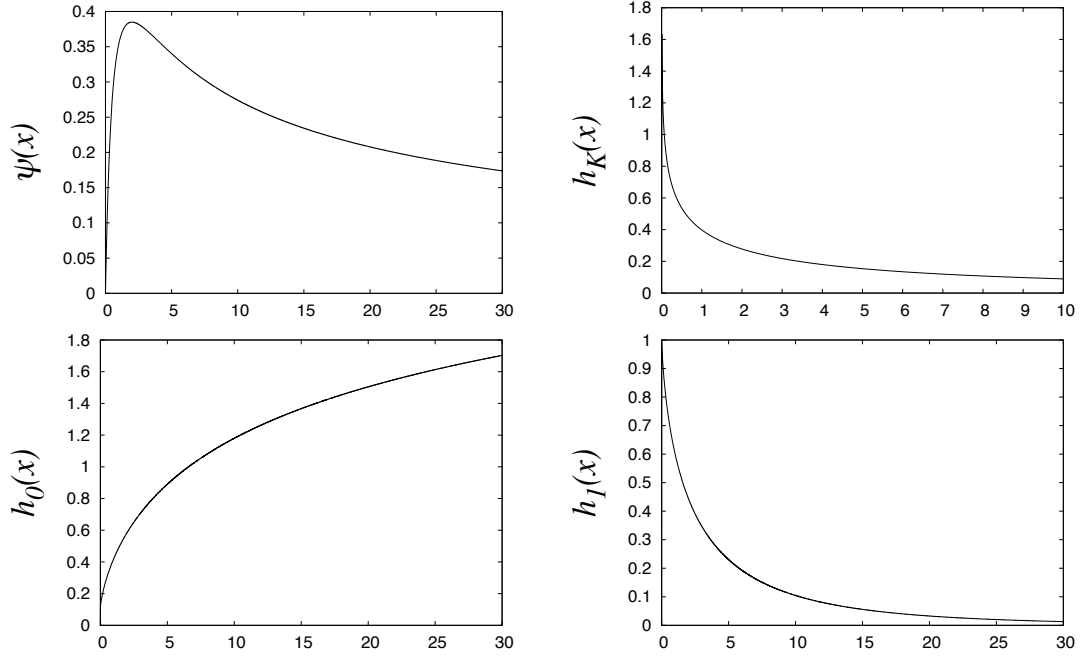


Figure 5: The graphs of the functions $\psi(x)$, $h_K(x)$, $h_0(x)$ and $h_1(x)$ in Appendix C.

6. The function $h_0(x)$: The function $h_0(x) = \log \sqrt{x} + K_0(\sqrt{x})$ is monotone increasing and upward-convex, since we have

$$h'_0(x) = \frac{1}{2}h_K(x) > 0, \quad h''_0(x) = \frac{1}{2}h'_K(x) < 0. \quad (\text{C.7})$$

Moreover, it follows from (C.3) that

$$h_0(0) = -\beta + \log 2. \quad (\text{C.8})$$

References

- [1] Anderson, C.: A vortex method for flows with slight density variations. J. Comput. Phys. **61**, 417-444 (1985)
- [2] Aref, H.: Motion of three vortices. Phys. Fluids, **22**(3), 393-400 (1979)
- [3] Batchelor, G. K.: Computation of the energy spectrum in homogeneous two-dimensional turbulence. Phys. Fluids Suppl. II, **12**, 233-239 (1969)

- [4] Benedetto, D., Marchioro, C. and Pulvirenti, M.: On the Euler flow in $\mathbb{R}^2$. Arch. Rat. Mech. Anal. **123**, 377-386 (1993)
- [5] Buckmaster, T.: Onsager's conjecture almost everywhere in time. Comm. Math. Phys. **333**, 1175-1198 (2015)
- [6] Buckmaster, T., De Lellis, C., Isett, P. and Székelyhidi, L., Jr: Anomalous dissipation for $1/5$ -Hölder Euler flows. Ann. of Math. **181**, 127-172 (2015)
- [7] Buckmaster, T., De Lellis, C. and Székelyhidi, L. Jr: Dissipative Euler flows with Onsager-critical spacial regularity. Comm. Pure Appl. Math., **69**, 1613-1670 (2016)
- [8] Chen, S., Foias, C., Holm, D. D., Olson, E., Titi, E. S. and Wynne, S.: Camassa-Holm equations as a closure model for turbulent channel and pipe flow. Phys. Rev. Lett. **81**(24), 5338-5341 (1998)
- [9] Chen, S., Holm, D. D., Margolin, L. G. and Zhang, R.: Direct numerical simulations of the Navier-Stokes alpha model. Physica D **133**, 66-83 (1999)
- [10] Cheskidov, A., Constantin, P., Friedlander, S., and Shvydkoy, R.: Energy conservation and Onsager's conjecture for the Euler equations. Nonlinearity **21**, 1233-1252 (2008).
- [11] Chorin, A. J.: Numerical study of slightly viscous flow. J. Fluid Mech. **57**, 785-796 (1973)
- [12] Chorin, A. J. and Bernard, P. S.: Discretization of a vortex sheet, with an example of roll-up. J. Comput. Phys. **13**, 423-429 (1973)
- [13] Constantin, P., E, W., and Titi, E. S.: Onsager's Conjecture on the Energy Conservation for Solutions of Euler's Equation. Comm. Math. Phys. **165**, 207-209 (1994).
- [14] De Lellis, C. and Székelyhidi, L. Jr: The Euler equations as a differential inclusion. Ann. of Math. **170**, 1417-1436 (2009)
- [15] De Lellis, C. and Székelyhidi, L. Jr: On admissibility criteria for weak solutions of the Euler equations. Arch. Ration. Mech. Anal. **195** 225-260 (2010)
- [16] De Lellis, C. and Székelyhidi, L. Jr: Dissipative continuous Euler flows. Invent. Math. **193**, 377-407 (2013)

- [17] De Lellis, C. and Székelyhidi, L. Jr: Dissipative Euler flows and Onsager ' s conjecture. J. Eur. Math. Soc. (JEMS) **16**, 1467-1505 (2014)
- [18] Delort, J.-M.: Existence de nappe de tourbillon en dimension deux. J. Amer. Math. Soc., **4**, 553-586 (1991)
- [19] Diperna, R. J. and Majda, A. J.: Concentrations in regularizations for 2-D incompressible flow. Comm. Pure Appl. Math., **40**, 301-345 (1987)
- [20] Duchon, J. and Robert, R.: Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations. Nonlinearity **13**, 249-255 (2000)
- [21] Evans, L. C. and Müller, S.: Hardy spaces and the two-dimensional Euler equations with nonnegative vorticity. J. Amer. Math. Soc. **7**, 199-218 (1994)
- [22] Eyink, G. L.: Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer. Physica D **78**, 222-240 (1994)
- [23] Eyink, G. L.: Dissipation in turbulent solutions of 2D Euler equations. Nonlinearity, **14**, 787-802 (2001)
- [24] Eyink, G. L.: Local 4/5-law and energy dissipation anomaly in turbulence. Nonlinearity **21**, 1233-1252 (2003).
- [25] Eyink, G. L. and Sreenivasan, K. R.: Onsager and the theory of hydrodynamic turbulence. Rev. Mod. Phys. **78**, 87-135 (2006).
- [26] Foias, C., Holm, D. D. and Titi, E. S.: The Navier-Stokes-alpha model of fluid turbulence. Physica D **152-153**, 505-519 (2001)
- [27] Foias, C., Holm, D. D. and Titi, E. S.: The three dimensional viscous Camassa-Holm equations, and their relation to the Navier-Stokes equations and turbulence theory. J. Dyn. Diff. Equ. **14**, 1-35 (2002)
- [28] Frisch, U.: Turbulence. Cambridge University Press, Cambridge, (1995)
- [29] Gotoda, T. and Sakajo, T.: Enstrophy variations in the incompressible 2D Euler flows and α point vortex system. Mathematical Fluid Dynamics, Present and Future, Springer Proceedings in Mathematics & Statistics **183**, (2015)
- [30] Gotoda, T. and Sakajo, T.: Distributional enstrophy dissipation via the collapse of triple point vortices. J. Nonlinear Sci., **26**, 1525-1570 (2016)

- [31] Holm, D. D.: Variational principles for Lagrangian-averaged fluid dynamics. J. Phys. A: Math. Gen. **35**, 679-688 (2002)
- [32] Holm, D. D., Marsden, J. E. and Ratiu, T. S.: Euler-Poincaré models of ideal fluids with nonlinear dispersion. Phys. Rev. Lett., **80**, 4173-4177 (1998)
- [33] Holm, D. D., Marsden, J. E. and Ratiu, T. S.: Euler-Poincaré equations and semidirect products with applications to continuum theories. Adv. Maths., **137**, 1-81 (1998)
- [34] Holm, D. D., Nitsche, M. and Putkaradze, V.: Euler-alpha and vortex blob regularization of vortex filament and vortex sheet motion. J. Fluid Mech. **555**, 149-176 (2006)
- [35] Hou, T. Y. and Li, C.: On global well-posedness of the Lagrangian averaged Euler equations. SIAM J. Math. Anal. **38**(3), 782-794 (2006)
- [36] Isett, P.: Hölder continuous Euler flows in three dimensions with compact support in time. Preprint, 2012. arXiv:1211.4065
- [37] Isett, P. and Oh, S.-J.: On nonperiodic Euler flows with Hölder regularity. Arch. Rat. Mech. Anal., **221**, 725-804 (2016)
- [38] Kimura, Y.: Similarity solution of two-dimensional point vortices. J. Phys. Soc. Japan., **56**, 2024-2030 (1987)
- [39] Kraichnan, R. H.: Inertial ranges in two-dimensional turbulence. Phys. Fluids, **10**, 1417-1423 (1967)
- [40] Krasny, R.: Desingularization of periodic vortex sheet roll-up. J. Comput. Phys. **65**, 292-313 (1986)
- [41] Kudela, H.: Collapse of n -point vortices in self-similar motion. Fluid Dyn. Res., **46**, 031414, (2014)
- [42] Kudela, H.: Self-similar collapse of n point vortices. J. Nonlinear Sci., **24**, 913-933, (2014)
- [43] Kolmogorov, A. N.: The local structure of turbulence in incompressible viscous fluid for very large Reynolds number. Dokl. Akad. Nauk SSSR **30**, 301-305 (1941).

- [44] Leith, C. E.: Diffusion approximation for two-dimensional turbulence. *Phys. Fluids*, **11**, 671-673 (1968)
- [45] Leonard, A.: Vortex methods for flow simulation. *J. Comput. Phys.* **37**, 289-335 (1980)
- [46] Liu, J. G. and Xin, Z.: Convergence of vortex methods for weak solutions to the 2-D Euler equations with vortex sheet data. *Comm. Pure Appl. Math.* **48**, 611-628 (1995)
- [47] Lunasin, E., Kurien, S., Taylor, M. A. and Titi, E. S.: A study of the Navier-Stokes- α model for two-dimensional turbulence. *J. Turbulence.*, **8**, 1-21 (2007)
- [48] Majda, A. J.: Remarks on weak solutions for vortex sheets with a distinguished sign. *Indiana Univ. Math. J.*, **42**, 921-939 (1993)
- [49] Marchioro, C. and Pulvirenti, M.: Mathematical theory of incompressible non-viscous fluids. *Applied Mathematical Sciences*, **96**, Springer, New York (1994)
- [50] Marsden, J. E. and Shkoller, S.: The anisotropic Lagrangian averaged Euler and Navier-Stokes equations. *Arch. Rat. Mech. Anal.*, **166**, 27-46 (2003)
- [51] Masaki, S.: Local existence and WKB approximation of solutions to Schrödinger-Poisson system in the two-dimensional whole space. *Comm. Part. Diff. Equ.*, **35**, 2253-2278 (2010)
- [52] McGrath, F. J.: Nonstationary plane flow of viscous and ideal fluids. *Arch. Rat. Mech. Anal.* **27**, 329-348 (1968)
- [53] Mohseni, K., Kosović, B., Shkoller, S. and Marsden, J. E.: Numerical simulations of the Lagrangian averaged Navier-Stokes equations for homogeneous isotropic turbulence. *Phys. Fluids* **15**(2), 524-544 (2003)
- [54] Newton, P. K.: *The N-Vortex Problem, Analytical Techniques*. Springer, (2001)
- [55] Novikov, E. A.: Dynamics and statistic of a system of vortices. *Sov. Phys. JETP.*, **41**, 937-943 (1976)
- [56] Oliver, M. and Shkoller, S.: The vortex blob method as a second-grade non-Newtonian fluid. *Comm. Part. Diff. Equ.*, **26**, 295-314 (2001)
- [57] Onsager, L.: Statistical hydrodynamics. *Nouvo Cimento Suppl.* **6**, 287-297 (1949).

- [58] Sakajo, T.: Instantaneous energy and enstrophy variations in Euler-alpha point vortices via triple collapse. *J. Fluid Mech.*, **702**, 188-214 (2012)
- [59] Scheffer, V.: An inviscid flow with compact support in space-time. *J. Geom. Anal.* **3**, 343-401 (1993)
- [60] Shkoller, S.: Geometry and curvature of diffeomorphism groups with H^1 metric and mean hydrodynamics. *J. Funct. Anal.*, **160**, 337-365 (1998)
- [61] Shnirelman, A.: On the nonuniqueness of weak solution of the Euler equation. *Comm. Pure Appl. Math.* **50**, 1261-1286 (1997)
- [62] Shnirelman, A.: Weak solutions with decreasing energy of incompressible Euler equations. *Comm. Math. Phys.* **210**, 541-603 (2000)
- [63] Shvydkoy, R.: Lectures on the Onsager conjecture. *Discr. Conti. Dyn. Syst. Ser.S* **3**(3), 473-496 (2010)
- [64] Synge, J. L.: On the motion of three vortices. *Can. J. Math.*, **1**, 257-270 (1949)
- [65] Watson, G. N.: A treatise on the theory of Bessel functions. Cambridge Mathematical Library. Cambridge University Press, Cambridge. (1995)
- [66] Yudovich, V. I.: Nonstationary motion of an ideal incompressible liquid. *USSR Comp. Math. Phys.*, **3**, 1407-1456 (1963)