

# Coefficient Inequalities for Ceratin Analytic Functions

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**Abstract.** For real  $\alpha (\alpha > 1)$ , subclasses  $M(\alpha)$  and  $N(\alpha)$  of analytic fuctions  $f(z)$  with  $f(0) = 0$  and  $f'(0) = 1$  in  $U$  are introduced. The object of the present paper is to consider the coefficient inequalities for functions  $f(z)$  to be in the classes  $M(\alpha)$  and  $N(\alpha)$ . Further, the bounds of  $\alpha$  for functions  $f(z)$  to be starlike in  $U$  are considered.

## 1 Introduction

Let  $A$  denote the class of functions  $f(z)$  of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk  $U = \{z \in \mathbb{C} : |z| < 1\}$ . Let  $M(\alpha)$  be the subclass of  $A$  consisting of functions  $f(z)$  which satisfy

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < \alpha \quad (z \in U)$$

for some  $\alpha (\alpha > 1)$ . And let  $N(\alpha)$  be the subclass of  $A$  consisting of functions  $f(z)$  which satisfy

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \alpha \quad (z \in U)$$

for some  $\alpha (\alpha > 1)$ . Then, we see that  $f(z) \in N(\alpha)$  if and only if  $zf'(z) \in M(\alpha)$ . Let us give examples of functions  $f(z)$  in the classes  $M(\alpha)$  and  $N(\alpha)$ .

**Remark** For  $1 < \alpha \leq \frac{4}{3}$ , the classes  $M(\alpha)$  and  $N(\alpha)$  were introduced by Uralegaddi, Ganigi and Sarangi [2].

**Example** (i)  $f(z) = z(1 - z)^{2(\alpha-1)} \in M(\alpha)$

(ii)  $g(z) = \frac{1}{2\alpha-1} \{1 - (1 - z)^{2\alpha-1}\} \in N(\alpha)$

*Proof.* Since  $f(z) \in M(\alpha)$  if and only if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < \alpha,$$

we can write

$$\frac{\alpha - \frac{zf'(z)}{f(z)}}{\alpha - 1} = \frac{1+z}{1-z},$$

which is equivalent to

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{2(\alpha-1)}{1-z}.$$

Integrating both sides of the above equality, we have

$$f(z) = z(1 - z)^{2(\alpha-1)} \in M(\alpha).$$

Next, since  $g(z) \in N(\alpha)$  if and only if  $zg'(z) \in M(\alpha)$ ,

$$zg'(z) = z(1 - z)^{2(\alpha-1)}.$$

for function  $g(z) \in N(\alpha)$ . It follows that

$$g(z) = \frac{-1}{2\alpha-1}(1 - z)^{2\alpha-1} + \frac{1}{2\alpha-1}$$

$$= \frac{1}{2\alpha-1} \{1 - (1 - z)^{2\alpha-1}\} \in N(\alpha).$$

□

## 2 Coefficient inequalities for the classes $M(\alpha)$ and $N(\alpha)$

We try to derive sufficient conditions for  $f(z)$  which are given by using coefficient inequalities.

**Theorem 1.** If  $f(z) \in A$  satisfies

$$\sum_{n=2}^{\infty} \{(n-1) + |n-2\alpha+1|\} |a_n| \leq 2(\alpha-1)$$

for some  $\alpha(\alpha > 1)$ , then  $f(z) \in M(\alpha)$ .

*Proof.* Let us suppose that

$$\sum_{n=2}^{\infty} \{(n-1) + |n-2\alpha+1|\} |a_n| \leq 2(\alpha-1)$$

for  $f(z) \in A$ .

It suffices to show that

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} - (2\alpha-1)} \right| < 1 \quad (z \in U).$$

We have

$$\begin{aligned} \left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} - (2\alpha-1)} \right| &\leq \frac{\sum_{n=2}^{\infty} (n-1) |a_n| |z|^{n-1}}{2(\alpha-1) - \sum_{n=2}^{\infty} |n-2\alpha+1| |a_n| |z|^{n-1}} \\ &< \frac{\sum_{n=2}^{\infty} (n-1) |a_n|}{2(\alpha-1) - \sum_{n=2}^{\infty} |n-2\alpha+1| |a_n|}. \end{aligned}$$

The last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty} (n-1) |a_n| \leq 2(\alpha-1) - \sum_{n=2}^{\infty} |n-2\alpha+1| |a_n|$$

which is equivalent to our condition

$$\sum_{n=2}^{\infty} \{(n-1) + |n-2\alpha+1|\} |a_n| \leq 2(\alpha-1)$$

of the theorem. This completes the proof of the theorem.  $\square$

By using Theorem 1, we have

**Corollary 1.** *If  $f(z) \in A$  satisfies*

$$\sum_{n=2}^{\infty} n\{(n-1) + |n-2\alpha+1|\}|a_n| \leq 2(\alpha-1)$$

*for some  $\alpha(\alpha > 1)$ , then  $f(z) \in N(\alpha)$ .*

*Proof.* From  $f(z) \in N(\alpha)$  if and only if  $zf'(z) \in M(\alpha)$ , replacing  $a_n$  by  $na_n$  in Theorem 1 we have the corollary.  $\square$

In view of Theorem 1 and Corollary 1, if  $1 < \alpha \leq 2$ , then  $n-2\alpha+1 \geq 0$  for all  $n \geq 2$ . Thus we have

**Corollary 2.** (i) *If  $f(z) \in A$  satisfies*

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \leq \alpha-1$$

*for some  $\alpha(1 < \alpha \leq 2)$ , then  $f(z) \in M(\alpha)$ .*

(ii) *If  $f(z) \in A$ , satisfies*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq \alpha-1$$

*for some  $\alpha(1 < \alpha \leq 2)$ , then  $f(z) \in N(\alpha)$ .*

### 3 Starlikeness for functions in $M(\alpha)$ and $N(\alpha)$

By Silverman [1], we know that if  $f(z) \in A$  satisfies

$$\sum_{n=2}^{\infty} n|a_n| \leq 1,$$

then  $f(z) \in S^*$ , where  $S^*$  denotes the subclasses of  $A$  consisting of all univalent and starlike functions  $f(z)$  in  $U$ . Thus we have

**Theorem 2.** *If  $f(z) \in A$  satisfies*

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \leq \alpha-1$$

for some  $\alpha(1 < \alpha \leq \frac{4}{3})$ , then  $f(z) \in S^* \cap M(\alpha)$ , therefore,  $f(z)$  is starlike in  $U$ . Further, if  $f(z) \in A$  satisfies

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq \alpha-1$$

for some  $\alpha(1 < \alpha \leq \frac{3}{2})$ , then  $f(z) \in S^* \cap N(\alpha)$ , therefore,  $f(z)$  is starlike in  $U$ .

*Proof.* Let us consider  $\alpha$  such that

$$\sum_{n=2}^{\infty} n|a_n| \leq \sum_{n=2}^{\infty} \frac{n-\alpha}{\alpha-1}|a_n| \leq 1.$$

Then we have  $f(z) \in S^* \cap M(\alpha)$  by means of Theorem1. This inequality holds true if

$$n \leq \frac{n-\alpha}{\alpha-1} \quad (n = 2, 3, 4, \dots).$$

Therefore, we have

$$1 < \alpha \leq 2 - \frac{2}{n+1} \quad (n = 2, 3, 4, \dots),$$

which shows  $1 < \alpha \leq \frac{4}{3}$ . Next, considering  $\alpha$  such that

$$\sum_{n=2}^{\infty} n|a_n| \leq \sum_{n=2}^{\infty} \frac{n(n-\alpha)}{\alpha-1}|a_n| \leq 1,$$

we have

$$n \leq \frac{n(n-\alpha)}{\alpha-1} \quad (n = 2, 3, 4, \dots).$$

which is equivalent to

$$1 < \alpha \leq \frac{n+1}{2} \quad (n = 2, 3, 4, \dots).$$

This implies that  $1 < \alpha \leq \frac{3}{2}$ . □

Finally, by virtue of the result for convex functions by Silverman [1], we have if  $f(z) \in A$  satisfies

$$\sum_{n=2}^{\infty} n^2|a_n| \leq 1,$$

then  $f(z) \in K$ , where  $K$  denotes the subclass of  $A$  consisting of all univalent and convex functions  $f(z)$  in  $U$ . Using the same manner as in the proof of Theorem2, we derive

**Theorem 3.** If  $f(z) \in A$  satisfies

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq \alpha-1$$

for some  $\alpha(1 < \alpha \leq \frac{4}{3})$ , then  $f(z) \in K \cap N(\alpha)$ , therefore,  $f(z)$  is convex in  $U$ .

## 4 Bounds of $\alpha$ for starlikeness

Note that the sufficient for  $f(z)$  to be in the class  $M(\alpha)$  is given by

$$\sum_{n=2}^{\infty} \{(n-1) + |n-2\alpha+1|\} |a_n| \leq 2(\alpha-1).$$

Since, if  $f(z) \in A$  satisfies

$$\sum_{n=2}^{\infty} n|a_n| \leq 1,$$

then  $f(z) \in S^*$  (cf. [1]), it is interesting to find the bounds of  $\alpha$  for starlikeness of  $f(z) \in M(\alpha)$ .

To do this, we have to consider the following inequality

$$\begin{aligned} \sum_{n=2}^{\infty} n|a_n| &\leq \frac{1}{2(\alpha-1)} \sum_{n=2}^{\infty} \{(n-1) + |n-2\alpha+1|\} |a_n| \\ &\leq 1 \end{aligned}$$

which is equivalent to

$$\sum_{n=2}^{\infty} \{|n-2\alpha+1| + (3-2\alpha)n\} |a_n| \geq 0.$$

Let us define

$$F(n) = |n-2\alpha+1| + (3-2\alpha)n \quad (n \geq 2).$$

Then, if  $F(n)$  satisfies

$$\sum_{n=2}^{\infty} F(n)|a_n| \geq 0,$$

then  $f(z)$  belongs to  $S^*$ .

**Theorem 4.** Let  $f(z) \in A$  satisfy

$$\sum_{n=2}^{\infty} \{(n-1) + |n-2\alpha+1|\} |a_n| \leq 2(\alpha-1)$$

for some  $\alpha > 1$ . Further, let  $\delta_k$  be defined by

$$\delta_k = \sum_{n=k}^{\infty} F(n)|a_n|.$$

Then,

- (i) if  $1 < \alpha \leq \frac{3}{2}$ , then  $f(z) \in S^*$ ,
- (ii) if  $\frac{3}{2} \leq \alpha \leq \min\left(\frac{13}{8}, \frac{3+\delta_3}{2}\right)$ , then  $f(z) \in S^*$ ,
- (iii) if  $\frac{8}{3} \leq \alpha \leq \min\left(\frac{17}{10}, \frac{12-\delta_4 + \sqrt{\delta_4^2 + 48\delta_4 + 48}}{12}\right)$ , then  $f(z) \in S^*$ .

*Proof.* For  $1 < \alpha \leq \frac{3}{2}$ , we know that

$$n - 2\alpha + 1 \geq 3 - 2\alpha \geq 0 \quad (n \geq 2),$$

that is that  $F(n) \geq 0$  ( $n \geq 2$ ). Therefore, we have

$$\sum_{n=2}^{\infty} F(n)|a_n| \geq 0.$$

If  $\frac{3}{2} \leq \alpha \leq \frac{13}{8}$ , then  $F(2) = 3 - 2\alpha \leq 0$  and

$$\begin{aligned} F(n) &= 2n(2 - \alpha) + 1 - 2\alpha \\ &\geq 13 - 8\alpha \\ &\geq 0 \end{aligned}$$

for  $n \geq 3$ . Further, we know that

$$|a_n| \leq \frac{2(\alpha - 1)}{(n - 1) + |n - 2\alpha + 1|} \quad (n \geq 2),$$

which gives us that  $|a_2| \leq 1$ . therefore, we obtain that

$$\begin{aligned} \sum_{n=2}^{\infty} F(n)|a_n| &= F(2)|a_2| + \sum_{n=3}^{\infty} F(n)|a_n| \\ &\geq 3 - 2\alpha + \delta_3 \geq 0 \end{aligned}$$

for

$$\frac{3}{2} \leq \alpha \leq \min\left(\frac{13}{8}, \frac{3+\delta_3}{2}\right),$$

Furthermore, if  $\frac{13}{8} \leq \alpha \leq \frac{17}{10}$ , then

$$F(2) = 3 - 2\alpha \leq 0$$

$$\begin{aligned} F(3) &= |4 - 2\alpha| + 3(3 - 2\alpha) \\ &= 13 - 8\alpha \\ &\leq 0, \end{aligned}$$

and

$$\begin{aligned} F(n) &= |n - 2\alpha + 1| + (3 - 2\alpha)n \\ &= 4n + 1 - 2(n + 1)\alpha \\ &\geq \frac{3(n - 4)}{5} \\ &\geq 0 \end{aligned}$$

for  $n \geq 4$ . Noting that  $|a_2| \leq 1$  and  $|a_3| \leq \frac{\alpha - 1}{3 - \alpha}$ , we conclude that

$$\begin{aligned} \sum_{n=2}^{\infty} F(n)|a_n| &= F(2)|a_2| + F(3)|a_3| + \sum_{n=4}^{\infty} F(n)|a_n| \\ &\geq (3 - 2\alpha) + (13 - 8\alpha)\frac{\alpha - 1}{3 - \alpha} + \delta_4 \\ &\geq 0. \end{aligned}$$

for  $\alpha$  which satisfies

$$6\alpha^2 - (12 - \delta_4)\alpha + 4 - 3\delta_4 \leq 0.$$

This shows that

$$\frac{8}{3} \leq \alpha \leq \min \left( \frac{17}{10}, \frac{12 - \delta_4 + \sqrt{\delta_4^2 + 48\delta_4 + 48}}{12} \right),$$

This completes the proof of Theorem4.

Finally, by virtue of Theorem4, we may suppose Conjecture. If  $f(z) \in A$  satisfies

$$\sum_{n=2}^{\infty} \{(n - 1) + |n - 2\alpha + 1|\} |a_n| \leq 2(\alpha - 1)$$

for some  $1 < \alpha < 2$ , then  $f(z) \in S^*$ . □

## References

- [1] H.Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51**(1975),109-116.
- [2] B.A.Uralegaddi, M.D.Ganigi and S.M.Sarangi, *Univalent functions with positive coefficients*, Tamkang J. Math. **25**(1994),225-230.

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