

# THE PRIME NUMBER THEOREM FOR RANKIN-SELBERG $L$ -FUNCTIONS

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## ABSTRACT

In this article, we survey and announce a recent unconditional proof of the prime number theorem for Rankin-Selberg  $L$ -functions attached to automorphic cuspidal representations of  $GL_n$  over  $\mathbb{Q}$ . Applications of this prime number theorem to Selberg's orthogonality conjecture and factorization of automorphic  $L$ -functions will be given.

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## 1. PROBLEMS CONCERNING AUTOMORPHIC $L$ -FUNCTIONS

Let  $\pi$  be an irreducible unitary cuspidal representation of  $GL_m(\mathbb{Q}_A)$ , and  $s = \sigma + it \in \mathbb{C}$ . The global  $L$ -function attached to  $\pi$  is given by products of local factors for  $\sigma > 1$  (Godement and Jacquet [5]):

$$L(s, \pi) = \prod_p L_p(s, \pi_p),$$

$$\Phi(s, \pi) = L_\infty(s, \pi_\infty) L(s, \pi),$$

where

$$L_p(s, \pi_p) = \prod_{j=1}^m \left(1 - \frac{\alpha_\pi(p, j)}{p^s}\right)^{-1},$$

and

$$L_\infty(s, \pi_\infty) = \prod_{j=1}^m \Gamma_{\mathbb{R}}(s + \mu_\pi(j)).$$

Here  $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2)$ , and  $\alpha_\pi(p, j)$  and  $\mu_\pi(j)$ ,  $j = 1, \dots, m$ , are complex numbers associated with  $\pi_p$  and  $\pi_\infty$ , respectively, according to the Langlands correspondence. Denote by

$$a_\pi(p^k) = \sum_{1 \leq j \leq m} \alpha_\pi(p, j)^k \quad (1.1)$$

the Fourier coefficients of  $\pi$ . Then for  $\sigma > 1$ , we have

$$\frac{d}{ds} \log L(s, \pi) = - \sum_{n=1}^{\infty} \frac{\Lambda(n) a_\pi(n)}{n^s}, \quad (1.2)$$

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where  $\Lambda(n)$  is the von Mangoldt function:  $\Lambda(p^k) = \log p$ , and  $= 0$  elsewhere. If  $\pi'$  is an automorphic irreducible cuspidal representation of  $GL_{m'}(\mathbb{Q}_\mathbb{A})$ , we define  $L(s, \pi')$ ,  $\alpha_{\pi'}(p, i)$ ,  $\mu_{\pi'}(i)$ , and  $a_{\pi'}(p^k)$  likewise, for  $i = 1, \dots, m'$ . If  $\pi$  and  $\pi'$  are equivalent, then  $m = m'$  and  $\{\alpha_\pi(p, j)\} = \{\alpha_{\pi'}(p, i)\}$  for any  $p$ . Hence, by (1.1),  $a_\pi(n) = a_{\pi'}(n)$  for any  $n = p^k$ , when  $\pi \cong \pi'$ .

The Rankin-Selberg  $L$ -function  $L(s, \pi \times \tilde{\pi}')$  was developed by Jacquet [7], Jacquet, Piatetski-Shapiro, and Shalika [8], Shahidi [29], and Moeglin and Waldspurger [19], where  $\pi$  and  $\pi'$  are automorphic irreducible cuspidal representations of  $GL_m$  and  $GL_{m'}$ , respectively, with unitary central characters. In our case, this (finite-part)  $L$ -function is defined by

$$L(s, \pi \times \tilde{\pi}') = \prod_p L_p(s, \pi_p \times \tilde{\pi}'_p),$$

where

$$L_p(s, \pi_p \times \tilde{\pi}'_p) = \prod_{j=1}^m \prod_{k=1}^{m'} \left(1 - \frac{\alpha_\pi(p, j) \bar{\alpha}_{\pi'}(p, k)}{p^s}\right)^{-1}.$$

The Archimedean local factor  $L_\infty(s, \pi_\infty \times \tilde{\pi}'_\infty)$  is defined by

$$L_\infty(s, \pi_\infty \times \tilde{\pi}'_\infty) = \prod_{j=1}^m \prod_{k=1}^{m'} \Gamma_{\mathbb{R}}(s + \mu_{\pi \times \tilde{\pi}'}(j, k))$$

where the complex numbers  $\mu_{\pi \times \tilde{\pi}'}(j, k)$  satisfy the trivial bound

$$\operatorname{Re} \mu_{\pi \times \tilde{\pi}'}(j, k) > -1.$$

Denote

$$\Phi(s, \pi \times \tilde{\pi}') = L_\infty(s, \pi_\infty \times \tilde{\pi}'_\infty) L(s, \pi \times \tilde{\pi}').$$

Also, we have for  $\sigma > 1$  that

$$\frac{d}{ds} \log L(s, \pi \times \tilde{\pi}') = - \sum_{n=1}^{\infty} \frac{\Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n)}{n^s}. \quad (1.3)$$

The Prime Number Theorem (PNT) for Rankin-Selberg  $L$ -functions is the following

**Problem 1.1 (PNT for Rankin-Selberg  $L$ -functions).** *Let  $\pi$  and  $\pi'$  be irreducible unitary cuspidal representations of  $GL_m(\mathbb{Q}_\mathbb{A})$  and  $GL_{m'}(\mathbb{Q}_\mathbb{A})$ , respectively. Determine the asymptotic behavior of*

$$\sum_{n \leq x} \Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n). \quad (1.4)$$

PNT with  $\pi$  and  $\pi'$  being homomorphic cusp forms has been studied by several authors. Rankin [25] proved a PNT for  $\pi \cong \pi'$  being homomorphic cusp forms for the modular group, Perelli [24] generalized this to arithmetic progressions, while Laurinćikas and Matsumoto [13] proved a PNT in arithmetic progressions for  $\pi \cong \pi'$  being homomorphic cusp forms for congruence groups. Ichihara [6] established a PNT in arithmetic progressions for homomorphic cusp forms  $\pi$  and  $\pi'$ , not necessarily equivalent.

A statement easier than Problem 1.1 is the Selberg Orthogonality Conjecture (SOC); see Selberg [28] and Ram Murty [22] [23].

**Conjecture 1.2 (SOC).** *Let  $\pi$  and  $\pi'$  be given as in Problem 1.1. Then*

$$\sum_{n \leq x} \frac{\Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n)}{n} = \begin{cases} \log x + O(1) & \text{if } \pi' \cong \pi; \\ O(1) & \text{if } \pi' \not\cong \pi. \end{cases}$$

Problem 1.1 can be compared with the classical PNT

$$\sum_{n \leq x} \Lambda(n) \sim x, \quad (1.5)$$

while SOC is similar to Mertens' theorem that

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1). \quad (1.6)$$

It is known that (1.6) is weaker than (1.5).

The following statement is still weaker than SOC.

**Conjecture 1.3 (Weighted SOC).** *Let  $\pi$  and  $\pi'$  be given as in Problem 1.1. Then*

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) \frac{\Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n)}{n} = \begin{cases} \log x + O(1) & \text{if } \pi' \cong \pi; \\ O(1) & \text{if } \pi' \not\cong \pi. \end{cases} \quad (1.7)$$

Rudnick and Sarnak [26] proved Conjecture 1.3 in the case  $\pi' \cong \pi$ , and then deduced Conjecture 1.2 in the case  $\pi' \cong \pi$ , using the fact that the left side of (1.7) is a sum of non-negative terms.

## 2. WEIGHTED SOC AND FACTORIZATION OF AUTOMORPHIC $L$ -FUNCTIONS

In [15] and [16], we proved Conjecture 1.3 in the case  $\pi' \not\cong \pi$ .

**Theorem 2.1.** *For any automorphic irreducible cuspidal representations  $\pi$  and  $\pi'$  of  $GL_m(\mathbb{Q}_\mathbb{A})$  and  $GL_{m'}(\mathbb{Q}_\mathbb{A})$ , respectively,*

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) \frac{\Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n)}{n} \ll 1, \quad (2.1)$$

*if  $\pi$  is not equivalent to  $\pi'$ .*

In fact, when  $\pi$  and  $\pi'$  are not twisted equivalent, i.e., when  $\pi \not\cong \pi' \otimes \alpha^{it}$  for any  $t \in \mathbb{R}$ , where  $\alpha(g) = |\det(g)|$ , (2.1) was proved in [15]. In the remaining case when  $m = m'$  and  $\pi \cong \pi' \otimes \alpha^{i\tau_0}$  for some non-zero  $\tau_0 \in \mathbb{R}$ , (2.1) was established in [16].

It is a far-reaching conjecture of Langlands [12] that the most general  $L$ -function is indeed the  $L$ -function  $L(s, \Pi)$  attached to an automorphic representation  $\Pi$  of  $GL_n$  over an algebraic number field. It was further conjectured that this  $L(s, \Pi)$  is equal to a product of  $L$ -functions  $L(s, \pi_j)$  attached to automorphic irreducible cuspidal representations  $\pi_j$  of  $GL_{m_j}$  over  $\mathbb{Q}$  in a unique way:

$$L(s, \Pi) = L(s, \pi_1) \cdots L(s, \pi_k). \quad (2.2)$$

These  $L(s, \pi_j)$  are called principal or primitive  $L$ -functions over  $\mathbb{Q}$  in the sense that they are supposed to be  $L$ -functions that cannot be factorized further. They are believed to be the building blocks of all  $L$ -functions.

A known special case of the unique factorization (2.2) is for  $\Pi$  being an automorphic irreducible cuspidal representation of  $GL_n$  over a cyclic algebraic number field  $F$ , when  $\Pi$  is invariant under the action of the  $\text{Gal}(F/\mathbb{Q})$ . According to Arthur and Clozel [1], such a representation  $\Pi$  is the base change of  $\pi \otimes \eta$ , where  $\pi$  is an automorphic irreducible cuspidal representation of  $GL_n$  over  $\mathbb{Q}$ , and  $\eta$  is any idele class character of  $\mathbb{Q}$  trivial on  $N_{F/\mathbb{Q}}(F_{\mathbb{A}}^{\times})$ . In terms of  $L$ -functions, we have the factorization

$$L(s, \Pi) = \prod_{\eta} L(s, \pi \otimes \eta)$$

uniquely.

In [16], we proved the uniqueness of the factorization in (2.2). That is, if any general  $L$ -function can be written as a product of principal  $L$ -functions  $L(s, \pi_j)$  for  $GL_{m_j}$  over  $\mathbb{Q}$ , we showed that this factorization is unique.

**Theorem 2.2.** *Let  $\pi_j$  and  $\pi'_i$ ,  $j = 1, \dots, k$ ,  $i = 1, \dots, l$ , be automorphic irreducible cuspidal representations of  $GL_{m_j}(\mathbb{Q}_{\mathbb{A}})$  and  $GL_{m'_i}(\mathbb{Q}_{\mathbb{A}})$  with unitary central characters, respectively. Then*

$$L(s, \pi_1) \cdots L(s, \pi_k) = L(s, \pi'_1) \cdots L(s, \pi'_l) \quad (2.3)$$

*cannot hold, if there is a  $\pi_j$  that is not equivalent to any  $\pi'_i$ .*

By taking  $k = 1$ , Theorem 2.2 implies that  $L(s, \pi_1)$  cannot be factorized further.

**Corollary 2.3.** *The  $L$ -function  $L(s, \pi)$  attached to an automorphic irreducible cuspidal representation  $\pi$  of  $GL_m(\mathbb{Q}_{\mathbb{A}})$  cannot be factorized into  $L(s, \pi'_1) \cdots L(s, \pi'_l)$  nontrivially, where  $\pi'_i$  is an automorphic irreducible cuspidal representation of  $GL_{m'_i}(\mathbb{Q}_{\mathbb{A}})$  with unitary central character.*

Unique factorization of  $L$ -functions in the Selberg class (Selberg [28]) was studied by Conrey and Ghosh [2] and Ram Murty [22], under SOC. For automorphic  $L$ -functions, Ram Murty [23] proved that  $L(s, \pi)$  is primitive, i.e., cannot be factorized further, when  $\pi$  is an automorphic irreducible cuspidal representation of  $GL_2(\mathbb{Q}_{\mathbb{A}})$ , under the Generalized Ramanujan Conjecture (GRC, Conjecture 3.1 below). Our Theorem 2.2 and Corollary 2.3 are unconditional.

### 3. PNT AND SOC UNDER GRC

In this section, we will need GRC.

**Conjecture 3.1 (GRC).** *Let  $\pi$  be an irreducible unitary cuspidal representation of  $GL_m(\mathbb{Q}_{\mathbb{A}})$ . For any unramified  $p$ , we have*

$$|\alpha_{\pi}(p, j)| = 1.$$

Note that in Conjecture 3.1, we do not include the Archimedean Ramanujan conjecture,  $\text{Re } \mu_{\pi}(j) = 0$ .

As a consequence of GRC, we proved in [17] the following PNT. Denote  $\alpha(g) = |\det(g)|$ .

**Theorem 3.2.** *Assume GRC. Let  $\pi$  and  $\pi'$  be irreducible unitary cuspidal representations of  $GL_m(\mathbb{Q}_{\mathbb{A}})$  and  $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$ , respectively. Assume that  $\pi$  and  $\pi'$  are self contragredient:  $\pi \cong \bar{\pi}$*

and  $\pi' \cong \tilde{\pi}'$ . Then

$$\begin{aligned} & \sum_{n \leq x} \Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n) \\ &= \begin{cases} \frac{x^{1+i\tau_0}}{(1+i\tau_0)} + O\{x \exp(-c\sqrt{\log x})\} \\ \quad \text{if } \pi' \cong \pi \otimes |\det|^{i\tau_0} \text{ for some } \tau_0 \in \mathbb{R}; \\ O\{x \exp(-c\sqrt{\log x})\} \\ \quad \text{if } \pi' \not\cong \pi \otimes |\det|^{it} \text{ for any } t \in \mathbb{R}. \end{cases} \end{aligned}$$

Here and throughout,  $c$  is a positive constant, not necessarily the same at each occurrence.

**Corollary 3.3.** Assume GRC. Let  $\pi$  and  $\pi'$  be given as in Theorem 3.2. We have

$$\begin{aligned} & \sum_{n \leq x} \frac{\Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n)}{n} \\ &= \begin{cases} \log x + c_1 + O\{\exp(-c\sqrt{\log x})\} \\ \quad \text{if } \pi' \cong \pi; \\ \frac{x^{i\tau_0}}{i\tau_0(1+i\tau_0)} + c_2 + O\{\exp(-c\sqrt{\log x})\} \\ \quad \text{if } \pi' \cong \pi \otimes |\det|^{i\tau_0} \text{ for some } \tau_0 \in \mathbb{R}^\times; \\ c_2 + O\{\exp(-c\sqrt{\log x})\} \\ \quad \text{if } \pi' \not\cong \pi \otimes |\det|^{it} \text{ for any } t \in \mathbb{R}. \end{cases} \end{aligned}$$

Here  $c_1$  and  $c_2$  are constants depending on  $\pi$  and  $\pi'$ :

$$c_1 = \lim_{s \rightarrow 0} \left( -\frac{L'}{L}(s+1, \pi \times \tilde{\pi}') - \frac{1}{s} \right) - 1, \quad c_2 = -\frac{L'}{L}(1, \pi \times \tilde{\pi}').$$

A remarkable feature of this corollary is that it describes the orthogonality of  $a_\pi(n)$  and  $a_{\pi'}(n)$  in three cases with different main terms. It is thus in a more precise form than Selberg's Conjecture 1.2. Moreover, one can see from the last case of Corollary 3.3 that the Dirichlet series on the right side of (1.3) converges to  $L'/L(s, \pi \times \tilde{\pi}')$  on  $\text{Re } s = 1$ , when  $\pi$  and  $\pi'$  are not twisted equivalent.

Note that in Theorem 3.2 and Corollary 3.3, we have to assume that  $\pi \cong \tilde{\pi}$  and  $\pi' \cong \tilde{\pi}'$ . This is because a standard zero-free region of the type of de la Vallée Poussin is only available for self contragredient representations (Moreno [20] [21], Sarnak [27], and Gelbart, Lapid, and Sarnak [3]). On the other hand, our Theorems 2.1 and 2.2, together with Corollary 2.3, hold for all representations, not necessarily self contragredient, as we did not use zero-free regions in their proofs.

#### 4. SOC WITHOUT GRC

In [14], we proved SOC without GRC. To this end, we firstly proved the following weighted PNT.

**Theorem 4.1.** *Let  $\pi$  and  $\pi'$  be irreducible unitary cuspidal representations of  $GL_m(\mathbb{Q}_\mathbb{A})$  and  $GL_{m'}(\mathbb{Q}_\mathbb{A})$ , respectively. Assume that  $\pi$  and  $\pi'$  are self contragredient. Then*

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) \Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n) = \begin{cases} \frac{x^{1+i\tau_0}}{(1+i\tau_0)(2+i\tau_0)} + O\{x \exp(-c\sqrt{\log x})\} \\ \text{if } \pi' \cong \pi \otimes \alpha^{i\tau_0} \text{ for some } \tau_0 \in \mathbb{R}; \\ O\{x \exp(-c\sqrt{\log x})\} \\ \text{if } \pi' \not\cong \pi \otimes \alpha^{it} \text{ for any } t \in \mathbb{R}. \end{cases}$$

If  $\tau_0 = 0$ , i.e. if  $\pi \cong \pi'$ , then  $a_\pi(n) = a_{\pi'}(n)$ , and hence Theorem 4.1 states that

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) \Lambda(n) |a_\pi(n)|^2 = \frac{x}{2} + O\{x \exp(-c\sqrt{\log x})\}. \quad (4.1)$$

Now  $\Lambda(n)|a_\pi(n)|^2$  is non-negative. By a classical argument of de la Vallée Poussin, we can remove the weight  $1 - n/x$  from (4.1), to get the following PNT for automorphic representations.

**Corollary 4.2.** *Let  $\pi$  be as in Theorem 4.1. Then*

$$\sum_{n \leq x} \Lambda(n) |a_\pi(n)|^2 = x + O\{x \exp(-c\sqrt{\log x})\}.$$

In general, we could not remove the weight  $1 - n/x$  from Theorem 4.1. But similar to Theorem 4.1, in [14] we established

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) \frac{\Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n)}{n} = \begin{cases} \log x + c_1 + O\{\exp(-c\sqrt{\log x})\} \\ \text{if } \pi' \cong \pi; \\ \frac{x^{i\tau_0}}{i\tau_0(1+i\tau_0)} + c_2 + O\{\exp(-c\sqrt{\log x})\} \\ \text{if } \pi' \cong \pi \otimes \alpha^{i\tau_0} \text{ for some } \tau_0 \in \mathbb{R}^\times; \\ c_2 + O\{\exp(-c\sqrt{\log x})\} \\ \text{if } \pi' \not\cong \pi \otimes \alpha^{it} \text{ for any } t \in \mathbb{R}. \end{cases} \quad (4.2)$$

Here  $c_1$  and  $c_2$  are as in Corollary 3.3. This is more precise than Conjecture 1.3.

Using Corollary 4.2 and an idea of Landau [11], we were able to remove the weight  $1 - n/x$  from (4.2), to get SOC.

**Corollary 4.3.** *Conjecture 1.2 is true, provided that  $\pi$  and  $\pi'$  are self contragredient.*

The reason that we can remove  $1 - n/x$  from (4.2) is that now the main term is of order  $\log x$  when  $\pi' \cong \pi$ , which is substantially bigger than the  $O(1)$  of the case of  $\pi' \not\cong \pi$ .

## 5. PNT WITHOUT GRC

The classical Perron's formula gives us a formula for a sum of complex numbers  $a_n$ ,  $1 \leq n \leq x$ , in terms of their Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and bounds for individual terms  $a_n$ . Let  $A(x) > 0$  be non-decreasing such that  $a_n \ll A(n)$ , and let

$$B(\sigma) = \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} \quad (5.1)$$

for  $\sigma > \sigma_a$ , the abscissa of absolute convergence of  $\sum_{n=1}^{\infty} a_n/n^s$ . Then the classical Perron's formula (see e.g. Titchmarsh [30]) states that, for  $x = [x] + 1/2$ ,  $b > \sigma_a$  and  $T \geq 4$ ,

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + O\left(\frac{A(2x)x \log x}{T}\right) + O\left(\frac{x^b B(b)}{T}\right). \quad (5.2)$$

When applying (5.2) to the Riemann zeta-function or Dirichlet  $L$ -functions, bounds for  $a_n$  pose no problem. When applying this formula to other automorphic  $L$ -functions, however, bounds for  $a_n$  often require an assumption of GRC. Examples include our Theorem 3.2. In proving Theorem 3.2 by (5.2), we start from (1.3), and let

$$a_n = \Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n), \quad f(s) = -\frac{L'}{L}(s, \pi \times \bar{\pi}'). \quad (5.3)$$

Therefore, by Rudnick and Sarnak [26], the upper bound function  $A(n)$  for  $|a_n|$  can be taken as

$$A(n) = n^{1-1/(m^2+1)-1/(m'^2+1)}. \quad (5.4)$$

Obviously, (5.4) will make the first  $O$ -term in (5.2) too big. If we assume GRC, then instead of (5.4), we can take

$$A(n) = mm' \log n,$$

from which we deduce Theorem 3.2.

In [18], we proved a revised version of Perron's formula (Theorem 5.1 below). Different from the classical (5.2), the new Perron formula produces a formula for  $\sum_{n \leq x} a_n$  in terms of a sum of  $|a_n|$  over a short interval. While bounding individual Fourier coefficients  $|a_\pi(n)|$  of an automorphic cuspidal representation is hard and may require GRC, estimation of a sum of  $|a_\pi(n)|$  can usually be done by the Rankin-Selberg method. The new Perron's formula thus allows us to prove certain results for automorphic  $L$ -functions without assuming GRC.

**Theorem 5.1.** *Let  $\{a_n\}_{n=1}^{\infty}$  be complex numbers and let the series  $f(s) = \sum_{n=1}^{\infty} a_n/n^s$  be absolutely convergent for  $\sigma > \sigma_a$ . Let  $B(\sigma)$  be as in (5.1). Then, for  $x = [x] + 1/2$ ,  $b > \sigma_a$  and  $T \geq 4$ ,*

$$\begin{aligned} \sum_{n \leq x} a_n &= \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + O\left\{ \sum_{x-x/\sqrt{T} < n \leq x+x/\sqrt{T}} |a_n| \right\} \\ &\quad + O\left\{ \frac{x^b B(b)}{\sqrt{T}} \right\}. \end{aligned}$$

We remark that Theorem 5.1 can be used to derive the classical PNT. In fact, taking  $a_n = \Lambda(n)$ , we have

$$\sum_{x-x/\sqrt{T} < n \leq x+x/\sqrt{T}} |a_n| \ll \log x \quad \sum_{x-x/\sqrt{T} < n \leq x+x/\sqrt{T}} 1 \ll \frac{x \log x}{\sqrt{T}},$$

and, for  $\sigma > \sigma_a = 1$ ,

$$B(\sigma) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} \ll \frac{1}{\sigma-1}.$$

Therefore, Theorem 5.1 with  $b = 1 + 1/\log x$  gives

$$\sum_{n \leq x} \Lambda(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \left\{ -\frac{\zeta'(s)}{\zeta(s)} \right\} \frac{x^s}{s} ds + O\left\{ \frac{x \log x}{\sqrt{T}} \right\}.$$

We can take  $T = \exp(\sqrt{\log x})$ . The classical PNT now follows from the zero-free region of the Riemann zeta-function and a standard contour-integration argument.

As another application, we proved in [18] a prime number theorem (Theorem 5.2) unconditionally for Rankin-Selberg  $L$ -functions  $L(s, \pi \times \bar{\pi}')$ , by removing the assumption of GRC in Theorem 3.2.

**Theorem 5.2.** *In Theorem 3.2, the assumption of GRC can be removed.*

To prove Theorem 5.2, we apply Corollary 4.2 to obtain a bound for the short sum

$$\sum_{x < n \leq x+y} \Lambda(n) |a_{\pi}(n)|^2 \ll y$$

for  $y \gg x \exp(-c\sqrt{\log x})$ . Let  $a_n$  be as in (5.3); then for the above  $y$ ,

$$\begin{aligned} \sum_{x < n \leq x+y} |a_n| &\ll \left\{ \sum_{x < n \leq x+y} \Lambda(n) |a_{\pi}(n)|^2 \right\}^{1/2} \left\{ \sum_{x < n \leq x+y} \Lambda(n) |a_{\pi'}(n)|^2 \right\}^{1/2} \\ &\ll y. \end{aligned}$$

Now let  $T = \exp(\sqrt{\log x})$ . Then

$$\sum_{x-x/\sqrt{T} < n \leq x+x/\sqrt{T}} |a_n| \ll \frac{x}{\sqrt{T}}. \quad (5.5)$$

On the other hand,

$$B(\sigma) = \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma}} \ll \left\{ \sum_{n=1}^{\infty} \frac{\Lambda(n) |a_{\pi}(n)|^2}{n^{\sigma}} \right\}^{1/2} \left\{ \sum_{n=1}^{\infty} \frac{\Lambda(n) |a_{\pi'}(n)|^2}{n^{\sigma}} \right\}^{1/2}. \quad (5.6)$$

By Corollary 4.2,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\Lambda(n) |a_{\pi}(n)|^2}{n^{\sigma}} &= \int_1^{\infty} \frac{1}{u^{\sigma}} d\left\{ \sum_{n \leq u} \Lambda(n) |a_{\pi}(n)|^2 \right\} \\ &= \int_1^{\infty} \frac{du}{u^{\sigma}} + \int_1^{\infty} \frac{1}{u^{\sigma}} dr(u), \end{aligned} \quad (5.7)$$

where  $r(u) \ll u \exp(-c\sqrt{\log u})$ . The last integral in (5.7) is  $O(1)$ , while the first one is  $O\{1/(\sigma-1)\}$ . Consequently (5.6) gives us

$$B(\sigma) \ll \frac{1}{\sigma-1}. \quad (5.8)$$

Without loss of generality, we may assume  $x = [x] + 1/2$ . Now we may apply Theorem 5.1 with  $b = 1 + 1/\log x$  and  $T = \exp(\sqrt{\log x})$  to  $\sum_{n \leq x} a_n$ . By (5.5) and (5.8), we get

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \left\{ -\frac{L'}{L}(s, \pi \times \tilde{\pi}') \right\} \frac{x^s}{s} ds + O\{x \exp(-c\sqrt{\log x})\}. \quad (5.9)$$

Now we can shift the contour in (5.9) to the left, apply the zero-free region of Moreno [20] [21], Sarnak [27], and Gelbart, Lapid, and Sarnak [3], and estimate the resulting sums over zeros and poles. Theorem 5.2 then follows.

**Corollary 5.3.** *In Corollary 3.3, the assumption of GRC can be removed.*

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