

Arithmetical properties of solutions of certain q -difference equations

Masaaki Amou (Gunma University)

天羽雅昭 (群馬大工)

In the present note we report certain results obtained by the author with Keijo Väänänen on the linear independence of the values of theta series and Tschakaloff functions.

1

Let $\Theta(q, z)$ be the theta series defined by

$$(1) \quad \Theta(q, z) = \sum_{n=-\infty}^{\infty} q^{n^2} z^n, \quad |q| < 1,$$

and let $T_q(z)$ be the Tschakaloff function defined by

$$(2) \quad T_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{q^{\binom{n}{2}}}, \quad |q| > 1.$$

If $qq = 1$, then these functions are connected by the equation

$$(3) \quad \Theta(q, z) = T_{q^2}(q^{-1}z) + T_{q^2}(q^{-1}z^{-1}) - 1.$$

It was shown in [3] and [5] that Nesterenko's result [6] on the values of Ramanujan functions implies the transcendence of $\Theta(q, 1)$ for all algebraic q , $0 < |q| < 1$ and that of $T_q(1)$ for all algebraic q , $|q| > 1$. However there are no such results for $\Theta(q, z)$ with general algebraic q , even if we take some special value $q = 1/q$, $q \in \mathbf{Z} \setminus \{0, \pm 1\}$ for q . On the other hand, since the paper [8] of Tschakaloff there are a lot of works containing linear independence results on the values of $T_q(z)$ and its derivatives, a good overview is given in [4]. In particular, it was proved in [8] if q is an integer in an imaginary quadratic field $\mathcal{I}$, $|q| > 1$, and $\alpha_1, \dots, \alpha_\ell$ are nonzero elements of $\mathcal{I}$ satisfying $\alpha_i/\alpha_j \notin q^{\mathbf{Z}}$ for all $i \neq j$, then the numbers $1, T_q(\alpha_1), \dots, T_q(\alpha_\ell)$ are linearly independent over $\mathcal{I}$, see also [9] for some results on more general algebraic number

fields. By (3), one can obtain linear independence of $1, \Theta(\mathbf{q}, \alpha_1), \dots, \Theta(\mathbf{q}, \alpha_\ell)$ under some extra conditions on α_i , see [7], Korollar 2.

In the present note we are interested in the linear independence of the numbers

$$1, \Theta(\mathbf{q}_1, \alpha_1), \dots, \Theta(\mathbf{q}_\ell, \alpha_\ell)$$

with different $\mathbf{q}_i$. As far as we know there are no earlier results of this type.

2

In the following we assume that $q \in \mathbb{Z} \setminus \{0, \pm 1\}$.

Theorem 1. *Let ℓ and L be positive integers with $L \geq 2(\ell - 1)$. Then there exists an effectively computable positive constant $\gamma_1(\ell, L)$ such that, for any positive integers $s_1, \dots, s_\ell$ satisfying*

$$\gamma_1(\ell, L) \leq s_1 < s_2 < \dots < s_\ell \leq s_1 + L/2$$

and for any nonzero rational numbers $\alpha_1, \dots, \alpha_\ell$ satisfying

$$\alpha_i \notin -q^{s_i(1+2\mathbb{Z})} \quad (i = 1, \dots, \ell),$$

the numbers

$$1, \Theta(q^{-s_1}, \alpha_1), \dots, \Theta(q^{-s_\ell}, \alpha_\ell)$$

are linearly independent over the rationals.

This result is a consequence of the following result on the values of Tschakaloff functions.

Theorem 2. *Let ℓ and L be positive integers with $L \geq \ell - 1$, and let $m_1, \dots, m_\ell$ be positive integers. Then there exists an effectively computable positive constant $\gamma_2(m, L)$ with $m = m_1 + \dots + m_\ell$ such that, for any positive integers $s_1, \dots, s_\ell$ satisfying*

$$\gamma_2(m, L) \leq s_1 < s_2 < \dots < s_\ell \leq s_1 + L$$

and for any nonzero rational numbers α_{ij} ($i = 1, \dots, \ell$; $j = 1, \dots, m_i$) satisfying

$$\alpha_{ij_1} / \alpha_{ij_2} \notin q^{s_i \mathbb{Z}} \quad (i = 1, \dots, \ell; j_1 \neq j_2),$$

the numbers

$$1, T_{q^{s_i}}(\alpha_{ij}) \quad (i = 1, \dots, \ell; j = 1, \dots, m_i)$$

are linearly independent over the rationals.

We here deduce Theorem 1 from Theorem 2. Denoting $q_i = q^{s_i}$, we have by (3)

$$\Theta(q_i^{-1}, \alpha_i) = T_{q_i^2}(q_i^{-1}\alpha_i) + T_{q_i^2}(q_i^{-1}\alpha_i^{-1}) - 1 \quad (i = 1, \dots, \ell).$$

Since

$$\frac{q_i^{-1}\alpha_i}{q_i^{-1}\alpha_i^{-1}} = \alpha_i^2 \notin (q_i^2)^{\mathbf{Z}} \iff \alpha_i \notin \pm q_i^{\mathbf{Z}},$$

the statement of Theorem 1 follows from Theorem 2 under the right-hand conditions in the above. Therefore, our task is to relax these conditions into the conditions given in the theorem. To this aim let us fix i and assume $\alpha_i \in \pm q_i^{\mathbf{Z}}$. Then we have $\alpha_i = \epsilon q_i^{-n}$ for some nonnegative integer n assuming $|\alpha_i| \leq 1$ without loss of generality, where ϵ is 1 or -1 . (We may replace the role of α_i by that of α_i^{-1} if necessary.) By denoting $\beta_i = q_i^{-1}\alpha_i$ this gives $q_i^{-1}\alpha_i^{-1} = \beta_i q_i^{2n}$. Since the repeated application of the functional equation

$$T_{q_i^2}(q_i^2 z) = q_i^2 z T_{q_i^2}(z) + 1$$

for $T_{q_i^2}(z)$ implies

$$T_{q_i^2}(q_i^{2n} z) = q_i^{2\binom{n+1}{2}} z^n T_{q_i^2}(z) + P_{in}(z), \quad P_{in}(z) \in \mathbf{Z}[z],$$

on noting $\beta_i = \epsilon q_i^{-(n+1)}$, we have

$$T_{q_i^2}(q_i^{-1}\alpha_i^{-1}) = q_i^{2\binom{n+1}{2}} \beta_i^n T_{q_i^2}(\beta_i) + P_{in}(\beta_i) = \epsilon^n T_{q_i^2}(\beta_i) + P_{in}(\beta_i).$$

Hence,

$$\Theta(q_i^{-1}, \alpha_i) = \begin{cases} P_{in}(\beta_i) - 1, & \epsilon = -1 \text{ and } n \text{ is odd,} \\ 2T_{q_i^2}(\beta_i) + P_{in}(\beta_i) - 1, & \text{otherwise,} \end{cases}$$

which completes the deduction of Theorem 1 from Theorem 2.

For the proof of Theorem 2 we use the method given in [2] and [10]. More precisely, we define

$$f_{ij}(z) = \sum_{n=0}^{\infty} \frac{z^{s_i n}}{q^{s_i \binom{n+1}{2}}} \alpha_{ij}^n \quad (i = 1, \dots, \ell; j = 1, \dots, m_i),$$

which satisfy the q -difference equations

$$\alpha_{ij} z^{s_i} f_{ij}(z) = f_{ij}(qz) - 1.$$

Since $f_{ij}(q) = T_{q^{s_i}}(\alpha_{ij})$, the statement of Theorem 2 follows from the linear independence over the rationals of the numbers

$$1, f_{ij}(q) \quad (i = 1, \dots, \ell; j = 1, \dots, m_i).$$

To prove this statement we construct Padé-type approximations of the second kind for $f_{ij}(z)$ taking the conditions $s_1 < s_2 < \dots < s_\ell$ into account (see [1] for details).

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