

ERROR ESTIMATES OF THE REAL INVERSION FORMULAS OF THE LAPLACE TRANSFORM(abstract)

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INTRODUCTION AND RESULTS

For any $q > 0$, we let L_q^2 be the class of all square integrable functions with respect to the measure $t^{1-2q}dt$ on the half line $(0, \infty)$. Then we consider the Laplace transform

$$[\mathcal{L}F](x) = \int_0^\infty F(t)e^{-xt}dt \quad (x > 0)$$

for $F \in L_q^2$. Then we have

Proposition 1 ([2, 5]). *For any fixed $q > 0$ and for any function $F \in L_q^2$, put $f = \mathcal{L}F$. Then the inversion formula*

$$(1) \quad F(t) = s - \lim_{N \rightarrow \infty} \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)dx \quad (t > 0)$$

is valid, where the limit is taken in the space L_q^2 and the polynomials $P_{N,q}$ are given by the formulas

$$P_{N,q}(\xi) = \sum_{0 \leq \nu \leq n \leq N} \frac{(-1)^{\nu+1}\Gamma(2n+2q)}{\nu!(n-\nu)!\Gamma(n+2q+1)\Gamma(n+\nu+2q)} \xi^{n+\nu+2q-1} \\ \times \left\{ \frac{2(n+q)}{n+\nu+2q} \xi^2 - \left(\frac{2(n+q)}{n+\nu+2q} + 3n+2q \right) \xi + n(n+\nu+2q) \right\}.$$

Moreover the series

$$(2) \quad \sum_{n=0}^\infty \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n [xf'(x)]|^2 x^{2n+2q-1} dx$$

converges and the truncation error is estimated by the inequality

$$(3) \quad \left\| F(t) - \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)dx \right\|_{L_q^2}^2 \\ \leq \sum_{n=N+1}^\infty \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n [xf'(x)]|^2 x^{2n+2q-1} dx.$$

Some characteristics of the strong singularity of the polynomials $P_{N,1}(\xi)$ and some effective algorithms for the real inversion formula in Proposition 1 are examined by J. Kajiwara and M. Tsuji [3, 4] and K. Tsuji [6]. Furthermore they gave numerical experiments by using computers.

In connection with the integral in (2) we have

Proposition 2 ([5], Chapter 5). *Let $q > 0$ be arbitrary and let $F \in L_q^2$. For the Laplace transform $\mathcal{L}F = f$, we have the isometrical identity*

$$(4) \quad \int_0^\infty |F(t)|^2 t^{1-2q} dt = \sum_{n=0}^\infty \frac{1}{n! \Gamma(n+2q+1)} \int_0^\infty |\partial_x^n [x f'(x)]|^2 x^{2n+2q-1} dx.$$

Moreover the image $f = \mathcal{L}F$ belongs to the Bergman-Selberg space $H_q(R^+)$ on the right half complex plane $R^+ = \{ \operatorname{Re} z > 0 \}$ admitting the reproducing kernel

$$K_q(z, \bar{u}) = \frac{\Gamma(2q)}{(z + \bar{u})^{2q}}$$

and comprising analytic functions on R^+ . For $q > \frac{1}{2}$, we can characterize

$$H_q(R^+) = \{ f : f \text{ analytic on } R^+, \\ \frac{1}{\Gamma(2q-1)\pi} \iint_{R^+} |f(z)|^2 (2x)^{2q-2} dx dy < \infty \}$$

and for $q = \frac{1}{2}$

$$H_{\frac{1}{2}}(R^+) = \{ f : f \text{ analytic on } R^+, \\ \lim_{x \rightarrow +0} \frac{1}{2\pi} \int_{-\infty}^\infty |f(x+iy)|^2 dy < \infty \}.$$

Moreover for any $q > 0$, we have the representation of the norm in $H_q(R^+)$

$$(5) \quad \|f\|_{H_q(R^+)}^2 = \sum_{n=0}^\infty \frac{1}{n! \Gamma(n+2q+1)} \int_0^\infty |\partial_x^n (x f'(x))|^2 x^{2n+2q-1} dx.$$

Now we can state our main results.

Theorem 1. *We assume that*

$$(6) \quad \max\left(\frac{1}{2}, 2q-1\right) < \alpha < 1,$$

and

$$(7) \quad \alpha \leq \beta < q + \frac{\alpha}{2}.$$

If $f \in H_q(R^+)$ and

$$(8) \quad f(z)z^\beta \in H_{q+\frac{\alpha}{2}-\beta}(R^+),$$

then the following error estimate holds

$$(9) \quad \left| F(t) - \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)dx \right| = t^{q-1+\frac{\alpha}{2}} o\left(N^{\frac{1-2\alpha}{4}}\right)$$

as $N \rightarrow \infty$.

Next we give a sufficient condition for F whose Laplace transform satisfies (8).

Theorem 2. *Let us assume (7). We further assume*

$$(10) \quad q + \frac{\alpha}{2} > 1.$$

If

$$(11) \quad F \in C^2[0, \infty),$$

$$(12) \quad F(0) = F'(0) = 0,$$

and

$$(13) \quad F'(t) = O(t^{-\delta}), \quad t > 0$$

for

$$(14) \quad 2 - q - \frac{\alpha}{2} < \delta < 1,$$

then (8) holds.

Note that from (12) and (13)

$$(15) \quad \lim_{t \rightarrow \infty} e^{-xt} F(t) = \lim_{t \rightarrow \infty} e^{-xt} F'(t) = 0, \quad x > 0.$$

Finally, we characterize F whose Laplace transform satisfies (8).

Theorem 3. *If $f = \mathcal{L}F$ satisfies (8), then there exists $h \in L^2_{q+\frac{\alpha}{2}-\beta}$ such that (7) is true and*

$$(16) \quad F(t) = \int_0^t h(x)(t-x)^{\beta-1} dx.$$

A real inversion formula for the Laplace transform is known (eg. Widder [7], page 386), which is different from ours. However it seems that no error estimates in the truncation are known.

PRELIMINARIES

First we shall give

Lemma. *If $f \in C^\infty(0, \infty)$ and*

$$(17) \quad I_{q,\alpha}(f) := \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n+2q+1)} \int_0^{\infty} |\partial_x^n [x f'(x)]|^2 x^{2n+2q-1+\alpha} dx < \infty,$$

for fixed

$$(18) \quad \max\left(\frac{1}{2}, 2q-1\right) < \alpha,$$

then

$$(19) \quad \left| \sum_{n=N+1}^{\infty} \frac{1}{n! \Gamma(n+2q+1)} \times \int_0^{\infty} \partial_x^n [x f'(x)] \partial_x^n (x \partial_x (e^{-tx})) x^{2n+2q-1} dx \right| \\ = t^{\frac{\alpha-2q}{2}} o\left(N^{\frac{1-2\alpha}{4}}\right),$$

as $N \rightarrow \infty$.

CONCLUDING REMARKS

(1) The conditions (12) and (13) are not essential if we know $F(0)$ and $F'(0)$, and we can assume that

$$(20) \quad |F(t)|, |F'(t)| \leq O(e^{kt}) \quad \text{for } t > 0 \quad \text{with } k > 0.$$

In fact, we set

$$(21) \quad \tilde{F}(t) = (F(t) - F(0) - F'(0)t)e^{-2kt}, \quad t > 0.$$

Then $\tilde{F}$ satisfies (12) and (13).

On the other hand,

$$(22) \quad (\mathcal{L}\tilde{F})(z) = f(z+2k) - \frac{F(0)}{z+2k} - \frac{F'(0)}{(z+2k)^2}.$$

Thus we first apply Theorems 1 and 2 to this function (22) so that we can obtain approximations $\tilde{F}_N(t)$ for $\tilde{F}(t)$:

$$(23) \quad |\tilde{F}(t) - \tilde{F}_N(t)| = t^{q-1+\frac{q}{2}} o(N^{\frac{1-2\alpha}{4}}).$$

We set

$$(24) \quad \hat{F}_N(t) = \tilde{F}_N(t)e^{2kt} + F(0) + F'(0)t, \quad \text{for } t > 0.$$

Then we have

$$(25) \quad |F(t) - \hat{F}_N(t)| = e^{2kt} |\tilde{F}(t) - \tilde{F}_N(t)| = e^{2kt} t^{q-1+\frac{q}{2}} o(N^{\frac{1-2\alpha}{4}}).$$

Thus we can obtain error estimates in any finite interval in t , which however breaks as $t \rightarrow \infty$.

(2) Since a typical member of the Bergman-Selberg space $H_q(R^+)$ is the reproducing kernel $K_q(z, \bar{u})$, we see that typical functions f satisfying (17) are given by

$$(26) \quad f(z) = \frac{z^{-\beta}}{(z + \bar{u})^{2q+\alpha-2\beta}}, \quad \text{Re } u > 0$$

for α and β satisfying (7). From the identities (16) and

$$K_{q+\frac{q}{2}-\beta}(z, \bar{u}) = \int_0^\infty e^{-tz} e^{-t\bar{u}} t^{2q+\alpha-2\beta-1} dt,$$

we see that the Laplace transform of the functions

$$(27) \quad \int_0^t e^{-x\bar{u}} x^{2q+\alpha-2\beta-1} (t-x)^{\beta-1} dx, \quad \text{Re } u > 0, \beta > 1$$

satisfies the property (17).

(3) As functions F where $f = \mathcal{L}F$ satisfies the conditions in Theorem 1, we consider Dirichlet series

$$(28) \quad F(t) = \sum_{k=1}^{\infty} C_k t^{\gamma-1} e^{-a_k t} \quad (a_k > 0, \gamma \geq 1),$$

where

$$(29) \quad \sum_{k=1}^{\infty} |C_k| a_k^{q-\gamma} < \infty, \quad \sum_{k=1}^{\infty} |C_k| a_k^{q+\frac{q}{2}-\gamma} < \infty, \quad \gamma > q + \frac{\alpha}{2} > 0.$$

Then $F \in L_q^2$ and $f = \mathcal{L}F$ satisfies (8) for β satisfying (7).

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