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INTERPOLATING CARLITZ ZETA VALUES

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This note is a survey of some results obtained in collaboration with B. Anglès and F. Tavares Ribeiro [5] on a new class of L-series arising in the theory of function fields of positive characteristic recently introduced in [14]. Complete proofs and wider investigations can be found in [4, 5].

1. PRELIMINARIES.

We set $A=\mathbb{F}_q[\theta]$, $K=\mathbb{F}_q(\theta)$, $K_\infty=\mathbb{F}_q((\theta^{-1}))$ and we denote by $\mathbb{C}_\infty$ the completion of an algebraic closure of K_∞.

The *Carlitz zeta values* are the series

(1) $$\zeta_C(n):=\sum_{a\in A^+}a^{-n}\in K_\infty,\quad n\geq 1,$$

where the sum runs over the set A^+ of monic polynomials. In analogy with the *classical zeta values*

$$\zeta(n)=\sum_{i\geq 1}i^{-n}$$

with n integer (convergence occurs only if $n\geq 2$).

It was proved by Carlitz [8] that, if $n\equiv 0 \pmod{q-1}$,

(2) $$\zeta_C(n)\in K^\times\widetilde{\pi}^n,$$

where $\widetilde{\pi}$ is the value in $\mathbb{C}_\infty$ of an infinite product

(3) $$\widetilde{\pi}:=-(-\theta)^{\frac{q}{q-1}}\prod_{i=1}^\infty(1-\theta^{1-q^i})^{-1}\in(-\theta)^{\frac{1}{q-1}}K_\infty,$$

uniquely defined up to the multiplication by an element of $\mathbb{F}_q^\times=\mathbb{F}_q\setminus\{0\}$ (corresponding to the choice of a root $(-\theta)^{\frac{1}{q-1}}$). We notice that $v_\infty(\widetilde{\pi})=-\frac{q}{q-1}$, where v_∞ is the valuation of $\mathbb{C}_\infty$ (so that $v_\infty(\theta)=-1$).

The element $\widetilde{\pi}$ is a fundamental period of the *Carlitz exponential* $\exp_C$ (Goss, [11, §3.2]), that is, the unique surjective, entire, $\mathbb{F}_q$-linear function

$$\exp_C:\mathbb{C}_\infty\to\mathbb{C}_\infty$$

of kernel $\widetilde{\pi}\mathbb{F}_q[\theta]$ such that its first derivative satisfies $\exp_C'=1$.

We have the following arithmetical analogy between the Carlitz zeta values $\zeta_C(n)\in K_\infty^\times$ $(n\geq 1)$ and the special values $\zeta(n)$ $(n\geq 2)$ of Riemann's zeta function, which was pointed out by Lenny Taelman.

For $n\geq 1$, we have the functor of Quillen K-theory K_{2n-1}, which, evaluated at $\mathbb{F}_p$, gives the finite group $K_{2n-1}(\mathbb{F}_p)$, cyclic of cardinality p^n-1 (Quillen's theorem). Moreover, the evaluation $\mathrm{Lie}(K_{2n-1})(\mathbb{F}_p)$ of the functor $\mathrm{Lie}(K_{2n-1})$ [1] has cardinality $|\mathrm{Lie}(K_{2n-1})(\mathbb{F}_p)| = p^n$ (this can be deduced, for example, from the paper of Hesselholt and Madsen [12, Theorem E]). Now, this yields the Eulerian product

$$\zeta(n) = \prod_p \left(\frac{|\mathrm{Lie}(K_{2n-1})(\mathbb{F}_p)|}{|K_{2n-1}(\mathbb{F}_p)|} \right)$$

which diverges of course for $n = 1$. We note that the cardinalities above can also be viewed as positive generators of Fitting ideals of finite $\mathbb{Z}$-modules.

We set $A = \mathbb{F}_q[\theta]$. The *Carlitz module* C is the functor from A-algebras to A-modules which sends an A-algebra $\mathcal{A}$ to the unique A-module which has $\mathcal{A}$ as underlying abelian group, and such that the (left) multiplication by θ of an element x of $\mathcal{A}$ is $C_\theta(x) = \theta x + x^q$.

Let P be a *prime* of A (that is, a monic irreducible polynomial of A). To the A-algebra A/PA, we can associate the A-module $C(A/PA)$, which is a finite A-module, to which we can associate the unique monic generator of its Fitting ideal $[C(A/PA)]_A$. In virtue of Goss, [11, Theorem 3.6.3], we have

$$[C(A/PA)]_A = P - 1.$$

More generally, Anderson and Thakur have introduced in [3], for $n \geq 1$, a t-module called the *n-th tensor power* of the Carlitz module C, denoted by $C^{\otimes n}$, which allows to extend the above formula for $\zeta(n)$ in our framework. Indeed, for all $n \geq 1$ and P a prime of A, the A-module $C^{\otimes n}(A/PA)$ is finite and the monic generator of its Fitting ideal is $P^n - 1$ [3, Proposition 1.10.3]. Furthermore, $[\mathrm{Lie}(C^{\otimes n})(A/PA)]_A = P^n$ and

$$\zeta_C(n) = \prod_P \left(\frac{[\mathrm{Lie}(C^{\otimes n})(A/PA)]_A}{[C^{\otimes n}(A/PA)]_A} \right),$$

convergence being ensured even with $n \geq 1$.

The value $\zeta_C(1)$ is somewhat distinguished also because its classical counterpart $\zeta(1)$ is a divergent series. If $q = 2$, then Carlitz result (2) implies that $\zeta_C(1) \in K^\times \widetilde{\pi}$ so that $\exp_C(\zeta_C(1))$ is a torsion point for C in this case. A little computation shows that $\zeta_C(1) = \frac{\widetilde{\pi}}{\theta(\theta+1)}$ so that $\exp_C(\zeta_C(1))$ is a point of $\theta(\theta+1)$-torsion and in fact, we find $\exp_C(\zeta_C(1)) = 1$ (note that if $q = 2$, $C_{\theta(\theta+1)}(1) = (\theta^2 + \theta + 1 + (\theta^2 + \theta)\tau + \tau^2)(1) = 0$; if $q > 2$, 1 is always a point of infinite order).

In [8], Carlitz proves that

$$\exp_C(\zeta_C(1)) = 1 \tag{4}$$

for *all* q; this is a completely different relation, if compared with (2).

Taelman [17] recently exhibited an appropriate setting to interpret the above formula as an instance of the *class number formula*. He worked more generally in the framework of *Drinfeld modules* defined over the ring of integers R of a finite extension L of K.

Taelman associated to each such Drinfeld module ϕ a finite A-module $H(\phi/R)$ called the *class module* and a finitely generated A-module $U(\phi/R)$ called the *unit module*. Taelman also introduced, for each such Drinfeld module ϕ/R, an *L-series value*

$$L(\phi/R) = \prod_{\mathfrak{m}} \left(\frac{[\mathrm{Lie}(\phi)(R/\mathfrak{m}R)]_A}{[\phi(R/\mathfrak{m}R)]_A} \right),$$

[1] We recall that if F : (Rings) $\to$ (Ab. groups) is a functor, $\mathrm{Lie}(F)$ denotes the functor $\mathrm{Ker}(F(A[\epsilon]) \to F(A))$, $\epsilon \mapsto 0$, where $A[\epsilon]$ denotes the ring of dual numbers.

where the product runs over the maximal ideals of R (the convergence can be checked easily). Taelman fundamental Theorem [17, Theorem 1] states that

$$L(\phi/R) = [H(\phi/R)]_A \operatorname{Reg}(U(\phi/R)),$$

where $\operatorname{Reg}(U(\phi/R))$ denotes a *regulator* of the unit module defined by Taelman. It is easy to see that $L(\phi/R)$ becomes $\zeta_C(1)$ in the case of $\phi = C$ and $R = A$. In particular, since $\exp_C$ induces an isometry of the disk $\{z \in \mathbb{C}_\infty; v_\infty(z) > -\frac{q}{q-1}\}$, the class A-module

$$H(C/A) = \frac{C(K_\infty)}{\exp_C(K_\infty) + C(A)}$$

is trivial. For similar reasons, the unit A-module

$$U(C/A) = \{f \in K_\infty; \exp_C(f) \in C(A)\}$$

is the free A-submodule of K_∞ generated by $\log_C(1)$, the *Carlitz logarithm* evaluated at one (this is the local composition inverse of $\exp_C$ at 0 and converges at one). From this, Carlitz formula (4) follows.

A generalization of Taelman's Theorem was recently considered by Jiangxue Fang [10] to certain L-series values associated to *Anderson's t-modules.* If E is a t-module defined over R (the ring of integers of L a finite extension of K), the definition of $L(E/R)$ is formally the same as Taelman's for Drinfeld modules, and we have $L(C^{\otimes n}/A) = \zeta_C(n)$. Fang's Theorem [10, Theorem 1.7] states a generalization of Taelman's class number formula in this setting. His results makes a fundamental use of the machinery of shtukas as in Lafforgue's paper [13].

2. Results.

In the preprint [5], we have generalized the formulas (2) and (4) in a different direction. For the sake of simplicity, we are now going to present a particular case of our results. For this purpose, we are going to introduce a generalization of the Carlitz module functor.

2.1. **The Carlitz functor revisited.** Let $t_1, \ldots, t_s$ be indeterminates, let us denote by A_s the polynomial algebra $A[t_1, \ldots, t_s]$. Let $\mathbb{T}_s$ be the standard Tate algebra of dimension s, that is, the completion of the polynomial algebra $\mathbb{C}_\infty[t_1, \ldots, t_s]$ for the Gauss norm $\|\cdot\|$ associated to the absolute value $|\cdot|$ of $\mathbb{C}_\infty$ uniquely normalized by setting $|\theta| = q^{-v_\infty(\theta)} = q$. We fix once and for all the embedding $A[t_1, \ldots, t_s] \subset \mathbb{T}_s$ determined by the embedding $A \subset \mathbb{C}_\infty$.

The Carlitz module $C(\mathbb{C}_\infty)$ over $\mathbb{C}_\infty$ extends in an unique way to an A_s-module $C(\mathbb{T}_s)$ (we allow a slight abuse of notation; $C(\mathbb{C}_\infty)$ is an A-module while $C(\mathbb{T}_s)$ is an A_s-module, but this will not lead to confusion). Explicitly, $C(\mathbb{T}_s)$ is the unique A_s-module having $\mathbb{T}_s$ with the usual multiplication as the underlying $\mathbb{F}_q[t_1, \ldots, t_s]$-module, and such that the (left) multiplication of an element $x \in \mathbb{T}_s$ by θ, denoted by $C_\theta(x)$, is $\theta x + \tau(x)$, where

$$\tau : \mathbb{T}_s \to \mathbb{T}_s$$

represents the $\mathbb{F}_q[t_1, \ldots, t_s]$-linear extension of $\tau : \mathbb{C}_\infty \to \mathbb{C}_\infty$.

To give a concrete example, let us consider $f = t_1 - \theta$, which belongs to A_1 hence to $\mathbb{T}_1$. Then, $\tau(f) = t_1 - \theta^q$ and $C_\theta(f) = t_1(\theta + 1) - (\theta^2 + \theta^q)$. In the case of $s = 1$, we also prefer to write $\mathbb{T} = \mathbb{T}_1$ and $t = t_1$.

Since τ induces a continuous automorphism of $\mathbb{T}_s$ for all s, there is an unique $\mathbb{F}_q[t_1, \ldots, t_s]$-linear extension

$$\exp_C : \mathbb{T}_s \to \mathbb{T}_s$$

of $\exp_C : \mathbb{C}_\infty \to \mathbb{C}_\infty$ which is a continuous, open $\mathbb{F}_q[t_1, \ldots, t_s]$-linear endomorphism of $\mathbb{T}_s$. We further have the following exact sequence of A_s-modules:

$$0 \to \widetilde{\pi} A_s \to \mathbb{T}_s \to C(\mathbb{T}_s) \to 0.$$

Here, the third arrow is $\exp_C$, and it is understood that

$$C_a(\exp_C(f)) = \exp_C(af)$$

for all $a \in A_s$.

2.2. **Torsion.** The above function $\exp_C$ has quite a rich torsion structure. If $f \in A_s$ is such that $f^{-1} \in \mathbb{T}_s$ (this means that f is a polynomial which has leading coefficient in $\mathbb{F}_q^\times$ as a polynomial in θ, or in other words, $f \in \mathbb{T}^\times$, group of units of $\mathbb{T}$), then

$$C_f\left(\exp_C\left(\frac{\widetilde{\pi}\theta^j}{f}\right)\right) = 0, \quad j = 0, \ldots, \deg_\theta(f) - 1.$$

It is easily seen, under the hypothesis that f is a unit of $\mathbb{T}_s$, that the functions $\exp_C\left(\frac{\widetilde{\pi}\theta^j}{f}\right)$ constitute an $\mathbb{F}_q[t_1, \ldots, t_s]$-basis of the submodule $\mathrm{Ker}(C_f) \subset C(\mathbb{T}_s)$, free of rank d.

These functions can be used to construct Galois representations

$$\mathrm{Gal}(K^{sep}/K) \to \mathrm{GL}_d(\mathbb{F}_q[[t_1, \ldots, t_s]])$$

(here, K^{sep} denotes the separable closure of K in $\mathbb{C}_\infty$). More generally, we can attach similar Galois representations to the torsion modules of *uniformizable Drinfeld modules of rank one defined over* A_s introduced in [5]. The simplest case is given by the *Anderson-Thakur function*, first introduced by Anderson and Thakur in [3]:

$$\omega = \exp_C\left(\frac{\widetilde{\pi}}{\theta - t}\right) \in \mathbb{T}^\times,$$

which is, by the above discussion, the generator of the $\mathbb{F}_q[t]$-module $\mathrm{Ker}(C_{\theta-t}) \subset \mathbb{T}$, free of rank one. Here, it is well known that the associated Galois representation

$$\mathrm{Gal}(K^{sep}/K) \to \mathrm{GL}_1(\mathbb{F}_q[[t]])$$

is surjective (use, for example [16, Theorem 0.2]). Since $\tau(\omega) = (t - \theta)\omega$ (this is equivalent to saying that $\omega \in \mathrm{Ker}(C_{\theta-t})$), we also deduce:

Proposition 1. *The following properties hold:*

1. *We have the product expansion*
$$\omega = (-\theta)^{\frac{1}{q-1}} \prod_{i \geq 0} \left(1 - \frac{t}{\theta^{q^i}}\right)^{-1},$$
convergent in $\mathbb{T}$.
2. ω, *as an element of* $\mathbb{T}$, *extends to a meromorphic function over* $\mathbb{C}_\infty$ *and has, as unique singularities, simple poles at the points* $t = \theta, \theta^q, \theta^{q^2}, \ldots$. *The residues can be explicitly computed. In particular, we have* $\mathrm{Res}_{t=\theta}(\omega) = -\widetilde{\pi}$.
3. *The function* $1/\omega$ *extends to an entire function* $\mathbb{C}_\infty \to \mathbb{C}_\infty$ *with unique zeros located at the poles of* ω.

2.3. **L-series values in $\mathbb{T}_s$.** We construct the *Carlitz zeta values* $\zeta_C(n;s)\in\mathbb{T}_s$, $n>0$. They are defined as follows for $n\geq 1$ an integer and $s\geq 0$:

$$\zeta_C(n;s)=\sum_{a\in A^+}a^{-n}a(t_1)\cdots a(t_s)\in\mathbb{T}_s\cap K_\infty[[t_1,\ldots,t_s]].$$

It is easy to show that $\zeta_C(n,s)\in\mathbb{T}_s^\times$ and that $\|\zeta_C(n;s)\|=1$. Carlitz zeta values are a special case of our construction with $s=0$. In [4] it is proved that, in terms of the variables $t_1,\ldots,t_s$, these series define entire functions $\mathbb{C}_\infty^s\to\mathbb{C}_\infty$. Therefore, evaluation at $t_i=\theta^{q^{k_i}}$, $i=1,\ldots,s$ and $k_i\in\mathbb{Z}$ makes sense and, for $n>0$,

$$\zeta_C(n)=\zeta_C(n;0)=\zeta_C(n+q^{k_1}+\cdots+q^{k_s};s)|_{t_i=\theta^{q^{k_i}}}.$$

In this respect, we can view these functions as *interpolations of Carlitz zeta values.*

In [5], we prove:

Theorem 2. *For $s\geq 0$, we have*

$$\exp_C(\zeta_C(1;s)\omega(t_1)\cdots\omega(t_s))=P_s\omega(t_1)\cdots\omega(t_s),$$

where $P_s\in A_s$. Moreover, for $s>1$, we have $P_s=0$ if and only if $s\equiv 1\pmod{q-1}$. In this case, we have

$$\zeta_C(1;s)=\frac{\widetilde{\pi}B_s}{\omega(t_1)\cdots\omega(t_s)},\tag{5}$$

with $B_s\in A_s$.

For $n=0$, we re-obtain Carlitz Theorem (4). The vanishing of P_s is equivalent to (5), since this means that $\zeta_C(1;s)\omega(t_1)\cdots\omega(t_s)$ is in the kernel of $\exp_C$. Of course, formula (5) can be viewed as a generalization of Carlitz Theorem (2) in the case $n=1$, but for various values of $s\equiv 1\pmod{q-1}$.

One of the ingredients of the proof of Theorem 2 is a variant of Taelman's *class number formula* for Drinfeld modules defined over integral closures of A in finite extensions of K [17, Theorem 1]. This was obtained by F. Demeslay and a particular case of his result (corresponding to what we need to prove Theorem 2) appears in the appendix of [5].

Demeslay's method is inspired by Taelman's proof in [17] and uses a generalization of the notion of Drinfeld module introduced in [5] (the extension to $\mathbb{T}_s$ of the Carlitz module is an example of this, but there exist many non-isomorphic Drinfeld modules of rank one over $\mathbb{T}_s$ as soon as $s\geq 1$).

In [5] we prove a generalization of Theorem 2 which holds for more general Drinfeld modules of rank one over $\mathbb{T}_s$, provided that they are defined over A_s. We point out that Demeslay is currently working on a generalisation of his class number formula which may well handle at once t-modules and Drinfeld A_s-modules (it would then encompass Fang's and Taelman's class number formulas).

Comparing (5) and (2) we are led to the following:

Question 3. Is it true that

$$\widetilde{\pi}^{-n}\zeta_C(n;s)\omega(t_1)\cdots\omega(t_s)\in K(t_1,\ldots,t_s)\tag{6}$$

if and only if $n\equiv s\pmod{q-1}$?

This question is also suggested by the results in [4], in which we prove that $s>1$ and $n\equiv s\pmod{q-1}$ imply (6). For example, in [14] it is proved that

$$\zeta_C(1;1)=\frac{\widetilde{\pi}}{(\theta-t)\omega(t)}.\tag{7}$$

Proposition 1, which provides analogies between Euler's gamma function and the function ω of Anderson and Thakur, also provides us (thanks to (7)) with the entire continuation $\mathbb{C}_\infty \to \mathbb{C}_\infty$ of $\zeta_C(1;1)$, and the whole phenomenology of the trivial zeros and the special values of $\zeta_C(1;1)$ (as in (2)).

This gives to the functional identity (7) a role similar to that of the functional equation of Riemann's zeta function and the second part of Theorem 2 gives a partial generalization of this. For further information, read [5].

2.3.1. *A transcendence question.* Let $(\mathcal{A},\nu)$ be an integral difference ring, that is, a domain $\mathcal{A}$ together with an endomorphism $\nu:\mathcal{A}\to\mathcal{A}$. A *$\nu$-polynomial in $X_1,\ldots,X_s$ over $\mathcal{A}$* is a polynomial of

$$\mathcal{A}[X_1,\ldots,X_s,\nu(X_1),\ldots,\nu(X_s),\nu^2(X_1),\ldots,\nu^2(X_s),\ldots]$$

(in infinitely many indeterminates $\nu^k(X_i)$, $k\geq 0$, $1\leq i\leq s$). Let $\mathcal{B}/\mathcal{A}$ be an integral difference ring extension. We say that elements $x_1,\ldots,x_n\in\mathcal{B}$ are *ν-independent over $\mathcal{A}$* if the only ν-polynomial in $X_1,\ldots,X_n$ over $\mathcal{A}$ vanishing at $(x_1,\ldots,x_n)$ is the zero polynomial.

We can give the question 3 a transcendental flavor by choosing $\mathcal{A}=A_\infty=A[t_1,t_2,\ldots]=\bigcup_s A_s$ with $\nu=\tau_p$ the unique $\mathbb{F}_p[t_1,t_2,\ldots]$-linear endomorphism such that $\tau_p(\theta)=\theta^p$ (here, p is the prime dividing q). We recall that, in [9], Chang and Yu have proved that the elements $\widetilde{\pi},\zeta_C(n)$ of $\mathbb{C}_\infty$, $n\geq 1$, $q-1\nmid n$, $p\nmid n$ are algebraically independent over K. The conditions on n allow us to avoid the Bernoulli-Carlitz relations (2) and the trivial relations $\zeta(pn)=\zeta(n)^p$.

We interpret the elements $\zeta_C(n;s)$ of $\mathbb{T}_s$ as a generalization of Carlitz' zeta values. This seems to legitimate the next question:

Question 4. Is it true that $\widetilde{\pi}$ and the series $\zeta_C(n;s)$ with $i\geq 1$, $s\geq 0$, $n\not\equiv s \pmod{q-1}$ and $p\nmid n$ are τ_p-independent over A_∞?

The conditions on n,s are required to avoid that the question has negative answer trivially. Indeed, if $n\equiv s \pmod{q-1}$, it is proved in [4] that (6) holds and we know that $x_i=\omega(t_i)$ is a solution of $\tau_p^e(X_i)=(t_i-\theta)X_i$ where e is such that $p^e=q$ so that $\widetilde{\pi}$ and $\zeta_C(n;s)$ are in this case τ-dependent (that is, not τ-independent). On the other hand, there is the trivial relation $\tau_p(\zeta_C(n;s))=\zeta_C(pn;s)$ that we want to equally avoid.

2.3.2. *Anderson log-algebraic Theorem revisited.* Theorem 2 can be applied to deduce an operator theoretic version of Anderson's log-algebraic Theorem (see [2]). We come back to the τ-polynomials of §2.3.1. The ring

$$\mathcal{A}_r=K[X_1,\ldots,X_s,\tau(X_1),\ldots,\tau(X_s),\tau^2(X_1),\ldots,\tau^2(X_s),\ldots]$$

is endowed with a structure of difference ring with the operator τ which sends $c\in\mathbb{C}_\infty$ to c^q and $\tau^k(X_i)$ to $\tau^{k+1}(X_i)$. In particular, we have, for all $d\in\mathbb{Z}$, the polynomials

$$w_d:=\sum_{a\in A^+}a^{-1}C_a(X_1)\cdots C_a(X_s)\in\mathcal{A}_r.$$

Let Z be a variable. We can define the formal series

$$\mathcal{L}_r:=\sum_{d\geq 0}Z^{q^d}w_d\in\mathcal{A}_r[[Z]].$$

We prove, in [5]:

Theorem 5. $\exp_C(\mathcal{L}_r)\in\mathcal{A}_r[Z]$.

The interest of this result relies on the fact that we can evaluate at X_i elements of $\mathbb{T}_s$; not just elements of $\mathbb{C}_\infty$ as in Anderson's original result.

3. Global L-series for φ-sheaves

3.1. **Settings.** We recall here the definition of the *global L-function* associated to a φ-sheaf. Our references are [6, 7, 11, 18].

We fix an absolutely irreducible smooth affine scheme Y over $\mathbb{F}_q$ (we call it the *coefficient scheme*). We denote by $\boldsymbol{A}$ the ring $H^0(Y, \mathcal{O}_Y)$. For any $\mathbb{F}_q$-scheme of finite type X (called the *base scheme*), we write

$$X_Y := X \times_{\mathbb{F}_q} Y.$$

If $X = \operatorname{Spec}(A)$ for some finitely presented $\mathbb{F}_q$-algebra as above, then X_Y is just $\operatorname{Spec}(A \otimes_{\mathbb{F}_q} \boldsymbol{A})$. We denote by

$$\sigma : X \to X$$

the map induced by the Frobenius morphism defined by $x \mapsto {}^\sigma x = x^q$ on the sheaf $\mathcal{O}_X$. We endow X_Y with the scheme endomorphism

$$\varphi = \sigma \times \mathrm{id}\,.$$

Definition 6 (Drinfeld). A *φ-sheaf* $\underline{\mathcal{F}}$ of rank r on X over $\boldsymbol{A}$ is a locally free $\mathcal{O}_{X_Y}$-module $\mathcal{F}$ of finite rank r, endowed with an injective morphism

$$\varphi : \sigma^* \mathcal{F} \to \mathcal{F}.$$

A morphism of φ-sheaves is an $\mathcal{O}_{X_Y}$-linear morphism with respect to the action of φ.

Compare with Definition 3.2.1 of Böckle and Pink book [7] where more general sheaves are considered. In the present note we always suppose that the underlying sheaf $\mathcal{F}$ is locally free.

3.1.1. *Example.* We choose $X = \mathbb{A}^1$ and $Y = \mathbb{A}^s$ (case in which $A = \mathbb{F}_q[\theta]$ and $\boldsymbol{A} = \mathbb{F}_q[t_1, \ldots, t_s]$). Then, $X_Y = \operatorname{Spec}(A[t_1, \ldots, t_s])$. For $\mathcal{F}$, we choose the structure sheaf of X_Y. Then, σ induces the map

$$P(\theta, t_1, \ldots, t_s) \in A[t_1, \ldots, t_s] \mapsto P(\theta^q, t_1, \ldots, t_s) \in A[t_1, \ldots, t_s].$$

Let $\underline{\mathcal{F}}$ be a φ-sheaf on X over $\boldsymbol{A}$. Then, $\underline{\mathcal{F}}$ can be identified with a projective $A[t_1, \ldots, t_s]$-module of finite rank r which can be injected in its *free extension by zero*, which is a free $A[t_1, \ldots, t_s]$-module of finite rank r' endowed with a σ-semilinear injective morphism acting as the Frobenius over A, and trivially on the variables t_i.

3.1.2. *Example.* Another important particular case is that of the base scheme $X = x = \operatorname{Spec}(k_x)$ where k_x is a finite extension of $\mathbb{F}_q$ of degree d_x, and coefficient scheme Y absolutely irreducible smooth affine scheme over $\mathbb{F}_q$. Let $\underline{\mathcal{F}}$ be a φ-sheaf on x over $\boldsymbol{A}$. Then, the *dual characteristic polynomial* (see Lemma-Definition 8.1.1 of [7])

$$\det_{\boldsymbol{A}}(\mathrm{id} - T\varphi | \underline{\mathcal{F}}) \in 1 + T\boldsymbol{A}[T]$$

is well defined (T denotes an indeterminate). In fact, it belongs to $1 + T^{d_x} A[t^{d_x}]$ [7, Lemma 8.1.4]. The *naive L-series* of $\underline{\mathcal{F}}_x$ is defined by

$$L(x, \underline{\mathcal{F}}, T) = \det_{\boldsymbol{A}}(\mathrm{id} - T\varphi | \underline{\mathcal{F}})^{-1} \in 1 + T^{d_x} \boldsymbol{A}[[T^{d_x}]] \tag{8}$$

(see [7, Definition 8.1.6]).

3.2. **Global L-functions.** Now, let us consider, more generally, a scheme of finite type X over $\mathbb{F}_q$ and a φ-sheaf $\underline{\mathcal{F}}$ on X over $\boldsymbol{A}$. We denote by $|X|$ the set of closed points of X. The choice of $x \in |X|$ determines a morphism $i : \mathrm{Spec}(k_x) \to X$ and we can construct a pull-back (*stalk* at x) $i_x^*\underline{\mathcal{F}}$ of $\underline{\mathcal{F}}$; a φ-sheaf on $\mathrm{Spec}(k_x)$ over $\boldsymbol{A}$ whose underlying sheaf is again locally free. We define, following [18, §2] or [7, Definition 8.1.8], the naive-L-function of $\underline{\mathcal{F}}$ on X over $\boldsymbol{A}$ as the product:

$$L(X, \underline{\mathcal{F}}, T) = \prod_{x\in|X|} L(x, i_x^*\underline{\mathcal{F}}, T) \in 1 + T\boldsymbol{A}[[T]].$$

The attribute of naive comes from the theory of crystals over function fields developed by Böckle and Pink [7]. They associate *crystalline L-series* canonically to *$\boldsymbol{A}$-crystals.* In the case in which the underlying sheaf is locally free and $\boldsymbol{A}$ is reduced, the naive and the crystalline L-series coincide (cf. [7, Corollary 9.4.3]). We recall that, among other results, Taguchi and Wan [18, Theorem 4.1] proved that ($\mathcal{F}$ being locally free and $\boldsymbol{A}$ the ring of integers of a finite extension of $\mathbb{F}_q(t)$) $L(X, \underline{\mathcal{F}}, T)$ is rational in T.

To define *a global L-function* (following Goss, Taguchi and Wan [18, §8] and Böckle [6, Definition 2.8]) we have to make an assumption. We require that there exists a morphism

$$f : X \to Y = \mathrm{Spec}(\boldsymbol{A}). \tag{9}$$

If $x \in |X|$, we set $\mathfrak{p}_x = f(x)$ and we have that the residue degree $d_{\mathfrak{p}_x}$ divides d_x.

3.2.1. *Exponentiation.* We now review Goss' idea of *exponentiation* of an ideal. The exponentiation takes place in $\boldsymbol{A}$ and for this reason, we need to suppose that $\dim_{\mathbb{F}_q}(\boldsymbol{A}) = 1$. Hence, we suppose, additionally, that $Y = \mathcal{C} = \overline{\mathcal{C}} \setminus \{\infty\}$, where $\overline{\mathcal{C}}$ is a smooth projective geometrically irreducible curve over $\mathbb{F}_q$, and ∞ is a point $\mathbb{F}_q$-rational on it (otherwise, the exponentiation of ideal becomes difficult to realize). In other words, $\boldsymbol{A} = H^0(\mathcal{C}, \mathcal{O}_\mathcal{C})$ is a Dedekind ring.

Goss' topological group of exponents is (for the ∞-adic theory)

$$\mathbb{S}_\infty = \mathbb{C}_\infty \times \mathbb{Z}_p,$$

where $\mathbb{C}_\infty$ is the completion of an algebraic closure of K_∞, the completion of the fraction field of $\boldsymbol{A}$ at the chosen infinity place.

Let I be a fractional ideal of $\boldsymbol{A}$ and $s = (z, n) \in \mathbb{S}_\infty$. The exponentiation of I by s is, by definition, the element of $\mathbb{C}_\infty$:

$$I^s = z^{\deg I}\langle I\rangle^n$$

($\langle I\rangle$ denotes the the *one unit part* of I, depending on a choice of uniformizer π of K_∞, see [11, §8.2] and $\langle I\rangle^n$ denotes the $\mathbb{Z}_p$-exponentiation by n). Then, the global L-series associated to the datum of $X, \mathcal{C}, f, \underline{\mathcal{F}}, \pi$ etc. is:

$$L^{glob}(X, \underline{\mathcal{F}}, s) = \prod_{x\in|X|} L(x, i_x^*\underline{\mathcal{F}}, T)|_{T^{d_{\mathfrak{p}_x}} = \mathfrak{p}_x^{-s}}.$$

This product converges on some half-plane of $\mathbb{S}_\infty$ to a $\mathbb{C}_\infty$-valued analytic function in the sense of Goss. Conjectures of Goss about meromorphy, essential algebraicity and entireness have been solved for these functions by Taguchi and Wan with a variant of Dwork's method in [18, Theorem 8.1] when $\boldsymbol{A} = \mathbb{F}_q[t]$ and later by Böckle in [6], for $\boldsymbol{A}$ the ring of regular functions of a smooth projective curve over $\mathbb{F}_q$ minus a point ∞, in a way which is closer to Grothendieck's approach.

3.2.2. *Example.* If $X = \mathrm{Spec}(A)$ with $A = \mathbb{F}_q[\theta]$ and $Y = \mathcal{C} = \mathrm{Spec}(\boldsymbol{A})$ with $\boldsymbol{A} = \mathbb{F}_q[t]$ with the map f in (9) corresponding to

$$\boldsymbol{A} \to A, \quad a(t) \mapsto a(\theta). \tag{10}$$

We set $\underline{\mathcal{F}}$ to be the structure sheaf of X_Y with $\varphi = (t-\theta)(\sigma \times \mathrm{id})$. In this case, it is easy to see that, for all $x = (\mathfrak{p})$ closed point of X (ideal generated by a *prime* $\mathfrak{p}$, that is, a monic irreducible polynomial of A),

$$L(\mathfrak{p}, \underline{\mathcal{F}}, T) = \frac{1}{1-\mathfrak{p}T}$$

so that

$$L^{glob}(X, \underline{\mathcal{C}}, s) = \prod_{\mathfrak{p}} \left(1 - \frac{\mathfrak{p}}{\mathfrak{p}^s}\right)^{-1}$$

is the Goss' zeta function evaluated at $s - s_1$ (where $s_1 = (\pi^{-1}, 1) \in \mathbb{S}_\infty$).

3.3. **Alternative construction for a global L-series.** Here we suppose, additionally, that both X, Y are affine schemes and $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(\boldsymbol{A})$.

We want to propose a different construction taking into account certain hidden features. This will give the Carlitz zeta values of §2 as a special case.

We have more freedom on the choice of Y; so we do not restrict to the case of an affine curve. Similar hypotheses occur in the definition of *A-premotives* by Tamagawa (see [19, Definition, p. 155]).

Exponentiation of ideals is anyway still needed, and will be performed now in the ring A so that we suppose X to be an affine curve: $X = \mathcal{C}$ with $\mathcal{C}, \overline{\mathcal{C}}, \infty, A, K_\infty, \mathbb{C}_\infty$ etc. as in §3.2.1 (K_∞ is now the completion of the fraction field of A at ∞). There is an injective homomorphism of groups

$$s : \mathbb{Z} \to \mathbb{S}_\infty, \quad n \mapsto (\pi^{-n}, n),$$

determined by the choice of π. Instead of letting the variable s varying in the group $\mathbb{S}_\infty$, we can even reduce ourselves to choose $s = s_n$ in the image of $\mathbb{Z}_{>0}$.

We choose $\boldsymbol{A} = \mathbb{F}_q[t_1, \ldots, t_n]/\mathcal{P}$, with $\mathcal{P}$ a prime ideal, or (0). We choose a norm $|\cdot|_\infty$ on $\mathbb{C}_\infty$. Then, the ring $\mathbb{C}_\infty \otimes_{\mathbb{F}_q} \boldsymbol{A}$ is endowed with the norm induced by the Gauss norm on $\mathbb{C}_\infty \otimes_{\mathbb{F}_q} \mathbb{F}_q[t_1, \ldots, t_n]$. We denote by $\mathcal{T}$ the affinoid Tate algebra $\widehat{\mathbb{C}_\infty \otimes_{\mathbb{F}_q}} \boldsymbol{A}$ obtained by completing $\mathbb{C}_\infty \otimes_{\mathbb{F}_q} \boldsymbol{A}$ for this Gauss norm. We denote by $\|\cdot\|_\mathcal{T}$ the standard norm of $\mathcal{T}$.

We construct local factors of our new global L-functions. Let x be again a closed point of X represented by a prime ideal P and let us consider a φ-sheaf $\underline{\mathcal{F}}$ on X over $\boldsymbol{A}$.

We do a different substitution in the local factor (8). We define the local factor at x of our global L-series by setting:

$$\mathcal{L}(x, i_x^*\underline{\mathcal{F}}, n)^{-1} = \det_{\boldsymbol{A}}(\mathrm{id} - P^{-s_n}\varphi|_{i_x^*\underline{\mathcal{F}}})^{-1} \in 1 + P^{-s_n}\boldsymbol{A}[P^{-s_n}] \subset \mathbb{C}_\infty \otimes \boldsymbol{A}.$$

Since

$$\|\mathcal{L}(x, i_x^*\underline{\mathcal{F}}, n)\|_\mathcal{T} = 1, \quad \|\mathcal{L}(x, i_x^*\underline{\mathcal{F}}, n) - 1\|_\mathcal{T} \to 0$$

with the limit taken for the cofinite filter on $|X|$, the product

$$\mathcal{L}^{glob}(X, \underline{\mathcal{F}}, n) := \prod_{x \in |X|} \mathcal{L}(x, i_x^*\underline{\mathcal{F}}, n)$$

converges in $\mathcal{T}$ to our new global L-function. This can be viewed as a *function* in virtue of the fact that the elements of $\mathcal{T}$ themselves can be viewed as functions.

3.3.1. *Example.* When $X = \mathbb{A}^1$ and $Y = \mathbb{A}^s$ (case in which $A = \mathbb{F}_q[\theta]$ and $\boldsymbol{A} = \mathbb{F}_q[t_1, \ldots, t_s]$ so that the ring A_s that we have used in §2 is $A_s = A[t_1, \ldots, t_s] = \mathrm{Spec}(X \times_{\mathbb{F}_q} Y)$), we choose $\underline{\mathcal{F}}_s$ to be the structure sheaf of X_Y with φ defined by $(t_1 - \theta) \cdots (t_s - \theta)(\sigma \times \mathrm{id})$. In this case, for all $n > 0$, it is easy to check that

$$\mathcal{L}^{glob}(X, \underline{\mathcal{F}}_s, n) = \zeta_C(n; s).$$

Now, we can vary $t_1, \ldots, t_s$ in the polydisk $\{(t_1, \ldots, t_s) \in \mathbb{C}_\infty; |t_i| \leq 1\}$; this provides us with a different type of global L-function.

More generally, it is not difficult to show that, for ϕ a *Drinfeld module of rank one* with *parameter* $\alpha \in A_s$ as defined in [5], the L-series value $L(n, \phi)$ defined there is equal to $\mathcal{L}^{glob}(X, \underline{\mathcal{F}}, n)$, where φ now acts as $\alpha(\sigma \times \mathrm{id})$. Explicitly,

$$\mathcal{L}^{glob}(X, \underline{\mathcal{F}}_s, n) = L(n, \phi),$$

in the notation of [5].

4. Final remarks.

It is likely that the functions of §3.3 can be further generalized in yet another arithmetically interesting direction. To simplify, we focus on the function

$$L(\chi_t, 1) = \sum_{a \in A^+} \frac{a(t)}{a} \in \mathbb{T},$$

which corresponds to $X = Y = \mathbb{A}^1$ and $\underline{\mathcal{F}} = \mathcal{O}_{X_Y}$ with $\varphi = (t - \theta)(\sigma \times \mathrm{id})$. From now on, $A, K, K_\infty, \mathbb{C}_\infty$ are as in §1.

Let u, v be variables of $\mathbb{C}_\infty$ such that $|u| \leq 1 < |v|$. We can define the series of functions of two variables:

$$L^\sharp(u, v) = \sum_{a \in A^+} \frac{a(u)}{a(v)}.$$

We have $L^\sharp(t, \theta) = \zeta_C(1; 1)$. By (7), and the first part of Proposition 1, we observe that

$$L^\sharp(u, v) = \prod_{i>0} \frac{1 - \frac{u}{v^{q^i}}}{1 - \frac{v}{v^{q^i}}}.$$

In particular, if x, y, z are variables of $\mathbb{C}_\infty$ such that $|z| \leq 1$ and $|x|, |y| < 1$, we have

$$\frac{L^\sharp(z, 1/y)}{L^\sharp(z, 1/x)} = \prod_{i>0} \frac{1 - x^{q^i - 1}}{1 - y^{q^i - 1}} \prod_{i>0} \frac{1 - y^{q^i} z}{1 - x^{q^i} z}.$$

We now follow Anderson, in [1]. In his definition of the *function τ of an A-lattice*, he introduces, in loc. cit. §2.3, the auxiliary formal series in four variables

$$h(t, x, y, z) = (1 - tz) \prod_{i \geq 0} \frac{1 - y^{q^i} z}{1 - x^{q^i} z} = \sum_{k \geq 0} h_k(t, x, y) z^k \in \mathbb{F}_q[[t, x, y, z]].$$

This can be viewed as a generating series for *solitons*, and by the above observation, $h(t, x, y, z)$ is related to the ratio of two of our series $L^\sharp$ by:

$$h(t, x, y, z) = \frac{(1 - tz)(1 - xz)}{1 - yz} \frac{L^\sharp(z, 1/y)}{L^\sharp(z, 1/x)} \prod_{i>0} \frac{1 - y^{q^i - 1}}{1 - x^{q^i - 1}}.$$

Another reason to investigate these functions is suggested in [15]; looking at [15, Proposition 28], we need to introduce the functions of two variables $|u| \leq 1 < |v|$

$$\omega_\tau^\sharp(u,v) = v_0 \prod_{i\geq 0} \tau^i \left(1 - \frac{u}{v}\right)^{-1}$$

(where v_0 is solution of $\tau(v_0) + vv_0 = 0$) to make visible analogues of the multiplication relations and the cyclotomic relations:

$$\omega_\tau^\sharp(u,v) = \prod_{i=0}^{n-1} \tau^i(\omega_{\tau^n}(u,v)), \tag{11}$$

$$\omega_\tau^\sharp(u,v^n) = \prod_{i=0}^{n-1} \tau^i(\omega_\tau(u,\zeta^i v)). \tag{12}$$

In the formulas above, $n > 0$, ζ is a solution of $X^n = 1$ in $\mathbb{C}_\infty$ such that $\zeta^k \neq 1$ for all $1 < k < n$. This suggests the study of the analytic and the arithmetic properties of the two variable series

$$L_\tau^\sharp(u,v) = \sum_{a \in (\mathbb{F}_q^{ac})^\tau[\theta]^+} \frac{a(u)}{a(v)}.$$

The sum takes place in the subring of $\mathbb{F}_q^{ac}[\theta]$ whose elements are the monic polynomials with coefficients fixed by τ. Of course, $\mathbb{F}_q^{ac}$ denotes the algebraic closure of $\mathbb{F}_q$ in $\mathbb{C}_\infty$.

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