

Simultaneous linearization of hyperbolic and parabolic fixed points

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1 Statement of the result

This note is a summary of the preprint [8]. We will show that the Fatou coordinates (the solution to Abel equation) for a parabolic fixed point of holomorphic map of one variable can be obtained as a modified limit of the solution to Schröder equation for the perturbed hyperbolic maps. (An alternative proof is given by Kawahira [4].)

Let $\{f_\tau\}_\tau$ be a family, depending on the parameter τ , of holomorphic maps of the form

$$f_\tau(z) = \tau z + 1 + \frac{a_1(\tau)}{z} + \frac{a_2(\tau)}{z^2} + \dots$$

defined in a neighborhood of ∞ of the Riemann sphere $\widehat{\mathbb{C}}$.

For each τ with $|\tau| > 1$, we have a unique analytic function $\chi_\tau(z)$ in a neighborhood of ∞ satisfying the Schröder equation

$$\chi_\tau(f_\tau(z)) = \tau \chi_\tau(z)$$

and normalized so that

$$\lim_{z \rightarrow \infty} \frac{\chi_\tau(z)}{z} = 1.$$

We will show that, when τ tends to 1 non-tangentially within the domain $|\tau| > 1$, the sequence

$$\chi_\tau(z) - \frac{1}{\tau - 1} - a_1(\tau) \log(\tau - 1)$$

converges to a solution to the Abel equation $\varphi(z) \varphi(f_1(z)) = \varphi(z) + 1$, on a half plane $\{\operatorname{Re} z > R\}$ with sufficiently large R .

2 A family of linear maps

We begin with studying the family $\{\ell_\tau\}_\tau$ of linear maps

$$\ell_\tau(z) = \tau z + 1 \quad (1)$$

on the Riemann sphere $\widehat{\mathbb{C}}$ with a fixed point at ∞ .

We will investigate the uniformity, with respect to the parameter τ , of convergence of the sequence of the iterates $\{f_\tau^n\}_{n=1}^\infty$. Here, the parameter will be restricted in the closed sector

$$T_\alpha = \{\tau \in \mathbb{C} \mid \operatorname{Re} \tau - 1 \geq |\tau - 1| \cos \alpha\},$$

where α is a real number with $0 < \alpha < \pi/2$.

To measure the rate of convergence to ∞ , we define a function $N : \widehat{\mathbb{C}} \times T_\alpha - \{(\infty, 1)\} \rightarrow \mathbb{R} \cup \{\infty\}$ as follows.

$$\begin{aligned} N_\tau(z) &= \left| z - \frac{1}{1-\tau} \right| - \left| \frac{1}{1-\tau} \right| && \text{for } (z, \tau) \in \widehat{\mathbb{C}} \times (T_\alpha - \{1\}); \\ N_1(z) &= \sup_{|\theta| \leq \alpha} \operatorname{Re}(e^{i\theta} z) && \text{for } z \in \mathbb{C}. \end{aligned}$$

We will not define $N_1(\infty)$.

As is easily shown, $N_\tau(z)$ is upper semi-continuous and

$$N_1(z) = \limsup_{T \ni \tau \rightarrow 1} N_\tau(z).$$

Further the inequality

$$|N_\tau(z) - N_\tau(w)| \leq |z - w| \quad z, w \in \mathbb{C}, \tau \in T_\alpha$$

and, in particular,

$$N_\tau(z) \leq |z|, \quad z \in \mathbb{C}, \tau \in T_\alpha.$$

hold.

For a real number R , let

$$\mathcal{V}_\alpha(R) = \{(z, \tau) \in \widehat{\mathbb{C}} \times T_\alpha - \{(\infty, 1)\} \mid N_\tau(z) > R\}.$$

We note that $\mathcal{V}_\alpha(R)$ is not open. Slices of $\mathcal{V}_\alpha(R)$ by $\tau = \text{const.}$ are open sets given by

$$\begin{aligned} V_\tau(R) &= \{z \in \widehat{\mathbb{C}} \mid N_\tau(z) > R\} \quad (\tau \neq 1); \\ V_1(R) &= \{z \in \mathbb{C} \mid N_1(z) > R\} = \bigcup_{|\theta| \leq \alpha} \{\operatorname{Re}(e^{i\theta} z) > 0\}. \end{aligned}$$

Lemma 2.1 For $(z, \tau) \in \widehat{\mathbb{C}} \times T_\alpha - \{(\infty, 1)\}$, we have

$$N_\tau(\ell_\tau(z)) \geq |\tau|N_\tau(z) + \cos \alpha.$$

If $N_\tau(z) > 0$, we have $N_\tau(\ell_\tau(z)) \geq N_\tau(z) + \cos \alpha$. So we have the following.

Proposition 2.2 The sequence $\{\ell_\tau^n(z)\}_n$ converges to ∞ as $n \rightarrow \infty$ uniformly on the set $\mathcal{V}_\alpha(0)$.

3 Families of maps with attracting/parabolic fixed points — Domain of convergence

Now we consider a family of holomorphic maps $f_\tau : U \rightarrow \widehat{\mathbb{C}}$ of the form

$$f_\tau(z) = \tau z + 1 + \frac{a_1(\tau)}{z} + \frac{a_2(\tau)}{z^2} + \dots \quad (2)$$

defined on a neighborhood

$$U = \{z \in \widehat{\mathbb{C}} \mid R < |z| \leq \infty\}$$

of $\infty \in \widehat{\mathbb{C}}$. We suppose that f depends holomorphically on $\tau \in \Delta_\rho(1) = \{\tau \in \mathbb{C} \mid |\tau - 1| < \rho\}$. Let

$$A_\tau(z) = \frac{a_1(\tau)}{z} + \frac{a_2(\tau)}{z^2} + \dots$$

As in the previous section, we choose and fix α so that $0 < \alpha < \pi/2$ and let $\delta = \frac{1}{2} \cos \alpha$. By shrinking the neighborhoods U and W , we assume that there is a constant K_1 such

$$|A_\tau(z)| < \frac{K_1}{|z|} < \delta \quad (3)$$

for $(z, \tau) \in U \times W$. Further we assume that $f_\tau(z)$ is injective in z for every $\tau \in \Delta_\rho(1)$

Since $f_\tau(z)$ are approximated by linear maps $\ell_\tau(z)$, we have a result concerning the uniformity of convergence of $\{f_\tau^n(z)\}$. Let $T_{\alpha,\rho} = T_\alpha \cap \Delta_\rho(1)$.

Lemma 3.1 For $(z, \tau) \in U \times T_{\alpha,\rho}$ we have

$$N_\tau(f_\tau(z)) \geq |\tau|N_\tau(z) + \delta.$$

Now let $\mathcal{V} = \mathcal{V}_{\alpha,\rho}(R) = \{(z, \tau) \in \mathcal{V}_\alpha(R) \mid \tau \in T_{\alpha,\rho}\}$.

Proposition 3.2 If $(z, \tau) \in \mathcal{V}$, then $(f_\tau(z), \tau) \in \mathcal{V}$. The sequence $\{f_\tau^n(z)\}_n$ converges uniformly on $\mathcal{V}$ to ∞ as $n \rightarrow \infty$.

4 Schröder-Abel equation — special case

Here we consider the special case where the coefficient $a_1(\tau)$ in (2) vanishes identically.

Theorem 4.1 *There exists a function $\varphi_\tau(z)$ continuous on $\mathcal{V}$ such that*

(i) $\varphi_\tau(z)$ satisfies the functional equation

$$\varphi_\tau(f_\tau(z)) = \tau\varphi_\tau(z) + 1; \quad (4)$$

(ii) $\varphi_\tau(z)$ is injective in the variable z for each parameter $\tau \in T_{\alpha,\tau}$.

(iii) $\lim_{z \rightarrow \infty} \varphi_\tau(z)/z = 1$ as $z \rightarrow \infty$, when $|\tau| > 1$.

In fact $\varphi_\tau(z)$ is given by

$$\varphi_\tau(z) = \lim_{n \rightarrow \infty} \left\{ \frac{1}{\tau^n} f^n(z) - \sum_{k=1}^n \frac{1}{\tau^k} \right\} \quad (5)$$

In the case where $a_1(\tau)$ does not identically vanish, the expression in (5) is not convergent. So we have to modify (5) in order to yield convergence. For this purpose, we will introduce a function satisfying a difference equation in the next section.

5 Solution to a difference equation

We consider the difference equation

$$h_\tau(\ell_\tau(z)) - \tau h_\tau(z) = \frac{1}{z} + C_\tau. \quad (6)$$

where $\ell_\tau(z) = \tau z + 1$ with $|\tau| > 1$ or $\tau = 1$; and C_τ is a constant depending on τ , which will be given later.

A solution to this equation is given by

$$h_\tau(z) = -\frac{1}{\tau z} + \sum_{n=1}^{\infty} \frac{1}{\tau^{n+1}} \left\{ \frac{1}{\ell_\tau^n(0)} - \frac{1}{\ell_\tau^n(z)} \right\}. \quad (7)$$

Proposition 5.1 *The function $h_\tau(z)$ is continuous on $\mathcal{V}_\alpha(0)$.*

For a fixed τ with $|\tau| > 1$, the function $h_\tau(z)$ is meromorphic on $\hat{\mathbb{C}}$ except the essential singularity at $1/(1-\tau)$, and has poles at $(1-\tau^{-n})/(1-\tau)$, ($n = 0, 1, 2, \dots$). This function $h_\tau(z)$ is holomorphic at ∞ and we write

$$H_\tau = h_\tau(\infty) = \sum_{n=1}^{\infty} \frac{1}{\tau^{n+1} \ell_\tau^n(0)}. \quad (8)$$

For $\tau = 1$, we have $\ell^n(z) = z + n$ and

$$h_1(z) = -\frac{1}{z} + \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{z+n} \right\}.$$

This function is meromorphic on $\mathbb{C}$ and has poles at $0, -1, -2, \dots$. We note that

$$h_1(z) = \frac{\Gamma'(z)}{\Gamma(z)} + \gamma$$

where $\Gamma(z)$ denotes the gamma function and γ denotes the Euler constant

$$\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right).$$

Now we study the dependence of $h_\tau(z)$ on the parameter τ .

Corollary 1 *The constant C_τ is a continuous function of $\tau \in T_\alpha$.*

The function $h_\tau(z)$ satisfies the equation () with

$$C_\tau = (1 - \tau)H_\tau. \quad (9)$$

for $|\tau| > 1$ and with $C_1 = 0$ for $\tau = 1$. We have $C_\tau \rightarrow C_1 = 0$ ($\tau \rightarrow 1$), since $h_\tau(z)$ is continuous.

Proposition 5.2 *For any $\varepsilon > 0$, there is a constant M such that*

$$|h'_\tau(z)| \leq \frac{M}{N_\tau(z)} \quad \text{on } \mathcal{V}_\alpha(\varepsilon)$$

6 Behavior of H_τ

Now we look at the behavior of the function H_τ defined by (), when $\tau \rightarrow 1$ within the sector T . It is clear from the expression () that H_τ is unbounded, while $C_\tau = (1 - \tau)H_\tau$ tends to 0 by the corollary to Proposition 2.4. Here we give a more precise description of its behavior.

Proposition 6.1 *We have*

$$H_\tau = -\log(\tau - 1) + \gamma - 1 + o(1)$$

as $\tau \rightarrow 1$ within the sector T . Here γ denotes the Euler constant.

To show this, we write $\lambda = 1/\tau$. We have

$$H_{1/\lambda} = (1 - \lambda)L(\lambda) - \lambda.$$

Here $L(\lambda)$ denotes the Lambert series defined by

$$L(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{1 - \lambda^n}.$$

This series $L(\lambda)$ defines a holomorphic function on $|\lambda| < 1$, and is developed into the power series

$$L(\lambda) = \sum_{n=1}^{\infty} d(n)\lambda^n = \lambda + 2\lambda^2 + 2\lambda^3 + 3\lambda^4 + \dots,$$

where $d(n)$ denotes the number of divisors of n . Let

$$\frac{L(\lambda)}{1 - \lambda} = \sum_{n=1}^{\infty} D(n)\lambda^n$$

with

$$D(n) = d(1) + \dots + d(n).$$

The asymptotic behavior of $D(n)$ is given by a theorem of Dirichlet (see Apostol [1], Chandrasekharan [2]) :

$$D(n) = n \log n + (2\gamma - 1)n + O(\sqrt{n}) \quad (n \rightarrow \infty).$$

Using this estimate, we have

$$\frac{L(\lambda)}{1 - \lambda} = \sum_{n=1}^{\infty} D(n)\lambda^n = -\frac{\lambda \log(1 - \lambda)}{(1 - \lambda)^2} + \frac{\gamma\lambda}{(1 - \lambda)^2} + P(\lambda)$$

where $P(\lambda) = \sum_{n=1}^{\infty} p_n \lambda^n$. From the estimate of p_n we have

$$P(\lambda) = o((1 - \lambda)^{-2}) \quad \text{as } \lambda \rightarrow 1 \text{ non-tangentially}$$

Hence it follows that

$$H_{\tau} = -\log(\tau - 1) + \gamma - 1 + o(\tau - 1)$$

7 Schröder-Abel equation — general case

Now we treat the general case where $a_1(\tau)$ does not necessarily vanish. Let

$$B_\tau = 1 - a_1(\tau)C_\tau$$

we have the following result corresponding to Theorem ?

Theorem 7.1 *There exists a function $\varphi_\tau(z)$ continuous on $\mathcal{V}$ such that*
(i) $\varphi_\tau(z)$ satisfies the functional equation

$$\varphi_\tau(f_\tau(z)) = \tau\varphi_\tau(z) + B_\tau; \quad (10)$$

- (ii) $\varphi_\tau(z)$ is injective in the variable z for each parameter $\tau \in T_{\alpha,\tau}$.*
(iii) $\lim_{z \rightarrow \infty} \varphi_\tau(z)/z = 1$ as $z \rightarrow \infty$, when $|\tau| > 1$.

To define $\varphi_\tau(z)$, we let

$$\Phi_\tau(z) = z - a_1(\tau)h_\tau(z).$$

Then

$$\Phi_\tau(f_\tau(z)) = \tau\Phi_\tau(z) + B_\tau + \tilde{A}(z).$$

From this we can define

$$\varphi_\tau(z) = \lim_{n \rightarrow \infty} \left\{ \frac{1}{\tau^n} \Phi_\tau(f_\tau^n(z)) - B_\tau \sum_{k=1}^n \frac{1}{\tau^k} \right\} \quad (11)$$

8 Relation with the Schröder equation

When $|\tau| > 1$, the Schröder equation

$$\chi_\tau(f_\tau(z)) = \tau\chi_\tau(z).$$

has a unique solution $\chi_\tau(z)$ of the form

$$\chi_\tau(z) = z + c_0 + \frac{c_1}{z} + \dots$$

in a neighbourhood of ∞ .

Theorem 8.1 *For $\tau \in T_{\alpha,\rho} - \{1\}$ we have*

$$\varphi_\tau(z) = \chi_\tau(z) - \frac{B_\tau}{\tau - 1}.$$

Proof We can easily verify that $\varphi(z) + B_\tau/(\tau - 1)$ satisfies the Schröder equation. The assertion follows from the uniqueness of the solution. $\square$

Now recall that

$$\begin{aligned}\frac{B_\tau}{\tau - 1} &= \frac{1 - a_1 C_\tau}{\tau - 1} \\ &= \frac{1}{\tau - 1} - a_1 H_\tau \\ &= \frac{1}{\tau - 1} + a_1 \log(\tau - 1) + a_1(1 - \gamma) + o(1)\end{aligned}$$

Using this fact the theorem is reformulated as follows:

Theorem 8.2 *Let*

$$\varphi(z) = \chi(z) - \frac{1}{\tau - 1} - a_1 \log(\tau - 1)$$

for $\tau \in T - \{1\}$. Then $\varphi(z)$ converges to a solution to the Abel equation.

References

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