

Validated numerics and Hilbert's 16th problem

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Introduction

Determining the number and location of (isolated) limit cycles for planar polynomial ordinary differential equations was posed as a grand challenge in Hilbert's seminal address to the International Congress of Mathematicians in 1900. Of the 23 problems presented by Hilbert, this (the 16th) turned out to be one of the most persistent: despite more than a century of intense research, not even the quadratic case has been resolved. For an overview of the progress that has been made to solve this problem we refer to [5]. Partial results for the quadratic case, and a general introduction to the bifurcation theory of planar polynomial vector fields can be found in [6]. A classic result is that any given polynomial vector field can have only a finite number of limit cycles; this is proved in [3, 4].

A restricted version of Hilbert's 16th problem introduced by Arnol'd, see e.g. [1], known as the *weak* or *tangential* Hilbert's 16th problem, asks for the number of limit cycles that can bifurcate from a perturbation of a Hamiltonian system, see e.g. [2]. We will present a novel, computer-aided, technique for obtaining lower bounds on this number.

Abelian integrals

A classical method to prove the existence of limit cycles bifurcating from a continuous family of level curves of a Hamiltonian, $\gamma_h \subset H^{-1}(h)$, depending continuously on h , is to study Abelian integrals, or, more generally, the Melnikov function, see e.g. [2]. The closed level-curves of a polynomial Hamiltonian are called *ovals*. We denote the interior of an oval D_h , i.e. $\partial D_h = \gamma_h$. Given a Hamiltonian system and a perturbation,

$$\begin{cases} \dot{x} &= -H_y(x, y) + \epsilon f(x, y) \\ \dot{y} &= H_x(x, y) + \epsilon g(x, y), \end{cases} \quad (1)$$

The weak formulation of Hilbert's 16th problem can be formulated as follows: For given integers $m = \deg(H)$ and $n = \max(\deg(f, g)) \geq 2$, what is the maximum number $Z(n, m)$ of limit cycles the system (1) can have for all possible H, f, g ? It is natural to consider the case $m = n + 1$, and so we define $Z(n) = Z(n + 1, n)$.

The Abelian integral associated to (1), in general multi-valued, is defined as

$$I(h) = \int_{\gamma_h} f(x, y) dy - g(x, y) dx. \quad (2)$$

The most important property of Abelian integrals is described by the Poincaré-Pontryagin theorem.

Theorem 2.1 (Poincaré-Pontryagin) *Let P be the return map defined on some section transversal to the ovals of H , parametrised by the values h of H , where h is taken from some bounded interval (a, b) . Let $d(h) = P(h) - h$ be the displacement function. Then, $d(h) = \epsilon(I(h) + \epsilon\phi(h, \epsilon))$ as $\epsilon \rightarrow 0$, where $\phi(h, \epsilon)$ is analytic and uniformly bounded on a compact neighbourhood of $\epsilon = 0$, $h \in (a, b)$.*

As a consequence of the Poincaré-Pontryagin theorem, one can prove that a simple zero of $I(h)$ corresponds to a unique limit cycle bifurcating from the Hamiltonian system as $\epsilon \rightarrow 0$. In fact, to prove the existence of a limit cycle, it suffices to have a zero of odd order.

Results

We study the quintic Hamiltonian, described in [7] :

$$H(x, y) = \frac{x^2}{2} - \frac{9x^4}{8} + \frac{x^6}{3} + \frac{y^2}{2} - \frac{73y^4}{144} + \frac{2y^6}{27} \quad (3)$$

corresponding to the differential system,

$$\begin{cases} \dot{x} &= -y \left(1 - \frac{16y^2}{9}\right) \left(1 - \frac{y^2}{4}\right) \\ \dot{y} &= x \left(1 - 4x^2\right) \left(1 - \frac{x^2}{2}\right). \end{cases} \quad (4)$$

The system has 25 equilibrium points and 19 periodic annuli, appearing in 9 classes. Following [7], we study the following $\mathbf{Z}_2$ equivariant perturbation of the Hamiltonian system (4),

$$\begin{aligned} p(x, y) &:= \frac{\alpha_{00}}{2} + \frac{\alpha_{20}}{4}x^2 + \frac{\alpha_{02}}{4}y^2 + \frac{\alpha_{40}}{6}x^4 + \frac{\alpha_{22}}{6}x^2y^2 + \frac{\alpha_{04}}{6}y^4 \\ f(x, y) &:= xp(x, y) \\ g(x, y) &:= yp(x, y) \end{aligned} \quad (5)$$

We can now establish the following result:

Theorem 3.1 *Consider the Hamiltonian vector field (4), perturbed as in (5). Then one can choose α_{ij} , such that, as $\epsilon \rightarrow 0$, at least 27 limit cycles appear in the configuration,*

$$(\Gamma_1^2)^4(\Gamma_2)^2(\Gamma_3)^2(\Gamma_6)(\Gamma_8)^2(\Gamma_9^3)^4.$$

As a consequence, we have $Z(5) \geq 27$.

Turning to Hamiltonian vector fields of degree 7, we study the following system:

$$\begin{cases} \dot{x} = -y(y^2 - 1)(y^2 - 2)(y^2 - 3) \\ \dot{y} = x(x^2 - 1.1)(x^2 - 2.3)(x^2 - 3.6). \end{cases} \quad (6)$$

The system has 49 equilibrium points and 42 periodic annuli, appearing in 14 classes. We are interested in limit cycles bifurcating from the periodic solutions of (6). We study the following $\mathbf{Z}_2$ equivariant perturbation of the Hamiltonian system (6)

$$\begin{aligned} p(x, y) &:= \frac{\alpha_{00}}{2} + \frac{\alpha_{20}}{4}x^2 + \frac{\alpha_{02}}{4}y^2 + \frac{\alpha_{40}}{6}x^4 + \frac{\alpha_{22}}{6}x^2y^2 + \frac{\alpha_{04}}{6}y^4 + \frac{\alpha_{60}}{8}x^6 + \frac{\alpha_{42}}{8}x^4y^2 + \frac{\alpha_{24}}{8}x^2y^4 + \frac{\alpha_{06}}{8}y^6 \\ f(x, y) &:= xp(x, y) \\ g(x, y) &:= yp(x, y) \end{aligned} \quad (7)$$

We can now establish the following result:

Theorem 3.2 *Consider the Hamiltonian vector field (6), perturbed as in (7). Then one can choose α_{ij} , such that, as $\epsilon \rightarrow 0$, at least 53 limit cycles appear in the configuration,*

$$(\Gamma_2^3)^4(\Gamma_3^2)^4(\Gamma_4)^4(\Gamma_5^2)^4(\Gamma_6^3)^4(\Gamma_7)^4(\Gamma_8)^4(\Gamma_{14}),$$

As a consequence, we have $Z(7) \geq 53$.

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