

ON THE SPACE OF STRINGS WITH PARTIALLY SUMMABLE LABELS

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1. INTRODUCTION

In [3] we attached to any pair of a euclidean space V and a partial abelian monoid M a space $C(V, M)$ whose points are pairs (c, a) , where c is a finite subset of V and a is a map $c \rightarrow M$, but (c, a) is identified with (c', a') if $c \subset c'$, $a'|_c = a$, and $a'(v) = 0$ for $v \notin c$. If V is an orthogonal G -module and M admits a G -action compatible with partial sum operations, where G is a finite group, then $C(V, X)$ is a G -space with respect to the G -action $g(c, a) = (gc, gag^{-1})$, $g \in G$.

Let $I(\mathbb{R})$ be the partial abelian monoid consisting of bounded 1-dimensional submanifolds in the real line (see [1] and §3), and let

$$I(V, M) = C(V, I(\mathbb{R}) \wedge M)$$

for any partial abelian monoid with G -action M . The points of $I(V, M)$ can be represented by the pairs (P, a) , where P is a finite union of parallel intervals,

$$\coprod_{v \in c} \{v\} \times P(v) \subset V \times \mathbb{R} \quad (c \subset V, P(v) \in I(\mathbb{R}))$$

and a is a map $c \rightarrow M$.

The purpose of this note is to show the following:

If V is sufficiently large then there is a G -equivariant group completion map $C(V, M) \rightarrow I(V, M)$, and the correspondence $X \mapsto \{\pi_n I(V, X \wedge M); n \geq 0\}$ defines a G -equivariant generalized homology theory.

2. PARTIAL ABELIAN MONOIDS

Definition 1. A pointed G -space M is called a partial abelian monoid with G -action, or *G -partial monoid* for short, if there are G -invariant subspaces M_n of M^n ($n \geq 0$) and G -equivariant maps, called *partial sum operations*,

$$M_n \rightarrow M, \quad (a_1, \dots, a_n) \mapsto a_1 + \dots + a_n$$

satisfying the conditions below.

- (1) $M_0 = \{0\}$.
- (2) $M_1 \rightarrow M$ is the identity of M .

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- (3) Let $J_1, \dots, J_r$ be pairwise disjoint subsets of $\{1, \dots, n\}$ such that $\{1, \dots, n\} = J_1 \cup \dots \cup J_r$ holds. Let $(a_1, \dots, a_n) \in M^n$ and suppose

$$\sum_{j \in J_k} a_j = a_{\lambda_k(1)} + \dots + a_{\lambda_k(j_k)}$$

exists for each k , where $J_k = \{\lambda_k(1), \dots, \lambda_k(j_k)\}$, $\lambda_k(1) < \dots < \lambda_k(j_k)$. Then $(a_1, \dots, a_n) \in M_n$ if and only if $(\sum_{j \in J_1} a_j, \dots, \sum_{j \in J_r} a_j) \in M_r$ and we have

$$\sum_{1 \leq j \leq n} a_j = \sum_{j \in J_1} a_j + \dots + \sum_{j \in J_r} a_j$$

whenever either the right or the left hand side sum exists.

Among the examples we have

- Let A be a topological abelian monoid with G -action. Then any G -invariant subset M of A with $0 \in M$ can be regarded as an G -partial monoid by taking $M_n = \{(a_1, \dots, a_n) \in M^n \mid a_1 + \dots + a_n \in M\}$.
- Any pointed space X is a G -partial monoid with respect to the trivial partial sum operations, i.e. folding maps $X_n = X \vee \dots \vee X \rightarrow X$. In fact this is a special case of the previous example, as X is a subset of the infinite symmetric product $\text{SP}^\infty X$.
- Let V be an infinite dimensional real inner product space on which G acts through linear isometries. Then the Grassmannian $\text{Gr}(V)$ of finite-dimensional subspaces of V is a G -partial monoid with respect to the inner direct sum operations

$$\text{Gr}(V)_n = \{(W_1, \dots, W_n) \mid W_i \perp W_j, i \neq j\} \xrightarrow{\oplus} \text{Gr}(V)$$

Definition 2. For given G -partial monoids M and N , the smash product $M \wedge N$ is a G -partial monoid whose partial sums are generated by the distributivity relations:

$$\begin{aligned} c_1 \wedge d + \dots + c_k \wedge d &= (c_1 + \dots + c_k) \wedge d, & (c_1, \dots, c_k) \in M_k \\ c \wedge d_1 + \dots + c \wedge d_l &= c \wedge (d_1 + \dots + d_l), & (d_1, \dots, d_l) \in N_l \end{aligned}$$

Example 3. If X is a pointed space then for any G -partial monoid M we have

$$(X \wedge M)_n = X \wedge M_n, \quad n \geq 0$$

3. THE SPACE OF PARALLEL STRINGS WITH LABELS

If J is a bounded interval in the real line $\mathbb{R}$ then its endpoint, say a , is called a closed endpoint if $a \in J$, and an open endpoint otherwise. Thus J is a closed (resp. open) interval if its two endpoints are closed (resp. open), and is a half open interval if J has a closed endpoint and an open endpoint.

Following [1], we denote by $I(\mathbb{R})$ the space of all bounded 1-dimensional submanifolds of the real line $\mathbb{R}$, including the empty set. An element of $I(\mathbb{R})$ can be written as a union, say $P = J_1 \cup \dots \cup J_r$, of finite number of pairwise disjoint bounded intervals. Here we may suppose $J_i < J_{i+1}$ holds for $1 \leq i < r$, that is to say, $x \in J_i$ and $y \in J_{i+1}$ yields $x < y$. But the union $J_i \cup J_{i+1}$ in this expression can be replaced by a single interval J if $J_i \cup J_{i+1} = J$ is a connected interval, and

J_i can be removed if J_i is a half open interval of length 0. The latter means that half open intervals are collapsible to the empty set.

Let $I(\mathbb{R})_+$ be the subset of $I(\mathbb{R})$ consisting of those elements $J_1 \cup \dots \cup J_r$ such that every J_i is an closed interval. Then $I(\mathbb{R})$ is a partial abelian monoid with respect to the *superimposition*,

$$I(\mathbb{R})_n \rightarrow I(\mathbb{R}) \quad (P_1, \dots, P_n) \mapsto P_1 \cup \dots \cup P_n,$$

where $I(\mathbb{R})_n$ consists of those $(P_1, \dots, P_n) \in I(\mathbb{R})^n$ such that $P_i \cap P_j = \emptyset$, $i \neq j$, and $I(\mathbb{R})_+$ is a partial submonoid of $I(\mathbb{R})$.

Definition 4. For an orthogonal G -module V and a G -partial monoid M , we put

$$I(V, M) = C(V, I(\mathbb{R}) \wedge M), \quad I_+(V, M) = C(V, I(\mathbb{R})_+ \wedge M)$$

We call $I(V, M)$ the *space of parallel strings in V with labels in M* .

To relate $I(V, M)$ with $C(V, M)$, we introduce the map $b: I(\mathbb{R})_+ \rightarrow C(\mathbb{R})$ which takes $J_1 \cup \dots \cup J_r$ to the finite set $\{bJ_1, \dots, bJ_r\}$ consisting of the barycenters of J_i . One easily observes that the natural map

$$I_+(V, M) \rightarrow C(V, C(\mathbb{R}) \wedge M)$$

induced by $b: I(\mathbb{R})_+ \rightarrow C(\mathbb{R})$, is a homotopy equivalence.

We also have

Lemma 5. *If V is sufficiently large then the inclusion*

$$C(V, C(\mathbb{R}) \wedge M) \rightarrow C(V \times \mathbb{R}, M)$$

is a G -homotopy equivalence.

Proof. Let $i: \mathbb{R} \rightarrow V^G$ be a linear embedding, and define a homotopy $h: I \times \mathbb{R} \rightarrow V$ by $h(t, x) = (1 - t)i(x)$. If we write $h_t(x) = h(t, x)$ then $h_0 = i$ and h_1 is the constant map with value 0. Let l be a G -linear isometry $V \times V \rightarrow V$. Then there is a homotopy

$$H: I_+ \wedge C(V \times \mathbb{R}, M) \rightarrow C(V \times \mathbb{R}, M)$$

such that $H_t = H(t, -)$ is induced by the composite

$$V \times \mathbb{R} \xrightarrow{1 \times \text{diag.}} V \times \mathbb{R} \times \mathbb{R} \xrightarrow{1 \times h_t \times 1} V \times V \times \mathbb{R} \xrightarrow{l \times 1} V \times \mathbb{R},$$

One easily observes that

- (1) H restricts to a map $I_+ \wedge C(V, C(\mathbb{R}) \wedge M) \rightarrow C(V, C(\mathbb{R}) \wedge M)$,
- (2) $\text{Im } H_0 \subset C(V, C(\mathbb{R}) \wedge M)$, and
- (3) H_1 is induced by the linear isometry $l' \times 1: V \times \mathbb{R} \rightarrow V \times \mathbb{R}$, where l' is the composite $V = V \times \{0\} \subset V \times V \xrightarrow{l} V$.

Since the space of linear isometries of V is contractible, l' is equivariantly isotopic to the identity through G -linear isometries. Thus we have $H_0 \simeq H_1 \simeq 1$, hence H_0 induces a homotopy inverse to the inclusion $C(V, C(\mathbb{R}) \wedge M) \rightarrow C(V \times \mathbb{R}, M)$. $\square$

Corollary 6. *If V is sufficiently large then there is a map of Hopf G -spaces $\lambda: I_+(V, M) \rightarrow C(V, M)$, which is natural in M and is a G -homotopy equivalence.*

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Proof. Choose a G -linear isometry $l: V \times \mathbb{R} \cong V$, and define λ as the composite

$$I_+(V, M) \xrightarrow{b_*} C(V, C(\mathbb{R}) \wedge M) \xrightarrow{\subseteq} C(V \times \mathbb{R}, M) \xrightarrow{l_*} C(V, M)$$

□

4. MAIN RESULTS

To state the main results we introduce the G -category $\text{Top}(G)$ consisting of all pointed G -spaces and pointed maps, with G acting on morphisms by conjugation. As we showed in [2], any G -equivariant continuous functor T of $\text{Top}(G)$ into itself is accompanied with pairings

$$X \wedge TY \rightarrow T(X \wedge Y), \quad TX \wedge Y \rightarrow T(X \wedge Y)$$

natural in both X and Y . One easily observes from this that T preserves G -homotopies, and there is a suspension natural transformation $T(X) \rightarrow \Omega^W T(\Sigma^W X)$ defined for any finite dimensional orthogonal G -module W .

Suppose V is linearly and equivariantly isometric to the direct product of countably many copies of the regular representation of G over the real number fields. Such a G -module V is said to be *sufficiently large*. Then we have

Theorem 7. *In the diagram of Hopf G -spaces*

$$C(V, M) \xleftarrow{\lambda} I_+(V, M) \xrightarrow{\rho} I(V, M),$$

where ρ is induced by the inclusion $I(\mathbb{R})_+ \subset I(\mathbb{R})$, we have

- (1) λ is a G -homotopy equivalence.
- (2) ρ is a G -equivariant group completion, that is to say, $\rho^H: I_+(V, M)^H \rightarrow I(V, M)^H$ is a group completion for every subgroup H of G .

Theorem 8. *The correspondence $X \rightarrow I(V, X \wedge M)$ is a G -equivariant continuous functor of $\text{Top}(G)$ into itself and we have*

- (1) *For any orthogonal G -module W the suspension map $I(V, X \wedge M) \rightarrow \Omega^W I(V, \Sigma^W X \wedge M)$ is a weak G -homotopy equivalence.*
- (2) *There is an $RO(G)$ -graded G -homology theory $h_*^G(-)$ such that*

$$h_n^G(X) = \pi_n I(V, X \wedge M)^G$$

holds for any X and $n \geq 0$.

5. BRIEF OUTLINE OF THE PROOFS

Outline of the Proof of Theorem 7. We need to show that

$$(5.1) \quad \rho^H: I_+(V, M)^H \rightarrow I(V, M)^H \text{ is a group completion}$$

holds for every subgroup H of G .

Observe that V is an H -universe for any subgroup H of G . Hence (5.1) for general H follows from the special case $H = G$. that $\rho^H: I_+(V, M)^H \rightarrow I(V, M)^H$ is a group completion follows from the case $H = G$. But the usual argument using the notion of orbit type family enables us to reduce the proof of this problem to the case where G is trivial. Thus we may assume $G = 1$ and $V = \mathbb{R}^\infty$. Recall from

[3] that there is a weak equivalence of Hopf spaces $\Phi: D(M) \rightarrow C(\mathbb{R}^\infty, M)$, where $D(M)$ is the realization of the diagonal

$$[k] \mapsto S_k N_k \mathcal{Q}(M) = N_k \mathcal{Q}(S_k M)$$

associated with the bisimplicial set of the total singular complex of the nerve of a permutative category $\mathcal{Q}(M)$, whose space of objects is $\coprod_{p \geq 0} M^p$, and whose morphisms from $(a_i) \in M^p$ to $(b_j) \in M^q$ are maps of finite sets $\theta: \{1, \dots, p\} \rightarrow \{1, \dots, q\}$ such that $b_j = \sum_{i \in \theta^{-1}(j)} a_i$ holds for $1 \leq j \leq q$.

Since Φ is natural in M , Theorem 7 follows from

Proposition 9. *The natural map $D(I_+(\mathbb{R}) \wedge M) \rightarrow D(I(\mathbb{R}) \wedge M)$, induced by the inclusion $I_+(\mathbb{R}) \subset I(\mathbb{R})$, is a group completion.*

The rest of this section is devoted to the proof of this proposition.

Given a map of topological monoids $f: D \rightarrow D'$ let $B(D, D')$ denote the realization of the category $\mathcal{B}(D, D')$ whose space of objects is D' and whose space of morphisms is the product $D \times D'$, where $(d, d') \in D \times D'$ is regarded as a morphism from d' to $f(d) \cdot d'$. Then there is a sequence of maps

$$D' = B(0, D') \rightarrow B(D, D') \rightarrow B(D, 0) = BD$$

induced by the maps $0 \rightarrow D$ and $D' \rightarrow 0$ respectively. Observe that BD is the standard classifying space of the monoid D and $B(D, D)$ is contractible when f is the identity.

Let $D = D(I_+(\mathbb{R}) \wedge M)$, $D' = D(I(\mathbb{R}) \wedge M)$, and let $i: D \rightarrow D'$ be the map induced by the inclusion $I_+(\mathbb{R}) \rightarrow I(\mathbb{R})$. Then we have a commutative diagram

$$(5.2) \quad \begin{array}{ccccc} D & \longrightarrow & B(D, D) & \longrightarrow & BD \\ i \downarrow & & \downarrow B(1, i) & & \parallel \\ D' & \longrightarrow & B(D, D') & \longrightarrow & BD \end{array}$$

in which the upper and the lower sequences are associated with the identity and the inclusion $i: D \rightarrow D'$, respectively.

Lemma 10. *The natural map $D \rightarrow \Omega BD$ is a group completion.*

This follows from the fact that D is a homotopy commutative, hence admissible, monoid.

Lemma 11. *The lower sequence in the diagram (5.2) is a homotopy fibration sequence with contractible total space $B(D, D')$.*

Proposition 9 can be deduced from these lemmas, because $D \rightarrow D'$ is equivalent to the group completion map $D \rightarrow \Omega BD$ under the equivalence $D' \simeq \Omega BD$.

Outline of the Proof of Theorem 8. We need the following

Proposition 12. *Let T be a G -equivariant continuous functor of the category of pointed G -spaces and pointed maps to itself. Suppose T satisfies the following conditions.*

C1: $T* = *$

C2: $\|T(X_\bullet)\| \simeq_G T(\|X_\bullet\|)$ for any simplicial objects $X_\bullet$.

C3: $T(X \vee Y) \simeq_G TX \times TY$

C4: $T(G/H_+ \wedge X) \simeq_G \text{Map}(G/H, TX)$ for any subgroup H .

Suppose further that TX^H is grouplike for any X and any subgroup H of G . Then we have

- (a) $TX \simeq_G \Omega^W T(\Sigma^W X)$ for any real G -module W ,
- (b) $X \mapsto \{\pi_n TX^G\}$ defines an $RO(G)$ -gradable equivariant homology theory on the category of pointed G -spaces.

Proof. As TX^H is grouplike for any H the natural map $TX \rightarrow \Omega T(\Sigma X)$ is a weak G -equivalence. Hence by the argument similar to the proof of [3, Theorem 2.12]

$$TA \rightarrow TX \rightarrow T(X \cup CA)$$

is a G -fibration sequence up to weak G -equivalence for any pair of pointed G -spaces (X, A) . It also follows by the property of G -equivariant continuous functor that T preserves G -homotopies. Thus (a) implies (b).

To prove (a) we need only show that the correspondence $S \mapsto T(S \wedge X)$, where S is any pointed finite G -set, defines a special Γ_G -space in the sense of [2], that is, the natural map

$$T(S \wedge X) \rightarrow \text{Map}_0(S, TX)$$

is a G -equivalence. But this follows from the conditions C3 and C4. $\square$

Let $TX = I(V, X \wedge M)$. We shall show that T satisfies the conditions C1, C2, C3 and C4. This of course implies Theorem 8.

Clearly the condition C1 holds, and C2 is a routine exercise. That C3 holds is proved as follows.

To verify C4 we shall show that $T(G/H_+ \wedge X) \rightarrow \text{Map}(G/H, TX)$ has a G -homotopy inverse ρ defined as follows:

- (1) Choose a G -embedding $G/H \rightarrow V$ and a linear G -isometry $V \times V \rightarrow V$.
- (2) For given $f: G/H \rightarrow TX$ write $f(gH) = (c(gH), a(gH))$, where $c(gH) \subset V$, $a(gH): c(gH) \rightarrow X \wedge M \wedge I(\mathbb{R})$.
- (3) Define $\tilde{c}$ to be the image of $\bigcup \{gH\} \times c(gH)$ under the embedding

$$\iota: G/H \times V \subset V \times V \rightarrow V$$

- (4) Define $\tilde{a}: \tilde{c} \rightarrow G/H_+ \wedge X \wedge I(\mathbb{R}) \wedge M$ by

$$\tilde{a}(\iota(gH, \xi)) = gH \wedge a(gH)(\xi), \quad \xi \in c(gH)$$

- (5) $\rho: \text{Map}(G/H, TX) \rightarrow T(G/H_+ \wedge X)$ is given by $\rho(f) = (\tilde{c}, \tilde{a})$.

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