

HOMOTOPY TYPE OF THE UNITARY GROUP OF THE UNIFORM ROE ALGEBRA ON $\mathbb{Z}^n$

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ABSTRACT. We study the homotopy type of the space of the unitary group $U_1(C_u^*(|\mathbb{Z}^n|))$ of the uniform Roe algebra $C_u^*(|\mathbb{Z}^n|)$ of $\mathbb{Z}^n$. We show that the stabilizing map $U_1(C_u^*(|\mathbb{Z}^n|)) \rightarrow U_\infty(C_u^*(|\mathbb{Z}^n|))$ is a homotopy equivalence. Moreover, when $n = 1, 2$, we determine the homotopy type of $U_1(C_u^*(|\mathbb{Z}^n|))$, which is the product of the unitary group $U_1(C^*(|\mathbb{Z}^n|))$ (having the homotopy type of $U_\infty(\mathbb{C})$ or $\mathbb{Z} \times BU_\infty(\mathbb{C})$ depending on the parity of n) of the Roe algebra $C^*(|\mathbb{Z}^n|)$ and rational Eilenberg–MacLane spaces.

1. INTRODUCTION

For a C^* -algebra A , let $GL_d(A)$ and $U_d(A)$ denote the space of the invertible and unitary matrices with entries in A , respectively. It is well-known that they always have the same homotopy type. We will often refer only to $U_d(A)$ but most statements are valid for $GL_d(A)$ as well. There have been a lot of works on the homotopy theory of $U_d(A)$ and some of them have important applications. For finite-dimensional case, the complex-valued unitary matrices $U_d(\mathbb{C})$ is just the usual unitary group acting linearly on $\mathbb{C}^d$. For infinite-dimensional case, Kuiper [Kui65] proved that the space of all unitary operators on an infinite-dimensional Hilbert space is contractible. This result is basic in the Atiyah–Singer index theory. This kind of contractibility result has been extended to $U_d(A)$ of some other algebras A while A is all the bounded operators on a infinite dimensional Hilbert space in the original result. Of course it is not always the case for $U_d(A)$ of other infinite-dimensional C^* -algebras A . In general, it is hard to determine the homotopy type of $U_d(A)$.

Let us use the notation

$$GL_\infty(A) = \lim_{d \rightarrow \infty} GL_d(A) \quad \text{and} \quad U_\infty(A) = \lim_{d \rightarrow \infty} U_d(A).$$

It is well-known that $U_\infty(A)$ has the same homotopy type as $U_1(A \otimes \mathcal{K})$ where $\mathcal{K}$ is the space of compact operators. The K -theory $K_i(A)$ ($i = 0, 1$) is a basic homotopy invariant of A , which is characterized as

$$K_0(A) = \pi_1(U_\infty(A)) \quad \text{and} \quad K_1(A) = \pi_0(U_\infty(A)).$$

Since $U_d(A)$ is not necessarily homotopy equivalent to $U_\infty(A)$, $K_i(A)$ is not a so strong invariant in general. But sometimes the natural map $U_d(A) \rightarrow U_\infty(A)$, which we will call the *stabilizing map*, becomes a homotopy equivalence. Study on such stability can be traced back to the work of Bass [Bas64]. There have been a various works on this kind of stability. Rieffel introduced the *topological stable rank* in [Rie83] and applied it to show the stability of the non-commutative

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torus in [Rie87], which is a key tool in the present work. It is difficult in general to determine how stable a given C^* -algebra is.

In the present paper, we study the stability of the uniform Roe algebra $C_u^*(\mathbb{Z}^n)$ on $\mathbb{Z}^n$ and investigate its homotopy type. The *uniform Roe algebra* $C_u^*(X)$ of a metric space X is introduced by Roe in [Roe88] to establish an index theory on open manifolds, where the index lives in the K -theory $K_*(C_u^*(X))$. The algebra $C_u^*(X)$ itself is also important since it encodes a kind of “large scale geometry” of X . Studying the homotopy type of $U_d(C_u^*(X))$ will provide more insights from a homotopy theoretic viewpoint, which cannot be obtained only from its K -theory. But there are only a few works on the homotopy type of $U_d(C_u^*(X))$ yet. For example, Manuilov and Troitsky [MT21] studied some condition for $U_d(C_u^*(X))$ being contractible. In the present work, we observe the other extreme, that is, $U_d(C_u^*(\mathbb{Z}^n))$ has a highly nontrivial homotopy type.

We give some comment on the relation with our previous work [KKT] on the space $\mathcal{U}$ of finite propagation unitary operators on $\mathbb{Z}$. Note that $U_1(C_u^*(\mathbb{Z}))$ can be viewed as a kind of completion of $\mathcal{U}$. We determined the homotopy type of $\mathcal{U}$ there. But it is not clear whether $\mathcal{U}$ has the same homotopy type as $U_1(C_u^*(\mathbb{Z}))$. Actually, they turn out to have the same homotopy type (Theorem 1.2). Also, the method there does not seem to be extended to $\mathbb{Z}^n$ when $n \geq 2$. We employ rather operator algebraic technique in the present paper. Our method here reduces the problem to determine the homotopy type of $U_d(C_u^*(\mathbb{Z}^n))$ to the one to show the surjectivity of the homomorphism on K -theory $K_*(C_u^*(\mathbb{Z}^n)) \rightarrow K_*(C^*(\mathbb{Z}^n))$ induced from the inclusion (Proposition 7.4), where $C^*(\mathbb{Z}^n)$ denotes the Roe algebra of $\mathbb{Z}^n$.

For stability, we show the following theorem in Section 3.

Theorem 1.1. *For any integer $n \geq 1$, the stabilizing maps*

$$\begin{aligned} \mathrm{GL}_1(C_u^*(\mathbb{Z}^n)) &\rightarrow \mathrm{GL}_\infty(C_u^*(\mathbb{Z}^n)), & U_1(C_u^*(\mathbb{Z}^n)) &\rightarrow U_\infty(C_u^*(\mathbb{Z}^n)), \\ \mathrm{GL}_1(C^*(\mathbb{Z}^n)) &\rightarrow \mathrm{GL}_\infty(C^*(\mathbb{Z}^n)), & U_1(C^*(\mathbb{Z}^n)) &\rightarrow U_\infty(C^*(\mathbb{Z}^n)) \end{aligned}$$

between the spaces of invertible and unitary elements are homotopy equivalences. This implies that these maps induce the following isomorphisms on homotopy groups for all $i \geq 0$:

$$\begin{aligned} \pi_i(\mathrm{GL}_1(C_u^*(\mathbb{Z}^n))) &\cong \pi_i(U_1(C_u^*(\mathbb{Z}^n))) \cong \begin{cases} K_1(C_u^*(\mathbb{Z}^n)) & i \text{ is even,} \\ K_0(C_u^*(\mathbb{Z}^n)) & i \text{ is odd,} \end{cases} \\ \pi_i(\mathrm{GL}_1(C^*(\mathbb{Z}^n))) &\cong \pi_i(U_1(C^*(\mathbb{Z}^n))) \cong \begin{cases} K_1(C^*(\mathbb{Z}^n)) & i \text{ is even,} \\ K_0(C^*(\mathbb{Z}^n)) & i \text{ is odd.} \end{cases} \end{aligned}$$

Let $K(V, i)$ denote the Eilenberg–MacLane space of type (V, i) and $B\mathrm{U}_\infty(\mathbb{C})$ denote the classifying space of the unitary group $\mathrm{U}_\infty(\mathbb{C})$. Also, for based spaces X_i ($i = 1, 2, \dots$), define

$$\prod_{i \geq 1}^{\circ} X_i = \lim_{k \rightarrow \infty} (X_1 \times X_2 \times \cdots \times X_k).$$

For the homotopy type of $U_1(C_u^*(\mathbb{Z}^n))$, we show the following results when $n = 1, 2$.

Theorem 1.2. *There exist homotopy equivalences of infinite loop spaces*

$$\mathrm{GL}_1(C_u^*(\mathbb{Z})) \simeq U_1(C_u^*(\mathbb{Z})) \simeq \mathbb{Z} \times B\mathrm{U}_\infty(\mathbb{C}) \times \prod_{i \geq 1}^{\circ} K(\ell^\infty(\mathbb{Z}, \mathbb{Z})_S, 2i - 1).$$

Theorem 1.3. *There exist homotopy equivalences of infinite loop spaces*

$$\mathrm{GL}_1(C_u^*(|\mathbb{Z}^2|)) \simeq \mathrm{U}_1(C_u^*(|\mathbb{Z}^2|)) \simeq V_1 \times \mathrm{U}_\infty(\mathbb{C}) \times \prod_{i \geq 1}^{\circ} (K(V_0, 2i-1) \times K(V_1, 2i)),$$

where V_0, V_1 are the rational vector spaces given by

$$V_0 = \ker[K_0(C_u^*(|\mathbb{Z}^2|)) \rightarrow K_0(C^*(|\mathbb{Z}^2|))], \quad V_1 = K_1(C_u^*(|\mathbb{Z}^2|))$$

and the product factor V_1 is a discrete space.

More detailed descriptions of the vector spaces V_0 and V_1 appear in the proof of Lemma 7.7.

We will see the existence of a homotopy section of the inclusion $\mathrm{U}_1(C_u^*(|\mathbb{Z}|)) \rightarrow \mathrm{U}_1(B^{\mathrm{SW}})$ in Section 6, where $\mathrm{U}_1(B^{\mathrm{SW}})$ is the Segal–Wilson restricted unitary group [SW85] having the homotopy type of $\mathbb{Z} \times B\mathrm{U}_\infty(\mathbb{C})$. This implies Theorem 1.2. We also show in Section 7 that, for any integer $n \geq 1$, the inclusion $\mathrm{U}_1(C_u^*(|\mathbb{Z}^n|)) \rightarrow \mathrm{U}_1(C^*(|\mathbb{Z}^n|))$ admits a homotopy section if and only if the homomorphism $K_*(C_u^*(|\mathbb{Z}^n|)) \rightarrow K_*(C^*(|\mathbb{Z}^n|))$ is surjective (Proposition 7.4). Since we can see it is surjective when $n = 1, 2$, Theorems 1.2 again and 1.3 follows. If one could show the surjectivity for $n \geq 3$, then a similar homotopy decomposition will immediately follow.

This paper is organized as follows. We fix our notation in Section 2. In Section 3, we recall Rieffel’s results on stability and show Theorem 1.1. In Section 4, we recall the Bott periodicity realized as a $*$ -homomorphism. In Section 5, we recall the Segal–Wilson restricted unitary group and show its stability. In Section 6, we show Theorem 1.2 using the Segal–Wilson restricted unitary group. In Section 7, we discuss the homotopy type of $\mathrm{U}_d(C_u^*(|\mathbb{Z}^n|))$ for general $n \geq 1$ and show Theorems 1.2 again and 1.3.

2. NOTATION

The C^* -algebra of bounded operators on a Hilbert space V is denoted by $\mathcal{B}(V)$ and the subalgebra of compact operators by $\mathcal{K}(V)$. We write the operator norm of $T \in \mathcal{B}(V)$ as $\|T\|$. The Hilbert space of square summable sequences indexed by a discrete group Γ will be written as

$$\ell^2(\Gamma) = \{(v_g)_g \mid \sum_{g \in \Gamma} |v_g|^2 < \infty\}.$$

We also consider the tensor product Hilbert space $\ell^2(\Gamma) \otimes \mathcal{H}$ with an infinite dimensional separable Hilbert space $\mathcal{H}$.

A bounded operator $T \in \mathcal{B}(\ell^2(\Gamma))$ can be expressed in the matrix form as

$$T = (T_{g,h})_{g,h}, \quad T_{g,h} \in \mathbb{C}.$$

For $T \in \mathcal{B}(\ell^2(\Gamma) \otimes \mathcal{H})$, we also have a similar expression $T = (T_{g,h})_{g,h}$ with $T_{g,h} \in \mathcal{B}(\mathcal{H})$.

Definition 2.1. Let Γ be a finitely generated group and d denote the word metric with respect to some finite set of generators. We say that a bounded operator $T \in \mathcal{B}(\ell^2(\Gamma))$ has *finite propagation* if

$$\mathrm{prop}(T) = \sup\{d(g, h) \mid T_{g,h} \neq 0\}$$

is finite. We define finite propagation for $T = (T_{g,h})_{g,h} \in \mathcal{B}(\ell^2(\Gamma) \otimes \mathcal{H})$ similarly.

Example 2.2. The *shift* $S_x \in \mathcal{B}(\ell^2(\Gamma))$ by $x \in \Gamma$ is defined by

$$S_x = ((S_x)_{g,h})_{g,h}, \quad S_{g,h} = \begin{cases} 1 & g^{-1}h = x, \\ 0 & \text{otherwise.} \end{cases}$$

The operator S_x is a unitary operator with $\text{prop}(S_x) = d(x, 1)$.

It is easy to see that the definition of having finite propagation is independent of the choice of generators while the value of $\text{prop}(T)$ depends on the word metric. Since we have

$$\text{prop}(ST) \leq \text{prop}(S) + \text{prop}(T), \quad \text{prop}(T^*) = \text{prop}(T), \quad \text{prop}(1) = 0$$

for any finite propagation operators $S, T \in \mathcal{B}(\ell^2(\Gamma))$, the subset of finite propagation operators becomes a unital $*$ -subalgebra of $\mathcal{B}(\ell^2(\Gamma))$. Similar properties hold for finite propagation operators $S, T \in \mathcal{B}(\ell^2(\Gamma) \otimes \mathcal{H})$ such that the components $T_{g,h}$ and $S_{g,h}$ are compact operators.

Definition 2.3. The *uniform Roe algebra* $C_u^*(|\Gamma|)$ of Γ is the norm closure of the algebra of finite propagation operators in $\mathcal{B}(\ell^2(\Gamma))$.

Definition 2.4. The *Roe algebra* $C^*(|\Gamma|)$ of Γ is the norm closure of the algebra of finite propagation operators $T \in \mathcal{B}(\ell^2(\Gamma) \otimes \mathcal{H})$ such that each component $T_{g,h}$ is a compact operator.

Remark 2.5. We follow the usual notation $C^*(|\Gamma|)$ for the Roe algebra of Γ to distinguish it from the group C^* -algebra of Γ though we do not consider the latter here.

We will consider the uniform Roe algebra $C_u^*(|\Gamma|)$ is a subalgebra of the Roe algebra $C^*(|\Gamma|)$ with respect to some inclusion $\mathbb{C} \subset \mathcal{H}$.

We use the symbol $\ell^\infty(\Gamma, \mathbb{C})$ to express the Banach algebra of $\mathbb{C}$ -valued bounded sequences indexed by Γ rather than the simpler symbol $\ell^\infty(\Gamma)$ since we also consider the abelian group of $\mathbb{Z}$ -valued bounded sequences $\ell^\infty(\Gamma, \mathbb{Z})$.

The group Γ acts on the algebras $\ell^\infty(\Gamma, \mathbb{C})$ and $\ell^\infty(\Gamma, \mathcal{K}(\mathcal{H}))$ by right translation. The action by $x \in \Gamma$ is compatible with the conjugation by S_x through the diagonal inclusion $\ell^\infty(\Gamma, \mathbb{C}) \rightarrow C_u^*(|\Gamma|)$ or $\ell^\infty(\Gamma, \mathcal{K}(\mathcal{H})) \rightarrow C^*(|\Gamma|)$ given by

$$(t_g)_g \mapsto (T_{g,h})_{g,h}, \quad T_{g,h} = \begin{cases} t_g & g = h, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, this inclusion extends to the well-known isomorphisms

$$\ell^\infty(\Gamma, \mathbb{C}) \rtimes \Gamma \cong C_u^*(|\Gamma|), \quad \ell^\infty(\Gamma, \mathcal{K}(\mathcal{H})) \rtimes \Gamma \cong C^*(|\Gamma|)$$

from the reduced crossed products of C^* -algebras. For example, see [Roe03, Theorem 4.28].

The $d \times d$ -matrix algebra $M_d(A)$ of a C^* -algebra A is again a C^* -algebra. The spaces of invertible elements and unitary elements in $M_d(A)$ will be denoted as $\text{GL}_d(A)$ and $\text{U}_d(A)$. The *stabilizing maps* are given as

$$\text{GL}_1(A) \rightarrow \text{GL}_\infty(A) = \lim_{d \rightarrow \infty} \text{GL}_d(A), \quad \text{U}_1(A) \rightarrow \text{U}_\infty(A) = \lim_{d \rightarrow \infty} \text{U}_d(A),$$

where the inductive limits are taken along the inclusions $\text{GL}_d(A) \subset \text{GL}_{d+1}(A)$ and $\text{U}_d(A) \subset \text{U}_{d+1}(A)$. The inductive limit spaces $\text{GL}_\infty(A)$ and $\text{U}_\infty(A)$ are well-known to be homotopy equivalent to the spaces $\text{GL}_1(A \otimes \mathcal{K}(\mathcal{H}))$ and $\text{U}_1(A \otimes \mathcal{K}(\mathcal{H}))$.

3. STABILITY

The aim of this section is to prove Theorem 1.1. Once the assumption of the following result by Rieffel [Rie87] is verified, the theorem will immediately follow.

Theorem 3.1 (Rieffel). *Let A be a unital C^* -algebra. If A is tsr -boundedly divisible, then the stabilizing maps*

$$\text{GL}_1(A) \rightarrow \text{GL}_\infty(A) \quad \text{and} \quad \text{U}_1(A) \rightarrow \text{U}_\infty(A)$$

are homotopy equivalences.

Remark 3.2. The original statement of Theorem 4.13 in [Rie87] is involved only with homotopy groups. But what is actually proved there is slightly stronger as above.

For the definitions of the *topological stable rank* $\text{tsr}(A) \in \mathbb{Z}_{\geq 1}$, see [Rie83]. A C^* -algebra A is said to be *tsr-boundedly divisible* [Rie87] if there is a constant K such that for any integer m , there exists an integer $d \geq m$ such that A is isomorphic to $M_d(B)$ for some C^* -algebra B with $\text{tsr}(B) \leq K$. To verify the assumption, we need the following two lemmas.

Lemma 3.3. *The topological stable ranks of $C_u^*(|\mathbb{Z}^n|)$ and $C^*(|\mathbb{Z}^n|)$ are estimated as*

$$\text{tsr}(C_u^*(|\mathbb{Z}^n|)) \leq n + 1 \quad \text{and} \quad \text{tsr}(C^*(|\mathbb{Z}^n|)) \leq n + 1.$$

Remark 3.4. We will see that both $C_u^*(|\mathbb{Z}^n|)$ and $C^*(|\mathbb{Z}^n|)$ are tsr -boundedly divisible using this lemma. Thus we will actually obtain the estimates $\text{tsr}(C_u^*(|\mathbb{Z}^n|)) \leq 2$ and $\text{tsr}(C^*(|\mathbb{Z}^n|)) \leq 2$ by [Rie87, Proposition 4.6].

Proof. Let $A = \mathbb{C}$ or $\mathcal{K}(\mathcal{H})$. Since the invertible elements in $\ell^\infty(\mathbb{Z}^n, \mathbb{C})$ and $\mathbb{C} \oplus \ell^\infty(\mathbb{Z}^n, \mathcal{K}(\mathcal{H}))$ are dense, we have

$$\text{tsr}(\ell^\infty(\mathbb{Z}^n, A)) = 1$$

by [Rie83, Proposition 3.1]. Considering the restricted action of $\mathbb{Z}^m \subset \mathbb{Z}^n$ on the first m factors of $\mathbb{Z}^n$, we obtain the isomorphism

$$\ell^\infty(\mathbb{Z}^n, A) \rtimes \mathbb{Z}^{m+1} \cong (\ell^\infty(\mathbb{Z}^n, A) \rtimes \mathbb{Z}^m) \rtimes \mathbb{Z}.$$

Thus, by [Rie83, Theorem 7.1], we get the desired estimates on $\text{tsr}(C_u^*(|\mathbb{Z}^n|))$ and $\text{tsr}(C^*(|\mathbb{Z}^n|))$. $\square$

Lemma 3.5. *For any integer $d \geq 1$, there exist isomorphisms*

$$\phi: C_u^*(|\mathbb{Z}^n|) \cong M_d(C_u^*(|\mathbb{Z}^n|)) \quad \text{and} \quad \phi: C^*(|\mathbb{Z}^n|) \cong M_d(C^*(|\mathbb{Z}^n|)).$$

Proof. Let $V = \mathbb{C}$ or $\mathcal{H}$. According to the decomposition

$$\ell^2(\mathbb{Z}^n) \otimes V = \bigoplus_{(i_1, \dots, i_n) \in \mathbb{Z}^n} V_{(i_1, \dots, i_n)}, \quad V_{(i_1, \dots, i_n)} \cong V,$$

we have the matrix expression for $T \in \mathcal{B}(\ell^2(\mathbb{Z}^n) \otimes V)$

$$T = (T_{(i_1, \dots, i_n)(j_1, \dots, j_n)}(i_1, \dots, i_n)(j_1, \dots, j_n)), \quad T_{(i_1, \dots, i_n)(j_1, \dots, j_n)}: V_{(j_1, \dots, j_n)} \rightarrow V_{(i_1, \dots, i_n)}.$$

Consider the map $\phi: \mathcal{B}(\ell^2(\mathbb{Z}^n) \otimes V) \rightarrow M_d(\mathcal{B}(\ell^2(\mathbb{Z}^n) \otimes V))$ given by

$$\phi(T)_{(i_1, \dots, i_n)(j_1, \dots, j_n)} = \begin{pmatrix} T_{(di_1, \dots, di_n)(dj_1, \dots, dj_n)} & \cdots & T_{(di_1, \dots, di_n)(dj_1 + d - 1, \dots, dj_n)} \\ \vdots & \ddots & \vdots \\ T_{(di_1 + d - 1, \dots, di_n)(dj_1, \dots, dj_n)} & \cdots & T_{(di_1 + d - 1, \dots, di_n)(dj_1 + d - 1, \dots, dj_n)} \end{pmatrix} \in M_d(\mathcal{B}(V)).$$

The restrictions to $C_u^*(|\mathbb{Z}^n|) \subset \mathcal{B}(\ell^2(\mathbb{Z}^n))$ and $C^*(|\mathbb{Z}^n|) \subset \mathcal{B}(\ell^2(\mathbb{Z}^n) \otimes \mathcal{H})$ are desired isomorphisms. $\square$

Remark 3.6. When $n = 1$, the map ϕ is just taking the block matrix of which each block is a $d \times d$ -matrix.

Proof of Theorem 1.1. By Lemmas 3.3 and 3.5, we can apply Theorem 3.1 to $C_u^*(|\mathbb{Z}^n|)$ and $C^*(|\mathbb{Z}^n|)$. This completes the proof of the theorem. $\square$

4. BOTT PERIODICITY

Let us recall the Bott periodicity of C^* -algebras here. Let A be a C^* -algebra, which might be non-unital. The direct sum $\mathbb{C} \oplus A$ is considered as the unitization with unit $(1, 0) \in \mathbb{C} \oplus A$. Define the unitary group $U'_d(A)$ by

$$U'_d(A) = \{U \in U_n(\mathbb{C} \oplus A) \mid U - (I_d, 0) \in M_d(A)\}.$$

If A is already unital, we have a canonical isomorphism $U'_d(A) \cong U_d(A)$. So we use the same symbol $U_d(A)$ for $U'_d(A)$ even if A is not unital.

Consider the following space of continuous functions:

$$C_0(\mathbb{R}^m, A) = \{T: \mathbb{R}^m \rightarrow A \mid T \text{ is continuous and } \lim_{|z| \rightarrow \infty} T(z) = 0\}.$$

This is a C^* -algebra without unit. Notice that $C_0(\mathbb{R}^m, A)$ is isomorphic to the space $\Omega^m A$ of based maps from the m -sphere S^m to A where the basepoint $*$ in S^m is mapped to $0 \in A$.

Set the element

$$p_B(z) = \frac{1}{1 + |z|^2} \begin{pmatrix} |z|^2 & z \\ \bar{z} & 1 \end{pmatrix} \in M_2(\mathbb{C} \oplus C_0(\mathbb{R}^2, \mathbb{C})) \quad (z \in \mathbb{R}^2),$$

where we identify $\mathbb{R}^2 \cong \mathbb{C}$ in the matrix entries. The *Bott map* $\beta: A \rightarrow M_2(A \oplus C_0(\mathbb{R}^2, A))$ is a $*$ -homomorphism defined by

$$\beta(a) = p_B \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} p_B.$$

Then we have the commutative square of unital C^* -algebras

$$\begin{array}{ccc} \mathbb{C} \oplus A & \xrightarrow{\epsilon} & \mathbb{C} \\ \beta \downarrow & & \downarrow \eta \\ M_2(\mathbb{C} \oplus A \oplus C_0(\mathbb{R}^2, A)) & \xrightarrow{\epsilon} & M_2(\mathbb{C} \oplus A) \end{array}$$

where $\epsilon: \mathbb{C} \oplus A \rightarrow \mathbb{C}$ and $\epsilon: M_2(\mathbb{C} \oplus A \oplus C_0(\mathbb{R}^2, A)) \rightarrow M_2(\mathbb{C} \oplus A)$ are the projections and $\eta: \mathbb{C} \rightarrow M_2(\mathbb{C} \oplus A)$ is the unit map. This square induces the $*$ -homomorphism between the kernels of ϵ :

$$\beta: A \rightarrow M_2(C_0(\mathbb{R}^2, A)).$$

We call this β the *Bott map* as well. It is natural in the following sense: if $f: A \rightarrow B$ is a $*$ -homomorphism between C^* -algebras, then the following square commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \beta \downarrow & & \downarrow \beta \\ M_2(C_0(\mathbb{R}^2, A)) & \xrightarrow{f_*} & M_2(C_0(\mathbb{R}^2, B)) \end{array}$$

Proposition 4.1. *The Bott map $\beta: A \rightarrow M_2(C_0(\mathbb{R}^2, A))$ induces an isomorphism on K -theory.*

Remark 4.2. This can be seen as a formulation of the Bott periodicity. If you wish to deduce this proposition from the results appearing in [Bla86], it follows from the observation 9.2.10 on the generator of $K_0(C_0(\mathbb{R}^2, \mathbb{C}))$ and the Künneth theorem for tensor products (Theorem 23.1.3).

The Bott periodicity provides the natural homotopy equivalence

$$U_\infty(A) \xrightarrow{\beta} U_\infty(M_2(C_0(\mathbb{R}^2, A))) \simeq \Omega^2 U_\infty(A),$$

which is a group homomorphism. Thus we obtain the following proposition on infinite loop structure.

Proposition 4.3. *The unitary group $U_\infty(A)$ of a C^* -algebra A is equipped with a canonical infinite loop space structure such that the map $U_\infty(A) \rightarrow U_\infty(B)$ induced from a $*$ -homomorphism $A \rightarrow B$ is an infinite loop map. Moreover, the underlying loop structure of $U_\infty(A)$ coincides with the group structure of $U_\infty(A)$.*

Remark 4.4. The last sentence in the proposition means that the classifying space $BU_\infty(A)$ of the topological group $U_\infty(A)$ is homotopy equivalent to the identity component of $\Omega U_\infty(A)$.

5. SEGAL–WILSON RESTRICTED UNITARY GROUP

To study the homotopy type of $U_1(C_u^*(|\mathbb{Z}|))$, we will relate it with other spaces. One is the Segal–Wilson restricted unitary group $U_1(B^{\text{SW}})$ and the other is the unitary group of the Roe algebra $U_1(C^*(|\mathbb{Z}|))$. We recall the former in this section.

We have another matrix expression for $T \in \mathcal{B}(\ell^2(\mathbb{Z}))$ as

$$T = \begin{pmatrix} T_{--} & T_{-+} \\ T_{+-} & T_{++} \end{pmatrix},$$

where

$$\begin{aligned} T_{--} &: \ell^2(\mathbb{Z}_{<0}) \rightarrow \ell^2(\mathbb{Z}_{<0}), & T_{-+} &: \ell^2(\mathbb{Z}_{\geq 0}) \rightarrow \ell^2(\mathbb{Z}_{<0}), \\ T_{+-} &: \ell^2(\mathbb{Z}_{<0}) \rightarrow \ell^2(\mathbb{Z}_{\geq 0}), & T_{++} &: \ell^2(\mathbb{Z}_{\geq 0}) \rightarrow \ell^2(\mathbb{Z}_{\geq 0}). \end{aligned}$$

Definition 5.1. We define the C^* -algebra B^{SW} by

$$B^{\text{SW}} := \{T \in \mathcal{B}(\ell^2(\mathbb{Z})) \mid T_{-+}, T_{+-} \text{ are compact}\}.$$

The symbol “SW” stands for Segal–Wilson. The unitary group $U_1(B^{\text{SW}})$ is called the *restricted unitary group* in the work of Segal and Wilson [SW85]. They used it as a model of the infinite Grassmannian.

Lemma 5.2 (Segal–Wilson). *The space $U_1(B^{\text{SW}})$ has the homotopy type of $\mathbb{Z} \times BU_\infty(\mathbb{C})$. Moreover, the map*

$$\pi_0(U_1(B^{\text{SW}})) \rightarrow \mathbb{Z}, \quad [U] \mapsto \text{ind}(U_{++}),$$

is bijective, where $\text{ind}(U_{++})$ denotes the Fredholm index of the Fredholm operator U_{++} .

Let $S = S_{+1} \in B^{\text{SW}}$ the shift operator as in Example 2.2. We have $\text{ind } S^n = n$.

The goal of this section is to see the following.

Proposition 5.3. *The stabilizing maps*

$$GL_1(B^{\text{SW}}) \rightarrow GL_\infty(B^{\text{SW}}) \quad \text{and} \quad U_1(B^{\text{SW}}) \rightarrow U_\infty(B^{\text{SW}})$$

are homotopy equivalences.

To show this, we do not use a kind of stability as in Section 3.

Lemma 5.4. *For any integer $d \geq 1$, the inclusion*

$$U_1(B^{\text{SW}}) \rightarrow U_d(B^{\text{SW}})$$

induces an isomorphism on π_0 .

Proof. Consider the composite of the inclusion and the isomorphism $\phi: B^{\text{SW}} \rightarrow M_d(B^{\text{SW}})$ similar to the one in the proof of Lemma 3.5:

$$U_1(B^{\text{SW}}) \rightarrow U_d(B^{\text{SW}}) \xrightarrow{\phi^{-1}} U_1(B^{\text{SW}}).$$

It is easy to see that the image of the shift $S \in U_1(B^{\text{SW}})$ under this composite again has index 1. This implies the lemma. $\square$

Lemma 5.5. *The K -theory of B^{SW} is computed as*

$$K_i(B^{\text{SW}}) \cong \begin{cases} 0 & i = 0, \\ \mathbb{Z} & i = 1, \end{cases}$$

where $K_1(B^{\text{SW}})$ is generated by the shift $S \in U_1(B^{\text{SW}})$.

Proof. This follows from the isomorphisms

$$K_0(B^{\text{SW}}) \cong \lim_{d \rightarrow \infty} \pi_1(U_d(B^{\text{SW}})) \quad \text{and} \quad K_1(B^{\text{SW}}) \cong \lim_{d \rightarrow \infty} \pi_0(U_d(B^{\text{SW}}))$$

and Lemmas 5.2 and 5.4. $\square$

Lemma 5.6. *For any $i \geq 0$, there exists an integer $m \geq 1$ such that the iterated Bott map*

$$\beta^i: U_d(B^{\text{SW}}) \rightarrow U_d(M_{2^i}(C_0(\mathbb{R}^{2^i}, B^{\text{SW}})))$$

induces an isomorphism on π_0 if $d \geq m$.

Proof. From the isomorphisms

$$\pi_0(U_d(M_{2^i}(C_0(\mathbb{R}^{2^i}, B^{\text{SW}}))) \cong \pi_{2^i}(U_{2^i d}(B^{\text{SW}})) \cong \pi_{2^i}(U_1(B^{\text{SW}})) \cong \mathbb{Z}$$

and

$$K_1(M_{2^i}(C_0(\mathbb{R}^{2^i}, B^{\text{SW}}))) \cong \lim_{d \rightarrow \infty} \pi_0(U_d(M_{2^i}(C_0(\mathbb{R}^{2^i}, B^{\text{SW}}))) \cong K_1(B^{\text{SW}}) \cong \mathbb{Z},$$

we can find an integer $m \geq 1$ such that the stabilizing map

$$\pi_0(U_d(M_{2^i}(C_0(\mathbb{R}^{2i}, B^{\text{SW}}))) \rightarrow K_1(C_0(M_{2^i}(\mathbb{R}^{2i}, B^{\text{SW}})))$$

is an isomorphism if $d \geq m$. Consider the commutative diagram

$$\begin{array}{ccc} \pi_0(U_d(B^{\text{SW}})) & \xrightarrow{\cong} & K_1(B^{\text{SW}}) \\ (\beta^i)_* \downarrow & & \downarrow (\beta^i)_* \\ \pi_0(U_d(M_{2^i}(C_0(\mathbb{R}^{2i}, B^{\text{SW}})))) & \xrightarrow{\cong} & K_1(M_{2^i}(C_0(\mathbb{R}^{2i}, B^{\text{SW}}))) \end{array}$$

where the top arrow is an isomorphism by Lemma 5.4 and the right Bott map β^i is an isomorphism by Proposition 4.1. Then the lemma follows. $\square$

Proof of Proposition 5.3. Take an integer $i \geq 0$. We can find an integer $m \geq 1$ as in Proposition 5.6 and

$$\pi_{2i}(U_d(B^{\text{SW}})) \rightarrow \pi_{2i}(U_\infty(B^{\text{SW}})) \cong \mathbb{Z}$$

is an isomorphism if $d \geq m$. Consider the following commutative diagram:

$$\begin{array}{ccccc} U_1(C_0(\mathbb{R}^{2i}, B^{\text{SW}})) & \xrightarrow{\cong} & U_1(M_{2^i}(C_0(\mathbb{R}^{2i}, B^{\text{SW}}))) & \xleftarrow{\beta^i} & U_1(B^{\text{SW}}) \\ \downarrow & & \downarrow & & \downarrow \text{isom. on } \pi_0 \\ U_d(C_0(\mathbb{R}^{2i}, B^{\text{SW}})) & \xrightarrow{\cong} & U_d(M_{2^i}(C_0(\mathbb{R}^{2i}, B^{\text{SW}}))) & \xleftarrow{\beta^i} & U_d(B^{\text{SW}}) \end{array}$$

where the left horizontal arrows are the isomorphisms similar to the one in Lemma 3.5 and the vertical arrows are the inclusions. Since the composite

$$U_1(B^{\text{SW}}) \rightarrow U_d(B^{\text{SW}}) \xrightarrow{\beta^i} U_d(M_{2^i}(C_0(\mathbb{R}^{2i}, B^{\text{SW}})))$$

induces an isomorphism on π_0 , the middle vertical arrow

$$U_1(M_{2^i}(C_0(\mathbb{R}^{2i}, B^{\text{SW}}))) \rightarrow U_d(M_{2^i}(C_0(\mathbb{R}^{2i}, B^{\text{SW}})))$$

induces a surjection on π_0 . But it is indeed an isomorphism as their π_0 are isomorphic to $\mathbb{Z}$. Then the map

$$U_1(C_0(\mathbb{R}^{2i}, B^{\text{SW}})) \rightarrow U_d(C_0(\mathbb{R}^{2i}, B^{\text{SW}}))$$

induces an isomorphism on π_0 . This implies that the map

$$U_1(B^{\text{SW}}) \rightarrow U_d(B^{\text{SW}})$$

induces an isomorphism on π_{2i} . Thus the map

$$U_1(B^{\text{SW}}) \rightarrow U_\infty(B^{\text{SW}})$$

induces an isomorphism on π_{2i} . This completes the proof. $\square$

6. HOMOTOPY TYPE OF $U_1(C_u^*(|\mathbb{Z}|))$

The goal of this section is to prove Theorem 1.2. The components T_{-+} and T_{+-} of a finite propagation operator $T \in \mathcal{B}(H)$ are finite rank operators. This implies the inclusion

$$C_u^*(|\mathbb{Z}|) \subset B^{\text{SW}}.$$

This map is a key to the proof of Theorem 1.2.

We begin with computing the K -theory.

Proposition 6.1. *The following isomorphism holds:*

$$K_i(C_u^*(|\mathbb{Z}|)) \cong \begin{cases} \ell^\infty(\mathbb{Z}, \mathbb{Z})_S & i = 0, \\ \mathbb{Z} & i = 1, \end{cases}$$

where

$$\ell^\infty(\mathbb{Z}, \mathbb{Z})_S = \ell^\infty(\mathbb{Z}, \mathbb{Z}) / \{a - Sa \mid a \in \ell^\infty(\mathbb{Z}, \mathbb{Z})\}$$

is the coinvariant by the shift $S : \ell^\infty(\mathbb{Z}, \mathbb{Z}) \rightarrow \ell^\infty(\mathbb{Z}, \mathbb{Z})$.

Proof. Applying the Pimsner–Voiculescu exact sequence [PV80] to the crossed product

$$C_u^*(|\mathbb{Z}|) \cong \ell^\infty(\mathbb{Z}, \mathbb{C}) \rtimes \mathbb{Z},$$

we get the six-term cyclic exact sequence:

$$\begin{array}{ccccc} K_0(\ell^\infty(\mathbb{Z}, \mathbb{C})) & \xrightarrow{1-S} & K_0(\ell^\infty(\mathbb{Z}, \mathbb{C})) & \longrightarrow & K_0(C_u^*(|\mathbb{Z}|)) \\ \uparrow & & & & \downarrow \\ K_1(C_u^*(|\mathbb{Z}|)) & \longleftarrow & K_1(\ell^\infty(\mathbb{Z}, \mathbb{C})) & \xleftarrow{1-S} & K_1(\ell^\infty(\mathbb{Z}, \mathbb{C})) \end{array}$$

As is well-known, we have

$$K_i(\ell^\infty(\mathbb{Z}, \mathbb{C})) \cong \begin{cases} \ell^\infty(\mathbb{Z}, \mathbb{Z}) & i = 0, \\ 0 & i = 1, \end{cases}$$

where the induced homomorphism $S : \ell^\infty(\mathbb{Z}, \mathbb{Z}) \rightarrow \ell^\infty(\mathbb{Z}, \mathbb{Z})$ is the shift as well. Thus we can compute $K_*(C_u^*(|\mathbb{Z}|))$ by the previous exact sequence. $\square$

We saw the homotopy stabilities as in Theorem 1.1 and Proposition 5.3. Then it is sufficient to investigate the inclusion $U_\infty(C_u^*(|\mathbb{Z}|)) \rightarrow U_\infty(B^{\text{SW}})$.

Lemma 6.2. *The inclusion $U_\infty(C_u^*(|\mathbb{Z}|)) \rightarrow U_\infty(B^{\text{SW}})$ induces isomorphisms on π_{2i} for $i \geq 0$.*

Proof. By Lemma 5.5, $K_1(B^{\text{SW}})$ is isomorphic to $\mathbb{Z}$ and generated by the shift $S \in B^{\text{SW}}$. Since $S \in C_u^*(|\mathbb{Z}|)$ and $K_1(C_u^*(|\mathbb{Z}|)) \cong \mathbb{Z}$, the map $K_1(C_u^*(|\mathbb{Z}|)) \rightarrow K_1(B^{\text{SW}})$ is an isomorphism. Thus the map $\pi_{2i}(U_\infty(C_u^*(|\mathbb{Z}|))) \rightarrow \pi_{2i}(U_\infty(B^{\text{SW}}))$ is also an isomorphism. $\square$

Let F_1 be the homotopy fiber of the inclusion $U_\infty(C_u^*(|\mathbb{Z}|)) \rightarrow U_\infty(B^{\text{SW}})$.

Proposition 6.3. *The space F_1 has the homotopy type of the product of Eilenberg–MacLane spaces*

$$\prod_{i \geq 1}^{\circ} K(\ell^\infty(\mathbb{Z}, \mathbb{Z})_S, 2i - 1).$$

where $\ell^\infty(\mathbb{Z}, \mathbb{Z})_S$ is a rational vector space of uncountable dimension.

Proof. Observing the homotopy exact sequence

$$\cdots \rightarrow \pi_i(F_1) \rightarrow \pi_i(U_\infty(C_u^*(|\mathbb{Z}|))) \rightarrow \pi_i(U_\infty(B^{\text{SW}})) \rightarrow \pi_{i-1}(F_1) \rightarrow \cdots,$$

we can see that the homotopy fiber inclusion $F_1 \rightarrow U_\infty(C_u^*(|\mathbb{Z}|))$ induces an isomorphism on π_{2i-1} and $\pi_{2i}(F_1) = 0$ by Lemma 6.2 and the fact that $\pi_{2i-1}(U_1(B^{\text{SW}})) = 0$. By Proposition 6.1, we have $\pi_{2i-1}(F_1) \cong \ell^\infty(\mathbb{Z}, \mathbb{Z})_S$. The abelian group $\ell^\infty(\mathbb{Z}, \mathbb{Z})_S$ is a rational vector space of uncountable dimension as seen in [KKT, Section 5]. By [KKT, Lemma 5.4], F_1 has the homotopy type of the product of Eilenberg–MacLane spaces as above. $\square$

The following easy lemma is useful to study the homotopy type of the unitary group of a C^* -algebra. Let $p_r \in M_d(\mathbb{C})$ denote the projection of rank r .

Lemma 6.4. *Let A be a C^* -algebra, where we do not require the existence of unit. For any element $u \in K_0(A)$, there exists a (non-unital in general) $*$ -homomorphism $f: \mathbb{C} \rightarrow M_d(A)$ such that $f_*[p_1] \in K_0(M_d(A)) \cong K_0(A)$ equals to u .*

Proof. We can find a projection $p \in M_d(\mathbb{C} \oplus A)$ and $r \geq 0$ such that $u = [p] - [p_r]$ in $K_0(A)$. Define a $*$ -homomorphism $f: \mathbb{C} \rightarrow M_d(A)$ by $f(1) = p$. This is the desired map. $\square$

Proposition 6.5. *The inclusion $U_\infty(C_u^*(|\mathbb{Z}|)) \rightarrow U_\infty(B^{\text{SW}})$ admits a homotopy section, which is an infinite loop map.*

Proof. Consider the inclusion of based loop spaces $U_\infty(C_0(\mathbb{R}, C_u^*(|\mathbb{Z}|))) \rightarrow U_\infty(C_0(\mathbb{R}, B^{\text{SW}}))$. By Proposition 4.3, Lemma 6.4 and $K_0(C_0(\mathbb{R}, C_u^*(|\mathbb{Z}|))) \cong \mathbb{Z}$, there exists an infinite loop map $f: U_\infty(\mathbb{C}) \rightarrow U_\infty(C_0(\mathbb{R}, C_u^*(|\mathbb{Z}|)))$ which induces an isomorphism on π_{2i-1} for any $i \geq 1$. It follows from this and Lemma 6.2 that the composite

$$U_\infty(\mathbb{C}) \xrightarrow{f} U_\infty(C_0(\mathbb{R}, C_u^*(|\mathbb{Z}|))) \rightarrow U_\infty(C_0(\mathbb{R}, B^{\text{SW}}))$$

is a homotopy equivalence. Then the inclusion of based loop spaces $U_\infty(C_0(\mathbb{R}, C_u^*(|\mathbb{Z}|))) \rightarrow U_\infty(C_0(\mathbb{R}, B^{\text{SW}}))$ admits a homotopy section. This implies that the inclusion of the double loop space $U_\infty(C_0(\mathbb{R}^2, C_u^*(|\mathbb{Z}|))) \rightarrow U_\infty(C_0(\mathbb{R}^2, B^{\text{SW}}))$ also admits a homotopy section. Thus the inclusion $U_\infty(C_u^*(|\mathbb{Z}|)) \rightarrow U_\infty(B^{\text{SW}})$ admits a homotopy section by Bott periodicity, which is again an infinite loop map. $\square$

Proof of Theorem 1.2. By Proposition 6.5, we have a homotopy equivalence

$$U_\infty(C_u^*(|\mathbb{Z}|)) \simeq U_\infty(B^{\text{SW}}) \times F_1$$

as infinite loop spaces. The homotopy types of the spaces $U_\infty(B^{\text{SW}})$ and F_1 are determined in Lemma 5.2 and Proposition 6.3, respectively. Together with the homotopy stability in Theorem 1.1, this completes the proof of the theorem. $\square$

7. GENERALIZATION

In this section, we study the relation between the homotopy type of $U_1(C_u^*(|\mathbb{Z}^n|))$ and the inclusion $U_1(C_u^*(|\mathbb{Z}^n|)) \subset U_1(C^*(|\mathbb{Z}^n|))$ for general $n \geq 2$. In view of Theorem 1.2, we propose the following question.

Question 7.1. Does the inclusion $U_d(C_u^*(|\Gamma|)) \rightarrow U_d(C^*(|\Gamma|))$ admits a homotopy section? Are the homotopy groups of its homotopy fiber are rational vector spaces?

Let us see the case when $\Gamma = \mathbb{Z}^n$ in view of this question.

Lemma 7.2. *The K -theory of the Roe algebra $C^*(|\mathbb{Z}^n|)$ is computed as*

$$K_i(C^*(|\mathbb{Z}^n|)) \cong \begin{cases} \mathbb{Z} & i \equiv n \pmod{2}, \\ 0 & i \not\equiv n \pmod{2}. \end{cases}$$

Proof. Let

$$A_m = \ell^\infty(\mathbb{Z}^n, \mathcal{K}(\mathcal{H})) \rtimes \mathbb{Z}^m$$

with respect to the action of $\mathbb{Z}^m$ ($m \leq n$) on the first m factors of $\mathbb{Z}^n$. Let S_j denote the shift on the j -th factor. Then by the Pimsner–Voiculescu exact sequence

$$\begin{array}{ccccc} K_0(A_{m-1}) & \xrightarrow{1-S_m} & K_0(A_{m-1}) & \longrightarrow & K_0(A_m) \\ \uparrow & & & & \downarrow \\ K_1(A_m) & \longleftarrow & K_1(A_{m-1}) & \xleftarrow{1-S_m} & K_1(A_{m-1}) \end{array}$$

for $A_m = A_{m-1} \rtimes_{S_m} \mathbb{Z}$, we obtain the short exact sequence

$$(1) \quad 0 \rightarrow K_i(A_{m-1})_{S_m} \rightarrow K_i(A_m) \rightarrow K_{1-i}(A_{m-1})^{S_m} \rightarrow 0$$

for $i = 0, 1$, where $K_i(A_{m-1})_{S_m}$ and $K_i(A_{m-1})^{S_m}$ denote the coinvariant and the invariant by S_m , respectively. Since $A_0 = \ell^\infty(\mathbb{Z}^n, \mathcal{K}(\mathcal{H}))$ and we have the well-known isomorphism

$$K_i(\ell^\infty(\mathbb{Z}^n, \mathcal{K})) \cong \begin{cases} \mathbb{Z}^{\mathbb{Z}^n} & i = 0, \\ 0 & i = 1, \end{cases}$$

where $\mathbb{Z}^{\mathbb{Z}^n}$ is the group of all $\mathbb{Z}$ -valued sequences over $\mathbb{Z}^n$, we obtain

$$K_i(A_m) \cong \begin{cases} \mathbb{Z}^{\mathbb{Z}^{n-m}} & i \equiv m \pmod{2}, \\ 0 & i \not\equiv m \pmod{2}, \end{cases}$$

by induction on m . The lemma is just the case when $m = n$. $\square$

Together with the previous lemma, the homotopy type of $U_\infty(C^*(|\mathbb{Z}^n|))$ is determined by the following lemma.

Lemma 7.3. *Let A be a C^* -algebra, where we do not require the existence of unit. Consider the following two conditions on K -theory:*

$$(i) \quad K_i(A) \cong \begin{cases} \mathbb{Z} & i = 0, \\ 0 & i = 1, \end{cases} \quad (ii) \quad K_i(A) \cong \begin{cases} 0 & i = 0, \\ \mathbb{Z} & i = 1. \end{cases}$$

If (i) holds, then $U_\infty(A)$ has the homotopy type of $U_\infty(\mathbb{C})$ as an infinite loop space. If (ii) holds, then $U_\infty(A)$ has the homotopy type of $\mathbb{Z} \times BU_\infty(\mathbb{C})$ as an infinite loop space.

Proof. Suppose the condition (i). By Lemma 6.4, there exists a homotopy equivalence $f: U_\infty(\mathbb{C}) \rightarrow U_\infty(A)$, which is an infinite loop map. When the condition (ii) holds, apply the result for the condition (i) to the algebra $C_0(\mathbb{R}, A)$. This implies that $U_\infty(C_0(\mathbb{R}, A))$ is homotopy equivalent to $U_\infty(\mathbb{C})$. By the Bott periodicity, $U_\infty(A)$ is homotopy equivalent to $\Omega U_\infty(\mathbb{C}) \simeq \mathbb{Z} \times BU_\infty(\mathbb{C})$. $\square$

Proposition 7.4. *The inclusion $U_\infty(C_u^*(|\mathbb{Z}^n|)) \rightarrow U_\infty(C^*(|\mathbb{Z}^n|))$ admits a homotopy section as an infinite loop map if and only if the homomorphism $K_*(C_u^*(|\mathbb{Z}^n|)) \rightarrow K_*(C^*(|\mathbb{Z}^n|))$ is surjective.*

Proof. The only if part is obvious. For the if part, when n is odd, this follows from Lemma 7.2 and the same argument as in the proof of Proposition 6.5. When n is even, apply the same argument to the map on the based loop spaces $U_\infty(C_0(\mathbb{R}, C_u^*(|\mathbb{Z}^n|))) \rightarrow U_\infty(C_0(\mathbb{R}, C^*(|\mathbb{Z}^n|)))$. Then the proposition follows from the existence of the homotopy section of the map on the double loop spaces $U_\infty(C_0(\mathbb{R}^2, C_u^*(|\mathbb{Z}^n|))) \rightarrow U_\infty(C_0(\mathbb{R}^2, C^*(|\mathbb{Z}^n|)))$ and the Bott periodicity. $\square$

Now all we have to do is to see that the homomorphism $K_*(C_u^*(|\mathbb{Z}^n|)) \rightarrow K_*(C^*(|\mathbb{Z}^n|))$ is surjective. Let

$$B_m = \ell^\infty(\mathbb{Z}^n, \mathbb{C}) \rtimes \mathbb{Z}^m$$

with respect to the action $\mathbb{Z}^m$ ($m \leq n$) on the first m factors of $\mathbb{Z}^n$ and S_j denote the shift on the j -th factor. We obtain the short exact sequences similar to (1)

$$(2) \quad 0 \rightarrow K_i(B_{m-1})_{S_m} \rightarrow K_i(B_m) \rightarrow K_{1-i}(B_{m-1})^{S_m} \rightarrow 0$$

for $i = 0, 1$. For $n = 1, 2$, we can see the surjectivity as follows.

Lemma 7.5. *The homomorphism $K_1(C_u^*(|\mathbb{Z}|)) \rightarrow K_1(C^*(|\mathbb{Z}|))$ is an isomorphism.*

Proof. Consider the commutative square

$$\begin{array}{ccc} K_1(C_u^*(|\mathbb{Z}|)) & \xrightarrow{\cong} & \ell^\infty(\mathbb{Z}, \mathbb{Z})^S \\ \downarrow & & \downarrow \cong \\ K_1(C^*(|\mathbb{Z}|)) & \xrightarrow{\cong} & (\mathbb{Z}^\mathbb{Z})^S \end{array}$$

obtained from the exact sequences (1) and (2). Thus the lemma follows. $\square$

Lemma 7.6. *The homomorphism $K_0(C_u^*(|\mathbb{Z}^2|)) \rightarrow K_0(C^*(|\mathbb{Z}^2|))$ is surjective.*

Proof. When $n = 2$, we can compute $K_*(B_1)$ by the exact sequence (2) as follows:

$$K_i(B_1) \cong \begin{cases} \ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1} & i = 0, \\ \ell^\infty(\mathbb{Z}^2, \mathbb{Z})^{S_1} & i = 1. \end{cases}$$

Again by the exact sequences (1) and (2) for $m = 2$, we have the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1 S_2} & \longrightarrow & K_0(C_u^*(|\mathbb{Z}^2|)) & \longrightarrow & \ell^\infty(\mathbb{Z}^2, \mathbb{Z})^{S_1 S_2} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong \\ 0 & \longrightarrow & 0 & \longrightarrow & K_0(C^*(|\mathbb{Z}^2|)) & \xrightarrow{\cong} & (\mathbb{Z}^{\mathbb{Z}^2})^{S_1 S_2} \longrightarrow 0 \end{array}$$

Thus the homomorphism $K_0(C_u^*(|\mathbb{Z}^2|)) \rightarrow K_0(C^*(|\mathbb{Z}^2|))$ is surjective by the right square. $\square$

To determine the homotopy type of $C_u^*(|\mathbb{Z}^2|)$, we also need its K -theory.

Lemma 7.7. *The K -theory $K_1(C_u^*(|\mathbb{Z}^2|))$ and the kernel of the homomorphism $K_0(C_u^*(|\mathbb{Z}^2|)) \rightarrow K_0(C^*(|\mathbb{Z}^2|))$ are rational vector spaces of uncountable dimension.*

Proof. As seen in the proof of Lemma 7.6, the latter group is isomorphic to $\ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1 S_2}$. The coinvariant $\ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1}$ can be seen to be a rational vector space of uncountable dimension by the same argument as in [KKT, Section 5]. Then, since $S_2: \ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1} \rightarrow \ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1}$ is a

linear map on a rational vector space, the coinvariant $\ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1 S_2}$ is a rational vector space of uncountable dimension. For $K_1(C_u^*(|\mathbb{Z}^2|))$, we obtain the exact sequence

$$0 \rightarrow (\ell^\infty(\mathbb{Z}^2, \mathbb{Z})^{S_1})_{S_2} \rightarrow K_1(C_u^*(|\mathbb{Z}^2|)) \rightarrow (\ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1})^{S_2} \rightarrow 0$$

from (2). Since $(\ell^\infty(\mathbb{Z}^2, \mathbb{Z})^{S_1})_{S_2} \cong \ell^\infty(\mathbb{Z}, \mathbb{Z})_S$ and $\ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1}$ are rational vector spaces, $K_1(C_u^*(|\mathbb{Z}^2|))$ is also a rational vector space of uncountable dimension. $\square$

Proof of Theorem 1.3. By Proposition 7.4 and Lemma 7.6, the inclusion $U_\infty(C_u^*(|\mathbb{Z}^2|)) \rightarrow U_\infty(C^*(|\mathbb{Z}^2|))$ admits a homotopy section as an infinite loop map. Let F_2 be the homotopy fiber of the inclusion. Then we have a homotopy equivalence

$$U_\infty(C_u^*(|\mathbb{Z}^2|)) \simeq U_\infty(C^*(|\mathbb{Z}^2|)) \times F_2$$

as infinite loop spaces. By Lemmas 7.2 and 7.3, $U_\infty(C^*(|\mathbb{Z}^2|))$ is homotopy equivalent to $U_\infty(\mathbb{C})$ as an infinite loop space. By the naturality of the Bott maps

$$\beta: U_\infty(C_u^*(|\mathbb{Z}^2|)) \xrightarrow{\cong} U_\infty(C_0(\mathbb{R}^2, C_u^*(|\mathbb{Z}^2|))) \quad \text{and} \quad \beta: U_\infty(C^*(|\mathbb{Z}^2|)) \xrightarrow{\cong} U_\infty(C_0(\mathbb{R}^2, C^*(|\mathbb{Z}^2|))),$$

we have the homotopy equivalence

$$\tilde{\beta}: F_2 \xrightarrow{\cong} \Omega^2 F_2$$

as well. The homotopy group of F_2 can be computed by Lemma 7.6:

$$\pi_i(F_2) \cong \begin{cases} V_1 & i \text{ is even,} \\ V_0 & i \text{ is odd,} \end{cases}$$

where

$$V_0 = \ker[K_0(C_u^*(|\mathbb{Z}^2|)) \rightarrow K_0(C^*(|\mathbb{Z}^2|))], \quad V_1 = K_1(C_u^*(|\mathbb{Z}^2|))$$

are rational vector spaces by Lemma 7.7. Again as in the proof of [KKT, Lemma 5.4], we can find maps

$$\prod_{i \geq 1}^{\circ} K(V_0, 2i-1) \rightarrow F_2 \quad \text{and} \quad \prod_{i \geq 1}^{\circ} K(V_1, 2i-1) \rightarrow \Omega F_2$$

inducing isomorphisms on the odd degree homotopy groups. Taking the loop of the latter map, we get the composite

$$V_1 \times \prod_{i \geq 1}^{\circ} K(V_1, 2i) \rightarrow \Omega^2 F_2 \xrightarrow{\tilde{\beta}^{-1}} F_2,$$

which induces isomorphisms on the even degree homotopy groups and the group of path components. Thus, we obtain the homotopy equivalence

$$V_1 \times \prod_{i \geq 1}^{\circ} (K(V_0, 2i-1) \times K(V_1, 2i)) \rightarrow F_2.$$

This completes the proof of the theorem. $\square$

Moreover, Lemma 7.5 provides another proof of Theorem 1.2 in a similar manner.

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