

Sheaf Theoretic L^2 -Cohomology

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If M is a compact manifold, then we have the famous de Rham isomorphism ; $H_i(M) \cong \text{Hom}(H_{\text{DR}}^i(M), \mathbb{R})$. Our purpose here is to generalize this isomorphism to the so-called Thom-Mather's stratified space. More explicitly, we aim to show that, exchange the simplicial homology for the Goresky-MacPherson's intersection homology and the de Rham cohomology for the L^2 -cohomology of the non-singular part, then the isomorphism is still valid for such a situation.

We discussed this subject at this research institute two years ago ([4]). At that time, I constructed the isomorphism directly. Therefore, at this time, I will select the other plan of proof, in which we will pay little attention to how to construct the isomorphic map. That is, according to Goresky and MacPherson ([3]), I will show the isomorphism axiomatically by sheaf theoretic method.

§1. L^2 -cohomology and intersection homology : Main Theorem

From now on, X^n is an n -dimensional compact stratified space without boundary. We will fix a stratification

$$X = X_n \supset X_{n-1} = X_{n-2} (= \Sigma) \supset X_{n-3} \supset \dots \supset X_1 \supset X_0,$$

and the tubular neighborhood system and, moreover, the

PL-structure compatible with these structures.

Let g be a metric on $X-\Sigma$, and let d_i be the exterior derivative on $X-\Sigma$ with domain

$$\text{dom } d_i = \{ \omega \in \Lambda^i(X-\Sigma) \cap L^2 \Lambda^i(X-\Sigma) \mid d\omega \in L^2 \Lambda^{i+1}(X-\Sigma) \}.$$

The i -th cohomology group of the cochain complex $\{\text{dom } d_i\}$ is called the i -th L^2 -cohomology group, denoted by $H_{(2)}^i(X-\Sigma)$.

Next, taking account of the PL-structure of X , let's define the intersection homology. Let $\bar{p} = (p_2, p_3, \dots, p_n)$ be a perversity, i.e., a sequence of non-negative integers satisfying $p_2 = 0$ and $p_k \leq p_{k+1} \leq p_k + 1$ for all k . The perversities which are of particular importance are as follows:

$\bar{0} = (0, \dots, 0)$, the zero perversity,

$\bar{m} = (0, 0, 1, 1, 2, 2, \dots)$, $m_k = \lfloor \frac{k}{2} \rfloor - 1$, the (lower) middle perversity,

$\bar{t} = (0, 1, 2, 3, \dots)$, the top perversity.

By the way, $\bar{p} \leq \bar{q}$ means that $p_k \leq q_k$ for all k . And we set

$$\bar{p} + \bar{q} = (p_2 + q_2, p_3 + q_3, \dots).$$

The perversity $\bar{q}$ is said to be the complementary perversity of $\bar{p}$ if $\bar{p} + \bar{q} = \bar{t}$. Then, take an integer i . A subspace Y of X is called $(\bar{p}, i)$ -allowable if $\dim Y \leq i$ and $\dim(Y \cap X_{n-k}) \leq i - k + p_k$ for all k . For example, that Y is $(\bar{0}, \dim Y)$ -allowable means that Y and the strata are in general position. Now, let's set

$$IC_{\bar{p}}^i(X) = \left\{ \{ \} \in C_i(X) \mid \begin{array}{l} |\{ \}| \text{ is } (\bar{p}, i)\text{-allowable and} \\ |\partial \{ \}| \text{ is } (\bar{p}, i-1)\text{-allowable.} \end{array} \right\}.$$

Then the i -th homology group of the chain complex $\{IC_{\bar{p}}^i(X)\}$ is called the i -th intersection homology group with $\bar{p}$ and denoted

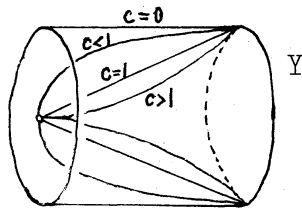
by $I\bar{H}_1^{\bar{p}}(X)$.

Now we may remark that the perversities which are interesting here or which we want to treat here are the perversities which are smaller than the middle perversity, i.e., $\bar{p} \leq \bar{m}$. This restriction is not essential; see the remark following Definition 1.1.

Here, once again, we will return to the L^2 -cohomology and define the metric associated to a given perversity $\bar{p}$ and then announce the main theorem explicitly.

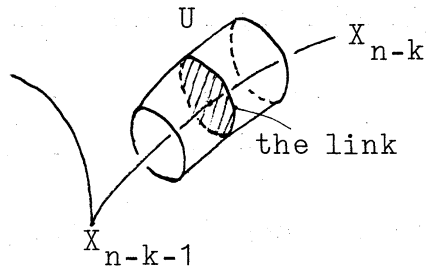
Let Y be a Riemannian manifold with metric g and let's take $c \geq 0$. Then we set

$$C^c(Y) = \text{" the Riemannian manifold } (0,1) \times Y \\ \text{with metric } dr \otimes dr + r^{2c} g \text{ " .}$$



Now, fix a sequence of non-negative real numbers $\bar{c} = (c_2, \dots, c_n)$. Then the metric g on $X - \Sigma$ is said to be associated to $\bar{c}$ if, for any point x of any non-empty stratum $X_{n-k} - X_{n-k-1}$, there exists a neighborhood $x \in U \subset X$ such that

$$U \cap (X - \Sigma) \underset{\text{quasi-isometry}}{\sim} C^{c_k} \left(\left(\text{the link of } X_{n-k} - X_{n-k-1} \right) \cap (X - \Sigma) \text{ with } g \right) \\ \times_{\text{product}} \left(U \cap (X_{n-k} - X_{n-k-1}) \text{ with Euclidean metric} \right).$$



Definition 1.1. The metric g on $X-\Sigma$ is said to be associated to the perversity $\bar{p} (\leq \bar{m})$ if g is associated to $\bar{c} = (c_2, \dots, c_n)$:

$$\begin{cases} (k-1-2p_k)^{-1} \leq c_k < (k-3-2p_k)^{-1} & ; 2p_k \leq k-3, \\ 1 \leq c_k < \infty & ; 2p_k = k-2. \end{cases}$$

If we want to treat the perversities which are larger than $\bar{m}$ or which cannot be comparable with $\bar{m}$, it will suffice to change (certain) c_k 's into suitable negative numbers. By the way, it is noteworthy that, if $1 \leq c_k < \infty$ for all k and the metric g is associated to $\bar{c}$, then g is associated to the important perversity $\bar{m}$: this case was treated by J. Cheeger ([1]).

We will use the notation $(X-\Sigma)_{\bar{p}}$ in order to express clearly that the metric on $X-\Sigma$ under consideration is associated to $\bar{p}$. Then we can now announce

Main Theorem. If $\bar{p} \leq \bar{m}$, then

$$IH_{\bar{p}}^{\bar{p}}(X) \cong \text{Hom}(H_{(2)}^1((X-\Sigma)_{\bar{p}}), \mathbb{R}).$$

The following two sections are the preparations for the proof of Main Theorem from the view point of sheaf theoretic method.

§2. Sheaf theoretic L^2 -cohomology and intersection homology

As above, the non-singular part $X - \Sigma$ is endowed with metric g . Let $\Omega^\cdot$ be the complex of sheaves on X which is defined by

$$\Gamma(U, \Omega^i) = \left\{ \omega \in \Lambda^i(U \cap (X - \Sigma)) \mid \begin{array}{l} \text{For any point } x \text{ of } U, \text{ there} \\ \text{exists a neighborhood } x \in V \subset U \\ \text{such that} \\ \int_{V \cap (X - \Sigma)} \omega \wedge * \omega < \infty, \\ \int_{V \cap (X - \Sigma)} d\omega \wedge * d\omega < \infty. \end{array} \right\}$$

with the sheaf maps $d: \Omega^i \rightarrow \Omega^{i+1}$ induced by the exterior derivative on $X - \Sigma$. In order to indicate that the metric g is associated to the given perversity $\bar{p}$, we will use the notation $\Omega_{\bar{p}}^\cdot$.

Next, paying attention to the PL-structure of X , we will define the complex of sheaves $\mathcal{IC}_{\bar{p}}^\cdot$. Before it, define the sheaf $\mathcal{C}_i$ by

$$\Gamma(U, \mathcal{C}_i) = \text{" the group of locally finite } i\text{-dimensional} \\ \text{simplicial chains with respect to the induced} \\ \text{PL-structure of } U \text{ " .}$$

For convenience, we set $\mathcal{C}^\cdot = \mathcal{C}_{-}$ and regard this as a complex of sheaves: the sheaf maps are induced by the simplicial boundary operator. Then we define its subcomplex $\mathcal{IC}_{\bar{p}}^\cdot$ by

$$\Gamma(U, \mathcal{IC}_{\bar{p}}^{i-1}) = \left\{ \xi \in \Gamma(U, \mathcal{C}^{i-1}) \mid \begin{array}{l} |\xi| \text{ is } (\bar{p}, i)\text{-allowable and } |\partial\xi| \\ \text{is } (\bar{p}, i-1)\text{-allowable with} \\ \text{respect to the induced} \\ \text{stratification of } U. \end{array} \right\}.$$

Now $\Omega_{\bar{p}}^{\cdot}$ and $\mathcal{H}_{\bar{p}}^{\cdot}$ are fine sheaves. Therefore we have

Lemma 2.1.

$$\mathcal{H}^i(X, \Omega_{\bar{p}}^{\cdot}) \cong H_{(2)}^i((X-\Sigma)_{\bar{p}}),$$

$$\mathcal{H}^{-i}(X, \mathcal{H}_{\bar{p}}^{\cdot}) \cong \mathrm{IH}_{\bar{i}}^{\bar{p}}(X).$$

Here $\mathcal{H}^*(X, \cdot)$ is the hypercohomology.

§3. Key Theorem due to Goresky and MacPherson

Let $\mathcal{S}^{\cdot}$ be a complex of sheaves on X which is constructible with respect to the given stratification $\{X_k\}$ (that is, for any j , $\mathcal{S}_{|X_j - X_{j-1}}^{\cdot}$ is cohomologically locally constant, i.e., its associate cohomology sheaves are locally constant).

Definition 3.1. We shall say $\mathcal{S}^{\cdot}$ satisfies the axiom $[\mathrm{AX}1]_{\bar{p}}$ provided:

- (a) $\mathcal{S}_{|X-\Sigma}^{\cdot} \cong \mathbb{R}[n]^{\cdot}$ (the isomorphism in the derived category),
- (b) $\mathcal{H}^i(\mathcal{S}^{\cdot}) = 0$ for all $i < -n$,
- (c) $\mathcal{H}^m(\mathcal{S}_{|X-X_{n-k-1}}^{\cdot}) = 0$ for all $m > p_k - n$,
- (d) the attaching maps (in the derived category)

$$\mathcal{H}^m(j_k^* \mathcal{S}_{|X-X_{n-k-1}}^{\cdot}) \longrightarrow \mathcal{H}^m(j_k^* \mathcal{R}_{i_k^*} i_k^* \mathcal{S}_{|X-X_{n-k-1}}^{\cdot})$$

are isomorphisms for all $m \leq p_k - n$.

Here $\mathcal{H}^*(\cdot)$ is the cohomology sheaf. And $i_k: X - X_{n-k} \rightarrow X - X_{n-k-1}$ and $j_k: X_{n-k} - X_{n-k-1} \rightarrow X - X_{n-k-1}$ are the inclusion maps.

Now, according to [3], we have

Key Theorem (Goresky and MacPherson).

- (1) The constructible complex of sheaves which satisfies the axiom $[AX1]_{\bar{p}}$ is unique up to isomorphism in the derived category.
- (2) $\mathcal{I}C_{\bar{p}}^{\bullet}$ is constructible and satisfies $[AX1]_{\bar{p}}$.
- (3) If $\bar{p} + \bar{q} = \bar{t}$, then $\mathcal{I}C_{\bar{p}}^{\bullet} \cong R\mathcal{H}om(\mathcal{I}C_{\bar{q}}^{\bullet}, \mathcal{O}_X^{\bullet})[n]$, where $\mathcal{O}_X^{\bullet}$ is the dualizing complex on X , i.e., $\mathcal{O}_X^{\bullet} = f^! \mathbb{R}_{pt}^{\bullet}$ with $f: X \rightarrow (\text{a point})$.

§4. Proof of Main Theorem

It suffices to prove

Assertion. $\Omega_{\bar{p}}^{\bullet}[n]$ is constructible and satisfies the axiom $[AX1]_{\bar{q}}$, where $\bar{q}$ is the complementary perversity, $\bar{p} + \bar{q} = \bar{t}$.

Actually we have

Proof of Main Theorem. From Key Theorem (1) and Assertion, we have

$$\Omega_{\bar{p}}^{\bullet}[n] \cong \mathcal{I}C_{\bar{q}}^{\bullet}.$$

Therefore, by substituting $\Omega_{\bar{p}}^{\bullet}[n]$ for $\mathcal{I}C_{\bar{q}}^{\bullet}$ at Key Theorem (3), we get

$$\mathcal{I}C_{\bar{p}}^{\bullet} \cong R\mathcal{H}om(\Omega_{\bar{p}}^{\bullet}, \mathcal{O}_X^{\bullet}).$$

Hence, by Verdier duality theorem, we have

$$\mathcal{H}^{-i}(X, \mathcal{I}C_{\bar{p}}^{\bullet}) \cong \text{Hom}(\mathcal{H}^i(X, \Omega_{\bar{p}}^{\bullet}), \mathbb{R}).$$

Thus, combined with Lemma 2.1, the proof is complete.

Now we will prove Assertion. It suffices to examine (a)-(d) of $[AX1]_{\bar{q}}$. The constructibility will be referred briefly later on

the way.

(a) Since $\Omega_{\bar{p}}^{\cdot}[n]|_{X-\Sigma}$ is a sheaf of C^{∞} -forms on $X-\Sigma$, we have $\Omega_{\bar{p}}^{\cdot}[n]|_{X-\Sigma} \cong \mathbb{R}_{X-\Sigma}^{\cdot}[n]$ because of the usual resolution.

(b) If $i < -n$, then $(\Omega_{\bar{p}}^{\cdot}[n])^i = \Omega_{\bar{p}}^{i+n} = 0$. Therefore $\mathcal{H}^i(\Omega_{\bar{p}}^{\cdot}[n]) = 0$ for all $i < -n$.

(Preparation for (c) and (d)) For a point x of $X_{n-k} - X_{n-k-1}$, take a suitable neighborhood U and the link L of the stratum at x . Then we have

$$(4.1) \quad \mathcal{H}^j(\Omega_{\bar{p}}^{\cdot})_x \cong H_{(2)}^j(U \cap (X - \Sigma)).$$

Strictly speaking, the right hand side of (4.1) should be the inductive limit $\varinjlim_U H_{(2)}^j(U \cap (X - \Sigma))$. But, for sufficiently small U , it is naturally isomorphic to $H_{(2)}^j(U \cap (X - \Sigma))$ because the L^2 -cohomology is invariant under the quasi-isometric transformation. Hence, also, $\Omega_{\bar{p}}^{\cdot}$ can be regarded as constructible. Moreover, (4.1) is isomorphic to

$$\begin{aligned} & H_{(2)}^j(C^{c_k}(L \cap (X - \Sigma))) \times (U \cap (X_{n-k} - X_{n-k-1})) \\ & \cong H_{(2)}^j(C^{c_k}(L \cap (X - \Sigma))) \\ & \cong \begin{cases} H_{(2)}^j(L \cap (X - \Sigma)) & ; j < \frac{1}{2}(k-1 + \frac{1}{c_k}), \\ 0 & ; j \geq \frac{1}{2}(k-1 + \frac{1}{c_k}), \end{cases} \end{aligned}$$

through the natural extension maps ([4], [5]). Hence

$$(4.2) \quad \mathcal{H}^j(\Omega_{\bar{p}}^{\cdot})_x \cong \begin{cases} H_{(2)}^j(L \cap (X - \Sigma)) & ; j \leq q_k, \\ 0 & ; j > q_k, \end{cases}$$

because $q_k < \frac{1}{2}(k-1 + \frac{1}{c_k}) \leq q_k + 1$.

(c) This is equivalent to the assertion that, if $j > q_k$, then $\mathcal{H}^j(\Omega_{\bar{p}}^\cdot)_x = 0$ for any point x of $X_{n-k} - X_{n-k-1}$. Hence, by (4.2), this is true.

(d) This is equivalent to the assertion that, if $j \leq q_k$, then the attaching maps

$$(4.3) \quad \mathcal{H}^j(\Omega_{\bar{p}|X-X_{n-k-1}}^\cdot)_x \longrightarrow \mathcal{H}^j(i_{k*}i_k^* \Omega_{\bar{p}|X-X_{n-k-1}}^\cdot)_x$$

are isomorphisms for any point x of $X_{n-k} - X_{n-k-1}$.

In order to prove this assertion, first remark that a cross section of $\Omega_{\bar{p}|X-X_{n-k-1}}^\cdot$ resp. $i_{k*}i_k^* \Omega_{\bar{p}|X-X_{n-k-1}}^\cdot$ is a smooth form which and whose image by the exterior derivative are square integrable near any point of $X - X_{n-k-1}$ resp. $X - X_{n-k}$. (For a cross section ω of $i_{k*}i_k^* \Omega_{\bar{p}|X-X_{n-k-1}}^\cdot$, it is not necessary to claim that ω and $d\omega$ are square integrable near any point of $X_{n-k} - X_{n-k-1}$.) Therefore we have the natural sheaf map

$$\Omega_{\bar{p}|X-X_{n-k-1}}^\cdot \longrightarrow i_{k*}i_k^* \Omega_{\bar{p}|X-X_{n-k-1}}^\cdot.$$

And this just induces the attaching map (4.3). Now, from the property of $i_{k*}i_k^* \Omega_{\bar{p}|X-X_{n-k-1}}^\cdot$ mentioned above, we have

$$(4.4) \quad \mathcal{H}^j(i_{k*}i_k^* \Omega_{\bar{p}|X-X_{n-k-1}}^\cdot)_x \cong \begin{cases} H_{(2)}^j(L \cap (X - \Sigma)) & ; j < k, \\ 0 & ; j \geq k, \end{cases}$$

for any point x of $X_{n-k} - X_{n-k-1}$. Hence, for $j \leq q_k$, the identity map from the right hand side of (4.2) to the right hand side of (4.2) is just the attaching map (4.3). Thus the proof of (d) is complete.

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