

Transformations of L -values

Wadim Zudilin*

School of Mathematical and Physical Sciences,
The University of Newcastle, Callaghan, NSW 2308, Australia

April 2012

Abstract

In our recent work with M. Rogers on resolving some Boyd's conjectures on two-variate Mahler measures, a new analytical machinery was introduced to write the values $L(E, 2)$ of L -series of elliptic curves as periods in the sense of Kontsevich and Zagier. Here we outline, in slightly more general settings, the novelty of our method with Rogers, and provide a simple illustrative example.

Throughout the note we keep the notation $q = e^{2\pi i\tau}$ for τ from the upper half-plane $\operatorname{Re} \tau > 0$, so that $|q| < 1$. Our basic constructor of modular forms and functions is Dedekind's eta-function

$$\eta(\tau) := q^{1/24} \prod_{m=1}^{\infty} (1 - q^m) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24}$$

with is modular involution

$$\eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau). \quad (1)$$

We also set $\eta_k := \eta(k\tau)$ for short.

We first describe a part of the general machinery from our joint works [6, 7] with M. Rogers on an example of computing the value $L(E_{32}, 2)$ of the L -series associated with a conductor 32 elliptic curve. It is known [3] that the corresponding cusp form in this case is $f_{32}(\tau) := \eta_4^2 \eta_8^2$, so that $L(E_{32}, s) = L(f_{32}, s)$. We choose the conductor 32 case here because it is not discussed in [6, 7].

Note the (Lambert series) expansion

$$\frac{\eta_8^4}{\eta_4^2} = \sum_{m \geq 1} \left(\frac{-4}{m} \right) \frac{q^m}{1 - q^{2m}} = \sum_{\substack{m, n \geq 1 \\ n \text{ odd}}} \left(\frac{-4}{m} \right) q^{mn}, \quad (2)$$

*This work is supported by Australian Research Council grant DP110104419. The text is loosely based on my talk "Mahler measures and L -series of elliptic curves" at the conference "Analytic number theory—related multiple aspects of arithmetic functions" (Research Institute for Mathematical Sciences, Kyoto University, Japan, October 31–November 2, 2011).

where $\left(\frac{-4}{m}\right)$ is the quadratic residue character modulo 4. In notation $\delta_{2|n} = 1$ if $2 \mid n$ and 0 if n is odd, we can write (2) as

$$\frac{\eta_8^4}{\eta_4^2} = \sum_{m,n \geq 1} a(m)b(n)q^{mn}, \quad \text{where } a(m) := \left(\frac{-4}{m}\right), \quad b(n) := 1 - \delta_{2|n}.$$

Then

$$\begin{aligned} f_{32}(it) &= \frac{\eta_8^4}{\eta_4^2} \frac{\eta_4^4}{\eta_8^2} \Big|_{\tau=it} = \frac{\eta_8^4}{\eta_4^2} \Big|_{\tau=it} \cdot \frac{1}{2t} \frac{\eta_8^4}{\eta_4^2} \Big|_{\tau=i/(32t)} \\ &= \frac{1}{2t} \sum_{m_1, n_1 \geq 1} a(m_1)b(n_1)e^{-2\pi m_1 n_1 t} \sum_{m_2, n_2 \geq 1} b(m_2)a(n_2)e^{-2\pi m_2 n_2/(32t)}, \end{aligned}$$

where $t > 0$ and the modular involution (1) was used.

Now,

$$\begin{aligned} L(E_{32}, 2) &= L(f_{32}, 2) = \int_0^1 f_{32} \log q \frac{dq}{q} = -4\pi^2 \int_0^\infty f_{32}(it)t dt \\ &= -2\pi^2 \int_0^\infty \sum_{m_1, n_1, m_2, n_2 \geq 1} a(m_1)b(n_1)b(m_2)a(n_2) \\ &\quad \times \exp\left(-2\pi\left(m_1 n_1 t + \frac{m_2 n_2}{32t}\right)\right) dt \\ &= -2\pi^2 \sum_{m_1, n_1, m_2, n_2 \geq 1} a(m_1)b(n_1)b(m_2)a(n_2) \\ &\quad \times \int_0^\infty \exp\left(-2\pi\left(m_1 n_1 t + \frac{m_2 n_2}{32t}\right)\right) dt. \end{aligned}$$

Here comes the crucial transformation of purely analytical origin: we make the change of variable $t = n_2 u / n_1$. It does not change the form of the integrand but affects the differential, and we obtain

$$\begin{aligned} L(E_{32}, 2) &= -2\pi^2 \sum_{m_1, n_1, m_2, n_2 \geq 1} \frac{a(m_1)b(n_1)b(m_2)a(n_2)n_2}{n_1} \\ &\quad \times \int_0^\infty \exp\left(-2\pi\left(m_1 n_2 u + \frac{m_2 n_1}{32u}\right)\right) du \\ &= -2\pi^2 \int_0^\infty \sum_{m_1, n_2 \geq 1} a(m_1)a(n_2)n_2 e^{-2\pi m_1 n_2 u} \\ &\quad \times \sum_{m_2, n_1 \geq 1} \frac{b(m_2)b(n_1)}{n_1} e^{-2\pi m_2 n_1/(32u)} du. \end{aligned}$$

What are the resulting series in the product? The first one corresponds to

$$\sum_{m,n \geq 1} a(m)a(n)n q^{mn} = \sum_{m,n \geq 1} \left(\frac{-4}{mn}\right) n q^{mn} = \sum_{n \geq 1} n \left(\frac{-4}{n}\right) \frac{n q^n}{1 + q^{2n}} = \frac{\eta_2^4 \eta_8^4}{\eta_4^4},$$

while the second one is

$$\begin{aligned} \sum_{m,n \geq 1} \frac{b(m)b(n)}{n} q^{mn} &= \sum_{m,n \geq 1} \frac{q^{mn}}{n} - \frac{q^{(2m)n}}{n} - \frac{q^{m(2n)}}{2n} + \frac{q^{(2m)(2n)}}{2n} \\ &= \frac{1}{2} \sum_{m,n \geq 1} \frac{2q^{mn} - 3q^{2mn} + q^{4mn}}{n} \\ &= -\frac{1}{2} \log \prod_{m \geq 1} \frac{(1-q^m)^2(1-q^{4m})}{(1-q^{2m})^3} = -\frac{1}{2} \log \frac{\eta_1^2 \eta_4}{\eta_2^3}, \end{aligned}$$

hence

$$L(E_{32}, 2) = \pi^2 \int_0^\infty \frac{\eta_2^4 \eta_8^4}{\eta_4^4} \Big|_{\tau=iu} \cdot \log \frac{\eta_1^2 \eta_4}{\eta_2^3} \Big|_{\tau=i/(32u)} du.$$

Applying the involution (1) to the eta quotient under the logarithm sign we obtain

$$L(E_{32}, 2) = \pi^2 \int_0^\infty \frac{\eta_2^4 \eta_8^4}{\eta_4^4} \log \frac{\sqrt{2} \eta_8 \eta_{32}^2}{\eta_{16}^3} \Big|_{\tau=iu} du.$$

Now comes the modular magic: assisted with Ramanujan's knowledge [1] we choose a particular modular function $x(\tau) := \eta_2^4 \eta_8^2 / \eta_4^6$, which ranges from 1 to 0 when $\tau \in (0, i\infty)$, and verify that

$$\frac{1}{2\pi i} \frac{x dx}{2\sqrt{1-x^4}} = -\frac{\eta_2^4 \eta_8^4}{\eta_4^4} d\tau \quad \text{and} \quad \left(\frac{\sqrt{2} \eta_8 \eta_{32}^2}{\eta_{16}^3} \right)^2 = \frac{1-x}{1+x}.$$

Thus,

$$L(E_{32}, 2) = \frac{\pi}{8} \int_0^1 \frac{x}{\sqrt{1-x^4}} \log \frac{1+x}{1-x} dx.$$

The result is a *period* in the sense of [2], and as such it can be compared with several other objects like values of generalized hypergeometric functions or even Mahler measures [4, 5]. This however involves a different set of routines which we do not touch here.

To summarize, in our evaluation of $L(E, 2) = L(f, 2)$ we first split $f(\tau)$ into a product of two Eisenstein series of weight 1 and at the end we arrive at a product of two Eisenstein(-like) series $g_2(\tau)$ and $g_0(\tau)$ of weights 2 and 0, respectively, so that $L(f, 2) = cL(g_2 g_0, 1)$ for some algebraic constant c . The latter object is doomed to be a period as $g_0(\tau)$ is a logarithm of a modular function, while $2\pi i g_2(\tau) d\tau$ is, up to a modular function multiple, the differential of a modular function, and finally any two modular functions are tied up by an algebraic relation over $\overline{\mathbb{Q}}$.

The method however can be formalized to even more general settings, and it is this extension which we attempt to outline below.

For two *bounded* sequences $a(m)$, $b(n)$, we refer to an expression of the form

$$g_k(\tau) = a + \sum_{m,n \geq 1} a(m)b(n)n^{k-1}q^{mn}, \quad q := e^{2\pi i \tau}, \quad (3)$$

as to an Eisenstein-like series of weight k , especially in the case when $g_k(\tau)$ is a modular form of certain level, that is, when it transforms sufficiently ‘nice’ under $\tau \mapsto -1/(N\tau)$ for some positive integer N . This automatically happens when $g_k(\tau)$ is indeed an Eisenstein series (for example, when $a(m) = 1$ and $b(n)$ is a Dirichlet character modulo N of designated parity, $b(-1) = (-1)^k$), in which case $\widehat{g}_k(\tau) := g_k(-1/(N\tau))(\sqrt{-N}\tau)^{-k}$ is again an Eisenstein series. It is worth mentioning that the above notion has perfect sense in case $k \leq 0$ as well. Indeed, modular units, or weak modular forms of weight 0, that are the logarithms of modular functions are examples of Eisenstein-like series $g_0(\tau)$. Also, for $k \leq 0$ examples are given by Eichler integrals, the $(1 - k)$ th τ -derivatives of holomorphic Eisenstein series of weight $2 - k$, a consequence of the famous lemma of Hecke [8, Section 5].

Suppose we are interested in the L -value $L(f, k_0)$ of a cusp form $f(\tau)$ of weight $k = k_1 + k_2$ which can be represented as a product (in general, as a linear combination of several products) of two Eisenstein(-like) series $g_{k_1}(\tau)$ and $\widehat{g}_{k_2}(\tau)$, where the first one vanishes at infinity ($a = g_{k_1}(i\infty) = 0$ in (3)) and the second one vanishes at zero ($\widehat{g}_{k_2}(i0) = 0$). (The vanishing happens because the product is a cusp form!) In reality, we need the series $g_{k_2}(\tau) := \widehat{g}_{k_2}(-1/(N\tau))(\sqrt{-N}\tau)^{-k_2}$ to be Eisenstein-like:

$$g_{k_1}(\tau) = \sum_{m,n \geq 1} a_1(m)b_1(n)n^{k_1-1}q^{mn} \quad \text{and} \quad g_{k_2}(\tau) = \sum_{m,n \geq 1} a_2(m)b_2(n)n^{k_2-1}q^{mn}.$$

We have

$$\begin{aligned} L(f, k_0) &= L(g_{k_1}\widehat{g}_{k_2}, k_0) = \frac{1}{(k_0 - 1)!} \int_0^1 g_{k_1}\widehat{g}_{k_2} \log^{k_0-1} q \frac{dq}{q} \\ &= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0 - 1)!} \int_0^\infty g_{k_1}(it)\widehat{g}_{k_2}(it)t^{k_0-1} dt \\ &= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0 - 1)! N^{k_2/2}} \int_0^\infty g_{k_1}(it)g_{k_2}(i/(Nt))t^{k_0-k_2-1} dt \\ &= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0 - 1)! N^{k_2/2}} \int_0^\infty \sum_{m_1, n_1 \geq 1} a_1(m_1)b_1(n_1)n_1^{k_1-1}e^{-2\pi m_1 n_1 t} \\ &\quad \times \sum_{m_2, n_2 \geq 1} a_2(m_2)b_2(n_2)n_2^{k_2-1}e^{-2\pi m_2 n_2/(Nt)}t^{k_0-k_2-1} dt \\ &= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0 - 1)! N^{k_2/2}} \sum_{m_1, n_1, m_2, n_2 \geq 1} a_1(m_1)b_1(n_1)a_2(m_2)b_2(n_2)n_1^{k_1-1}n_2^{k_2-1} \\ &\quad \times \int_0^\infty \exp\left(-2\pi\left(m_1 n_1 t + \frac{m_2 n_2}{Nt}\right)\right)t^{k_0-k_2-1} dt; \end{aligned}$$

the interchange of integration and summation is legitimate because of the exponential decrease of the integrand at the endpoints. After performing the change of variable

$t = n_2 u / n_1$ and interchanging back summation and integration we obtain

$$\begin{aligned}
L(f, k_0) &= \frac{(-1)^{k_0-1} (2\pi)^{k_0}}{(k_0-1)! N^{k_2/2}} \sum_{m_1, n_1, m_2, n_2 \geq 1} a_1(m_1) b_1(n_1) a_2(m_2) b_2(n_2) n_1^{k_1+k_2-k_0-1} n_2^{k_0-1} \\
&\quad \times \int_0^\infty \exp\left(-2\pi\left(m_1 n_2 u + \frac{m_2 n_1}{Nu}\right)\right) u^{k_0-k_2-1} du \\
&= \frac{(-1)^{k_0-1} (2\pi)^{k_0}}{(k_0-1)! N^{k_2/2}} \int_0^\infty \sum_{m_1, n_2 \geq 1} a_1(m_1) b_2(n_2) n_2^{k_0-1} e^{-2\pi m_1 n_2 u} \\
&\quad \times \sum_{m_2, n_1 \geq 1} a_2(m_2) b_1(n_1) n_1^{k_1+k_2-k_0-1} e^{-2\pi m_2 n_1 / (Nu)} u^{k_0-k_2-1} du \\
&= \frac{(-1)^{k_0-1} (2\pi)^{k_0}}{(k_0-1)! N^{k_2/2}} \int_0^\infty g_{k_0}(iu) g_{k_1+k_2-k_0}(i/(Nu)) u^{k_0-k_2-1} du.
\end{aligned}$$

Assuming a modular transformation of the Eisenstein-like series $g_{k_1+k_2-k_0}(\tau)$ under $\tau \mapsto -1/(N\tau)$, we can realize the resulting integral as $c\pi^{k_0-k_1} L(g_{k_0} \widehat{g}_{k_1+k_2-k_0}, k_1)$, where c is algebraic (plus extra terms when $g_{k_1+k_2-k_0}(\tau)$ is an Eichler integral). Alternatively, if $g_{k_0}(\tau)$ transforms under the involution, we perform the transformation and switch to the variable $v = 1/(Nu)$ to arrive at $c\pi^{k_0-k_1} L(\widehat{g}_{k_0} g_{k_1+k_2-k_0}, k_1)$. In both cases we obtain an identity which relates the starting L -value $L(f, k_0)$ to a different ‘ L -value’ of a modular-like object of the same weight.

The case $k_1 = k_2 = 1$ and $k_0 = 2$, discussed in [6, 7] and in our example above, allows one to reduce the L -values to periods. In our future work [9] we plan to address some examples with $k_0 > 2$.

Acknowledgements. I am thankful to the organizers of the RIMS conference “Analytic number theory — related multiple aspects of arithmetic functions” (Kyoto University, Japan, October 31–November 2, 2011) represented by Takumi Noda for invitation to give a talk at the meeting. Special thanks go to my host Yasuo Ohno and his team from the Kinki University (Osaka); they made my stay in Japan both culturally and scientifically enjoyable.

I am indebted to Anton Mellit and Mat Rogers for fruitful conversations on the subject, and to Don Zagier for his encouragement to isolate the transformation part from [6, 7].

References

- [1] B. C. BERNDT, *Ramanujan’s notebooks. Part I* (Springer-Verlag, New York, 1985); *Part II* (Springer-Verlag, New York, 1989); *Part III* (Springer-Verlag, New York, 1991); *Part IV* (Springer-Verlag, New York, 1994); *Part V* (Springer-Verlag, New York, 1998).
- [2] M. KONTSEVICH and D. ZAGIER, Periods, in: *Mathematics unlimited — 2001 and beyond* (Springer, Berlin, 2001), 771–808.

- [3] Y. MARTIN and K. ONO, Eta-quotients and elliptic curves, *Proc. Amer. Math. Soc.* **125** (1997), 3169–3176.
- [4] F. RODRIGUEZ-VILLEGAS, Modular Mahler measures I, in: *Topics in number theory* (University Park, PA, 1997), Math. Appl. **467** (Kluwer Acad. Publ., Dordrecht, 1999), 17–48.
- [5] M. ROGERS, Hypergeometric formulas for lattice sums and Mahler measures, *Intern. Math. Res. Not.* (2011), 4027–4058.
- [6] M. ROGERS and W. ZUDILIN, From L -series of elliptic curves to Mahler measures, *Compositio Math.* **148** (2012), 385–414.
- [7] M. ROGERS and W. ZUDILIN, On the Mahler measure of $1 + X + 1/X + Y + 1/Y$, *Preprint*, 18 pages; <http://arxiv.org/abs/1102.1153>.
- [8] A. WEIL, Remarks on Hecke’s lemma and its use, in: *Algebraic number theory* (Kyoto Internat. Sympos., Res. Inst. Math. Sci., Univ. Kyoto, Kyoto, 1976) (Japan Soc. Promotion Sci., Tokyo, 1977), 267–274.
- [9] W. ZUDILIN, Periodddness of L -values, *in preparation*.