

# REMARKS ON MULTI-COMPARATIVELY QUASI-CONTRACTIONS

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ABSTRACT. In 2025, the author introduced a new class of quasi-contractions, called multi-comparatively quasi-contractions, which includes many known multivalued contractions as special cases. In this work, new sufficient conditions for multi-comparatively quasi-contractions will be established.

## 1. INTRODUCTION

Let  $(X, d)$  be a metric space. For each  $a \in X$  and  $M \subseteq X$ , let

$$d(a, M) = \inf_{b \in M} d(a, b).$$

Denote by  $\mathcal{N}(X)$  the class of all nonempty subsets of  $X$ ,  $\mathcal{C}(X)$  the family of all nonempty closed subsets of  $X$  and  $\mathcal{CB}(X)$  the family of all nonempty closed and bounded subsets of  $X$ . A function  $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \rightarrow [0, \infty)$  defined by

$$\mathcal{H}(A, B) = \max \left\{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \right\}$$

is said to be the Hausdorff metric on  $\mathcal{CB}(X)$  induced by the metric  $d$  on  $X$ . Throughout this paper, we denote by  $\mathbb{N}$  and  $\mathbb{R}$ , the sets of positive integers and real numbers, respectively.

Recall that a multivalued mapping  $T : X \rightarrow \mathcal{N}(X)$  is called

- (i) a *Nadler's type contraction*, if there exists a number  $k \in [0, 1)$  such that

$$\mathcal{H}(Tx, Ty) \leq kd(x, y) \quad \text{for all } x, y \in X,$$

- (ii) a *Kannan's type contraction*, if there exists a number  $k \in [0, \frac{1}{2})$  such that

$$\mathcal{H}(Tx, Ty) \leq k(d(x, Tx) + d(y, Ty)) \quad \text{for all } x, y \in X,$$

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- (iii) a *Chatterjea's type contraction*, if there exists a number  $k \in [0, \frac{1}{2})$  such that

$$\mathcal{H}(Tx, Ty) \leq k(d(x, Ty) + d(y, Tx)) \quad \text{for all } x, y \in X,$$

- (iv) a *Mizoguchi-Takahashi's type contraction*, if there exists an  $\mathcal{MT}$ -function  $\alpha : [0, \infty) \rightarrow [0, 1)$  such that

$$\mathcal{H}(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \quad \text{for all } x, y \in X,$$

- (v) a *multivalued  $(\theta, L)$ -almost contraction* [1, 9, 10], if there exist two constants  $\theta \in (0, 1)$  and  $L \geq 0$  such that

$$\mathcal{H}(Tx, Ty) \leq \theta d(x, y) + Ld(y, Tx) \quad \text{for all } x, y \in X,$$

- (vi) a *Berinde-Berinde's type contraction* (or a *generalized multivalued almost contraction* [1, 9, 10], if there exists an  $\mathcal{MT}$ -function  $\alpha : [0, \infty) \rightarrow [0, 1)$  and  $L \geq 0$  such that

$$\mathcal{H}(Tx, Ty) \leq \alpha(d(x, y))d(x, y) + Ld(y, Tx) \quad \text{for all } x, y \in X.$$

It is worth noting that a Berinde-Berinde's type contraction is a real generalization of Mizoguchi-Takahashi's type contraction (see, e.g., [4, Example 2.1]).

A function  $p : X \times X \rightarrow [0, +\infty)$  is called a *w-distance* [12, 14, 16], if the following are satisfied:

- (w1)  $p(a, c) \leq p(a, b) + p(b, c)$  for any  $a, b, c \in X$ ;
- (w2) for any  $a \in X$ ,  $p(a, \cdot) : X \rightarrow [0, +\infty)$  is l.s.c.;
- (w3) for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $p(c, a) \leq \delta$  and  $p(c, b) \leq \delta$  imply  $d(a, b) \leq \varepsilon$ .

A function  $p : X \times X \rightarrow [0, +\infty)$  is said to be a  $\tau$ -function [2, 3, 13, 15], first introduced and studied by Lin and Du, if the following conditions hold:

- ( $\tau$ 1)  $p(a, c) \leq p(a, b) + p(b, c)$  for any  $a, b, c \in X$ ;
- ( $\tau$ 2) if  $a \in X$  and  $\{b_n\}$  in  $X$  with  $\lim_{n \rightarrow \infty} b_n = b$  such that  $p(a, b_n) \leq \beta$  for some  $\beta = \beta(a) > 0$ , then  $p(a, b) \leq \beta$ ;
- ( $\tau$ 3) for any sequence  $\{a_n\}$  in  $X$  with  $\limsup_{n \rightarrow \infty} \{p(a_n, a_m) : m > n\} = 0$ , if there exists a sequence  $\{b_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} p(a_n, b_n) = 0$ , then  $\lim_{n \rightarrow \infty} d(a_n, b_n) = 0$ ;
- ( $\tau$ 4) for  $a, b, c \in X$ ,  $p(a, b) = 0$  and  $p(a, c) = 0$  imply  $b = c$ .

It is well known that the metric  $d$  is a *w-distance* and any *w-distance* is a  $\tau$ -function, but the converse is not true (see [2, 13]). Note that not either of the implications  $p(x, y) = 0 \iff x = y$  necessarily holds and  $p$  is nonsymmetric in general.

In 2016, Du [6] introduced and studied the concept of essential distance as follows:

**Definition 1.1** (see [6, Definition 1.2]). Let  $(X, d)$  be a metric space. A function  $p : X \times X \rightarrow [0, +\infty)$  is called an *essential distance* (abbreviated as “*e-distance*”) if conditions ( $\tau$ 1), ( $\tau$ 2) and ( $\tau$ 3) hold.

It is very obvious that any  $\tau$ -function is an  $e$ -distance.

In 2019, Du introduced and studied the concept of  $e^0$ -distance [7] (see also [11]) as follows:

**Definition 1.2** (see [7, Definition 1.3]). Let  $(X, d)$  be a metric space. A function  $p : X \times X \rightarrow [0, +\infty)$  is called an  $e^0$ -distance if it is an  $e$ -distance on  $X$  with  $p(x, x) = 0$  for all  $x \in X$ .

**Example 1.3.** Let  $X = \mathbb{R}$  with the metric  $d(a, b) = |a - b|$ . Then  $(X, d)$  is a metric space. Define the function  $p : X \times X \rightarrow [0, +\infty)$  by

$$p(x, y) = \max\{5(y - x), 3(x - y)\}.$$

Therefore  $p$  is not a metric due to its asymmetry. It is easy to see that  $p$  is an  $e^0$ -distance on  $X$ .

The following definition of  $e^0$ -metric was studied by Du in [7] which generalizes the concept of Hausdorff metric.

**Definition 1.4** (see [7, Definition 1.4]). Let  $(X, d)$  be a metric space and  $p$  be an  $e^0$ -distance. For any  $E, F \in \mathcal{CB}(X)$ , define a function  $\mathcal{D}_p : \mathcal{CB}(X) \times \mathcal{CB}(X) \rightarrow [0, +\infty)$  by

$$\mathcal{D}_p(E, F) = \max\{\xi_p(E, F), \xi_p(F, E)\},$$

where  $\xi_p(E, F) = \sup_{x \in E} p(x, F)$ , then  $\mathcal{D}_p$  is said to be the  $e^0$ -metric on  $\mathcal{CB}(X)$  induced by  $p$ .

**Theorem 1.5** (see [7, Theorem 2.4] or [11, Theorem 6]). Let  $(X, d)$  be a metric space and  $\mathcal{D}_p$  be an  $e^0$ -metric defined as in Definition ?? on  $\mathcal{CB}(X)$  induced by an  $e^0$ -distance  $p$ . Then for  $E, F, G \in \mathcal{CB}(X)$ , the following hold:

- (i)  $\xi_p(E, F) = 0 \iff E \subseteq F$ ;
- (ii)  $\xi_p(E, F) \leq \xi_p(E, G) + \xi_p(G, F)$ ;
- (iii) Every  $e^0$ -metric  $\mathcal{D}_p$  is a metric on  $\mathcal{CB}(X)$ .

In 2025, Du introduced a new class of quasi-contractions, called multi-comparatively quasi-contractions (see Definition 2.3 below), which includes Nadler's type contractions, Kannan's type contractions, Chatterjea's type contractions, Mizoguchi-Takahashi's type contractions, multivalued  $(\theta, L)$ -almost contractions and Berinde-Berinde's type contractions as special cases. In this work, some new sufficient conditions for multi-comparatively quasi-contractions will be established.

## 2. PRELIMINARIES

In 2005, Uderzo [17] introduced directional multivalued  $k(\cdot)$ -contractions and established many new fixed point theorems for directional multivalued  $k(\cdot)$ -contractions (see [17] for more details).

**Definition 2.1** (Uderzo [17]). Let  $U$  be a nonempty subset of a metric space  $(X, d)$ . A multivalued map  $T : U \rightarrow \mathcal{CB}(X)$  is called a *directional multivalued  $k(\cdot)$ -contraction* if there exists  $\lambda \in (0, 1]$ ,  $a : (0, +\infty) \rightarrow [\lambda, 1]$

and  $k : (0, +\infty) \rightarrow [0, 1]$  such that for every  $x \in U$  with  $x \notin Tx$ , there is  $y \in U \setminus \{x\}$  satisfying the inequalities

$$a(d(x, y))d(x, y) + d(y, Tx) \leq d(x, Tx)$$

and

$$\sup_{z \in Tx} d(z, Ty) \leq k(d(x, y))d(x, y).$$

**Definition 2.2** (see [8, Definition 3.1]). Let  $(X, d)$  be a metric space and  $p$  be an  $e$ -distance.

- (i) For any  $a_1, a_2, a_3, a_4, a_5 \in [0, 1]$ , we define a real-valued mapping  $f_{[a_1, a_2, a_3, a_4, a_5]} : X \times X \rightarrow [0, +\infty)$  by
$$f_{[a_1, a_2, a_3, a_4, a_5]}(x, y) := a_1 p(x, y) + a_2 p(x, Tx) + a_3 p(y, Ty) + a_4 p(x, Ty) + a_5 p(y, Tx).$$
- (ii) We denote by  $\mathcal{O}$  the collection of all mappings  $f_{[a_1, a_2, a_3, a_4, a_5]}$  on  $X \times X$  satisfying  $0 < \sum_{i=1}^5 a_i \leq 1$ .
- (iii) For any  $(x, y) \in X \times X$ , we define
$$\mathcal{O}_{[x, y]} := \{f_{[a_1, a_2, a_3, a_4, a_5]}(x, y) \in [0, +\infty) : f_{[a_1, a_2, a_3, a_4, a_5]} \in \mathcal{O}\}.$$

Here, we shall give some examples of such mappings in  $\mathcal{O}$ :

$$f_{[1, 0, 0, 0, 0]}(x, y) = p(x, y), \quad f_{[0, 1, 0, 0, 0]}(x, y) = p(x, Tx), \quad f_{[0, 0, 1, 0, 0]}(x, y) = p(y, Ty),$$

$$f_{[0, 0, 0, 1, 0]}(x, y) = p(x, Ty), \quad f_{[0, 0, 0, 0, 1]}(x, y) = p(y, Tx),$$

$$f_{[0, \frac{1}{2}, \frac{1}{2}, 0, 0]}(x, y) = \frac{p(x, Tx) + p(y, Ty)}{2},$$

$$f_{[0, 0, 0, \frac{1}{2}, \frac{1}{2}]}(x, y) = \frac{p(x, Ty) + p(y, Tx)}{2},$$

$$f_{[0, \frac{1}{3}, 0, \frac{2}{3}, 0]}(x, y) = \frac{p(x, Tx) + 2p(x, Ty)}{3},$$

$$f_{[\frac{1}{4}, \frac{1}{4}, 0, 0, \frac{1}{4}]}(x, y) = \frac{p(x, y) + p(x, Tx) + p(y, Tx)}{4},$$

$$f_{[\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}]}(x, y) = \frac{p(x, y) + p(x, Tx) + p(y, Ty) + p(x, Ty) + p(y, Tx)}{7},$$

$$f_{[\frac{1}{4}, \frac{1}{8}, 0, \frac{1}{8}, \frac{1}{2}]}(x, y) = \frac{2p(x, y) + p(x, Tx) + p(x, Ty) + 4p(y, Tx)}{8},$$

and so on.

It is easy to see that for any  $(x, y) \in X \times X$ , we have

$$\sup \mathcal{O}_{[x,y]} = \max \{p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx)\}.$$

Very recently, Du [8] introduced the concept of multi-comparatively quasi-contractions to generalize and improve the concept of directional multivalued  $k(\cdot)$ -contractions.

**Definition 2.3** (see [8, Definition 3.3]). Let  $U$  be a nonempty subset of a metric space  $(X, d)$ ,  $p : X \times X \rightarrow [0, +\infty)$  and  $\phi : [0, +\infty) \rightarrow [0, 1)$  be functions. Assume that  $c \in [0, 1)$  and let  $\eta : [0, +\infty) \rightarrow (c, 1]$  be a function. A multivalued mapping  $T : U \rightarrow \mathcal{N}(X)$  is called a *multi-comparatively quasi-contraction* with respect to  $p, \phi, c$  and  $\eta$  (abbreviated as  $(p, \phi, c, \eta)$ -**MCQC**) if for any  $x \in U$  with  $x \notin Tx$ , there exist  $y \in U \setminus \{x\}$  and  $z \in Tx$  such that

$$\eta(p(x, y))p(x, y) + p(y, z) \leq p(x, Tx)$$

and

$$(2.1) \quad p(z, Ty) \leq \phi(p(x, y))G(x, y),$$

where  $\Omega$  is a nonempty subset of  $\mathcal{O}$ ,

$$\Omega_{[x,y]} := \{f_{[a_1, a_2, a_3, a_4, a_5]}(x, y) \in [0, +\infty) : f_{[a_1, a_2, a_3, a_4, a_5]} \in \Omega\}$$

and  $G(x, y) = \sup \Omega_{[x,y]}$  for  $(x, y) \in X \times X$ . In particular, if  $p \equiv d$ , then we use the notation  $(\phi, c, \eta)$ -**MCQC** instead of  $(d, \phi, c, \eta)$ -**MCQC**.

**Remark 2.4.** In Definition 2.3, if condition (2.1) is replaced by the following condition

$$p(z, Ty) \leq \phi(p(x, y))p(x, y),$$

Then  $T$  is called a directional hidden contraction with respect to  $p, \phi, c$  and  $\eta$ , which was studied by Du in [4, Definition 2.1]. It is obvious that a directional hidden contraction is  $(p, \phi, c, \eta)$ -**MCQC** with  $\Omega_{[x,y]} := \{d(x, y)\}$ .

**Theorem 2.5** (Du [8, Theorem 3.5]). *Any directional multivalued  $k(\cdot)$ -contraction is a multi-comparatively quasi-contraction.*

### 3. SUFFICIENT CONDITIONS FOR MULTI-COMPARATIVELY QUASI-CONTRACTIONS

**Theorem 3.1** (see [8, Theorem 3.6]). *Let  $(X, d)$  be a metric space,  $p$  be a  $e^0$ -distance,  $T : X \rightarrow \mathcal{C}(X)$  be a multivalued mapping and  $\varphi : [0, +\infty) \rightarrow [0, 1)$  be a function. Suppose that*

(H1) *there exists  $\beta \in [0, +\infty)$  such that*

$$\limsup_{t \rightarrow \beta^+} \varphi(t) < 1;$$

(H2) *for each  $x \in X$  with  $x \notin Tx$ , it holds*

$$p(y, Ty) \leq \varphi(p(x, y)) \max \{p(x, y), p(x, Tx), p(y, Ty), p(x, Ty)\} \quad \text{for all } y \in Tx.$$

*Then there exist  $c \in [0, 1)$  and functions  $\eta : [0, +\infty) \rightarrow (c, 1]$  and  $\phi : [0, +\infty) \rightarrow [0, 1)$  such that*

- (a)  $\limsup_{t \rightarrow \beta^+} \phi(t) \leq c < \liminf_{t \rightarrow \beta^+} \eta(t)$ ;  
 (b)  $T$  is a  $(p, \phi, c, \eta)$ -**MCQC**.

**Theorem 3.2** (see [8, Theorem 3.7]). *Let  $(X, d)$  be a metric space,  $p$  be a  $e^0$ -distance,  $\mathcal{D}_p$  be a  $e^0$ -metric on  $\mathcal{CB}(X)$  induced by  $p$ ,  $T : X \rightarrow \mathcal{CB}(X)$  be a multivalued mapping and  $\varphi : [0, +\infty) \rightarrow [0, 1)$  and  $h : X \times X \rightarrow [0, +\infty)$  be functions. Suppose that*

(K1) *there exists  $\beta \in [0, +\infty)$  such that*

$$\limsup_{t \rightarrow \beta^+} \varphi(t) < 1;$$

(K2)  $\mathcal{D}_p(Tx, Ty) \leq \varphi(p(x, y))S(x, y) + h(x, y)p(y, Tx)$  for all  $x, y \in X$  with  $x \neq y$ , where

$$S(x, y) = \max \left\{ p(x, y), \frac{p(x, Tx) + p(y, Ty)}{2}, \frac{p(x, Ty) + p(y, Tx)}{2} \right\}.$$

*Then there exist  $c \in [0, 1)$  and functions  $\eta : [0, +\infty) \rightarrow (c, 1]$  and  $\phi : [0, +\infty) \rightarrow [0, 1)$  such that*

- (a)  $\limsup_{t \rightarrow \beta^+} \phi(t) \leq c < \liminf_{t \rightarrow \beta^+} \eta(t)$ ;  
 (b)  $T$  is a  $(p, \phi, c, \eta)$ -**MCQC**.

The following result follows directly from Theorem 3.2.

**Theorem 3.3** (see [8, Theorem 3.8]). *Let  $(X, d)$  be a metric space and  $T : X \rightarrow \mathcal{CB}(X)$  be a multivalued mapping. Assume that one of the following conditions holds:*

- (1)  $T$  is a Berinde-Berinde's type contraction;
- (2)  $T$  is a multivalued  $(\theta, L)$ -almost contraction;
- (3)  $T$  is a Mizoguchi-Takahashi's type contraction;
- (4)  $T$  is a Nadler's type contraction;
- (5)  $T$  is a Kannan's type contraction;
- (6)  $T$  is a Chatterjea's type contraction.

*Then there exist  $c \in [0, 1)$  and functions  $\eta : [0, +\infty) \rightarrow (c, 1]$  and  $\phi : [0, +\infty) \rightarrow [0, 1)$  such that  $T$  is a  $(\phi, c, \eta)$ -**MCQC**.*

Finally, we establish the following new existence theorem which provides new sufficient conditions for multi-comparatively quasi-contractions.

**Theorem 3.4.** *Let  $(X, d)$  be a metric space,  $p$  be a  $e^0$ -distance,  $\mathcal{D}_p$  be a  $e^0$ -metric on  $\mathcal{CB}(X)$  induced by  $p$ ,  $T : X \rightarrow \mathcal{CB}(X)$  be a multivalued mapping and  $\varphi : [0, +\infty) \rightarrow [0, 1)$  and  $h : X \times X \rightarrow [0, +\infty)$  be functions. Suppose that*

(V1) *there exists  $\beta \in [0, +\infty)$  such that*

$$\limsup_{t \rightarrow \beta^+} \varphi(t) < 1;$$

(V2)  $\mathcal{D}_p(Tx, Ty) \leq \varphi(p(x, y))S(x, y) + h(x, y)p(y, Tx)$  for all  $x, y \in X$  with  $x \neq y$ , where

$$S(x, y) = \max \left\{ p(x, y), \frac{p(x, Tx) + p(y, Ty)}{2}, \frac{p(x, Ty) + p(y, Tx)}{2}, \frac{p(x, Tx) + p(y, Tx)}{2}, \right. \\ \frac{p(y, Tx) + p(y, Ty)}{2}, \frac{p(x, Tx) + p(y, Ty) + p(y, Tx)}{3}, \\ \frac{p(x, Tx) + p(x, Ty) + p(y, Tx)}{3}, \frac{p(y, Ty) + p(x, Ty) + p(y, Tx)}{3}, \\ \frac{p(x, y) + p(y, Ty) + p(x, Tx)}{3}, \frac{p(x, y) + p(x, Tx) + p(y, Tx)}{3}, \\ \frac{2p(y, Tx) + p(y, Ty) + p(x, Ty)}{4}, \frac{p(x, y) + p(y, Ty) + p(y, Tx)}{4}, \\ \frac{p(x, Tx) + p(y, Ty) + p(x, Ty) + p(y, Tx)}{4}, \\ \frac{p(x, y) + p(y, Ty) + p(x, Ty)}{4}, \frac{p(x, y) + 2p(x, Tx) + p(y, Ty)}{5}, \\ \left. \frac{p(x, y) + p(x, Tx) + p(y, Ty) + p(x, Ty) + p(y, Tx)}{5} \right\}.$$

Then there exist  $c \in [0, 1)$  and functions  $\eta : [0, +\infty) \rightarrow (c, 1]$  and  $\phi : [0, +\infty) \rightarrow [0, 1)$  such that

- (a)  $\limsup_{t \rightarrow \beta^+} \phi(t) \leq c < \liminf_{t \rightarrow \beta^+} \eta(t)$ ;
- (b)  $T$  is a  $(p, \phi, c, \eta)$ -**MCQC**.

**Remark 3.5.** (a) Theorem 3.4 is a real generalization of Theorem 3.2.  
 (b) In fact, Theorem 3.4 can lead to some new sufficient conditions for multi-comparatively quasi-contractions.

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