

On Higher Differentiability and Partial Regularity of the Minimizers in the Calculus of Variations

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1. Introduction

In this paper we shall treat with the following problem in the calculus of variations : Let n and N be positive integers and suppose that $\Omega \subset R^n$ is a bounded domain with the C^2 -class boundary. Then we consider the functional,

$$(1.1) \quad I[v] \equiv \sum_{\alpha, \beta=1}^n \sum_{i, j=1}^N \int_{\Omega} a_{i, j}^{\alpha, \beta}(x, v) D_{\alpha} v^i D_{\beta} v^j dx \quad \text{for } v : \Omega \mapsto R^N,$$

where $D_{\alpha} v^i = \frac{\partial v^i}{\partial x_{\alpha}}$ ($\alpha = 1, \dots, n$, $i = 1, \dots, N$) and $a_{i, j}^{\alpha, \beta}$ ($\alpha, \beta = 1, \dots, n$, $i, j = 1, \dots, N$) are continuously differentiable functions in $\Omega \times R^N$ satisfying the following : There exist positive numbers λ and Λ ($0 < \lambda \leq \Lambda < +\infty$) such that $a_{i, j}^{\alpha, \beta}$ ($\alpha, \beta = 1, \dots, n$, $i, j = 1, \dots, N$) satisfy for $\forall (x, v) \in \Omega \times R^N$

$$(1.2) \quad \lambda |\zeta|^2 \leq \sum_{i, j=1}^N \sum_{\alpha, \beta=1}^n a_{i, j}^{\alpha, \beta}(x, v) \zeta_{\alpha}^i \zeta_{\beta}^j \leq \Lambda |\zeta|^2 \quad \text{for } \forall (x, v) \in R^{n \times N} \text{ and } \forall \zeta \in R^{n \times N},$$

$$(1.3) \quad a_{i, j}^{\alpha, \beta} = a_{j, i}^{\beta, \alpha}.$$

In addition, since the coefficients $a_{i, j}^{\alpha, \beta}$ belong to $C^1(\Omega \times R^N; R)$, for positive numbers K_1 and K_2 there exists a positive number $L(K_1, K_2)$ such that

$$(1.4) \quad \begin{aligned} & \max_{\substack{1 \leq i, j \leq N \\ 1 \leq \alpha, \beta \leq n}} \max_{\substack{|x| \leq K_1 \\ |z| \leq K_2}} |a_{i, j}^{\alpha, \beta}(x, z)| + \max_{\substack{1 \leq i, j \leq N \\ 1 \leq \alpha, \beta \leq n}} \max_{\substack{|x| \leq K_1 \\ |z| \leq K_2}} \left| \frac{\partial a_{i, j}^{\alpha, \beta}}{\partial e}(x, z) \right| \\ & + \max_{\substack{1 \leq i, j, k \leq N \\ 1 \leq \alpha, \beta \leq n}} \max_{\substack{|x| \leq K_1 \\ |z| \leq K_2}} |a_{i, j, k}^{\alpha, \beta}(x, z)| \leq L(K_1, K_2) \end{aligned}$$

$$\text{where } a_{i, j, k}^{\alpha, \beta}(x, z) \equiv \frac{\partial a_{i, j}^{\alpha, \beta}}{\partial z_k}(x, z)$$

and

$\frac{\partial a_{i,j}^{\alpha,\beta}}{\partial e}(x, z)$ denotes the derivative in a direction of a vector e in R^n .

This implies the existence of at least a minimizer of the functional I in the Sobolev space $H^{1,2}(\Omega; R^N)$ and I is lower semicontinuous with respect to the weak topology of $H^{1,2}(\Omega; R^N)$ (see [Mo]) under an appropriate boundary condition.

First, we show that the first-derivatives of minimizers satisfies a modulus of uniform continuity in the norm $L^2_{loc}(\Omega; R^N)$.

Secondly, we mention a convergence theorem and a partial regularity result of the weak differentials of minimizers. However, we remark that the former theorem was proved in [Gm], [HKL] and [Mm].

We use the summation convention that Latin indices run from 1 to N and Greek indices run from 1 to n .

We conclude this introduction by recalling other notational conventions:

$$(1.5) \quad B_R(x_0) \equiv \{x \in R^N : |x - x_0| < R\}.$$

For a set $A \subset R^N$, we denote by $mes A$ and $|A|$ the n -dimensional Lebesgue measure of A .

For $u \in L^1(B_R(x_0); R^N)$, we define

$$(1.6) \quad u_{x_0, R} = \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u(x) dx.$$

For a sufficiently small number d , we define an open set

$$(1.7) \quad \Omega_d = \Omega - \{x \in \Omega : dist(x, \partial\Omega) \leq d\},$$

where $dist(x, \partial\Omega)$ means the Euclidean metric between x and $\partial\Omega$.

For a set A in R^n , $H^{(k)}(A)$ denotes the k -dimensional Hausdorff measure of A (for the definition, see [Gm]).

e_i ($i = 1, \dots, n$) means the unit vector in R^N parallel to the x_i -axis. We define a translate operator Δ_m ($m = 1, 2, \dots, n$) by

$$(1.8) \quad (\Delta_m f)(x) = f(x + h e_m) - f(x) \quad \text{for} \quad f \in L^p(\Omega; R^N).$$

2. Main Result

Under the above preparations, we can describe

THEOREM 1. *Let u be a minimizer of the functional I in $H^{1,2}(\Omega; R^N)$ and let us suppose that u is a bounded, namely there exists some positive constant M such that $ess.sup|u| \leq M$. Then, for any fixed domain $\tilde{\Omega}$ compactly contained in Ω , there exists positive number $\alpha = \alpha(n, N, \lambda, \Lambda, M, L)$ ($0 < \alpha \leq 1$) and $C = C(n, N, \lambda, \Lambda, \tilde{\Omega}, \Omega, M, L)$ such that for $h > 0$ with $h \leq dist(\tilde{\Omega}; \partial\Omega)$ u satisfies*

$$\int_{\tilde{\Omega}} |\Delta_m(\nabla u(x))|^2 dx \leq C \cdot h^\alpha \quad \text{for} \quad \forall m (m = 1, 2, \dots, n)$$

THEOREM 2. Suppose that $\{u_i\}_{i \geq 1}$ is a sequence of minimizers of I in the space $H^{1,2}(\Omega; R^N)$ such that $\{u_i\}_{i \geq 1}$ converges strongly to a function u_0 in $L^2_{loc}(\Omega; R^N)$. Then the function u_0 belongs to $H^{1,2}_{loc}(\Omega; R^N)$ and moreover a suitable subsequence of $\{u_i\}_{i \geq 1}$ converges strongly to u_0 in $H^{1,2}_{loc}(\Omega; R^N)$.

THEOREM 3. Let u be a minimizer of the functional I in $H^{1,2}(\Omega; R^N)$. Then, for a singular set defined by

$$(2.1) \quad S = \{x \in \Omega : \nexists \lim_{\rho \rightarrow +0} |(Du)_{x,\rho}|\} \cup \{x \in \Omega : \lim_{\rho \rightarrow +0} |(Du)_{x,\rho}| = +\infty\}$$

the following

$$(2.2) \quad H^{(\beta)}(S) = 0$$

holds for any positive number β satisfying $n - 2\alpha < \beta < n$.

Remark. In the following proof, the letter C_i ($i = 1, \dots, 14$) means a various constant depending only on $n, N, \lambda, \Lambda, \Omega, \tilde{\Omega}, M$ and L .

PROOF OF THEOREM 1

First, a minimizer u is a weak solution of the Euler-Lagrange equations of the functional I , u satisfies

$$(2.3) \quad \begin{aligned} & 2 \int_{\Omega} a_{i,j}^{\alpha,\beta}(u(x), x) D_{\alpha} u^i(x) D_{\beta} \phi^j(x) dx \\ & + \int_{\Omega} a_{i,j,k}^{\alpha,\beta}(u(x), x) D_{\alpha} u^i(x) D_{\beta} u^j(x) \phi^k(x) dx = 0 \end{aligned}$$

$$\text{for } \forall \phi(x) \in \mathring{H}^{1,2}(\Omega; R^N).$$

Next, let δ be a positive number satisfying $\delta < \frac{1}{8} \text{dist}(\tilde{\Omega}, \partial\Omega)$. For each number h ($0 < h < \delta$), the parallel transition along with x_m -axis ($m = 1, \dots, n$) leads to

$$(2.4) \quad \begin{aligned} & 2 \int_{\Omega} a_{i,j}^{\alpha,\beta}(u(x + he_m), x + he_m) D_{\alpha} u^i(x + he_m) D_{\beta} \phi^j(x) dx \\ & + \int_{\Omega} a_{i,j,k}^{\alpha,\beta}(u(x + he_m), x + he_m) D_{\alpha} u^i(x + he_m) D_{\beta} u^j(x + he_m) \phi^k(x) dx = 0 \end{aligned}$$

$$\text{for } \forall \phi(x) \in \mathring{C}^{\infty}(\Omega_{\delta}; R^N).$$

Then we have

$$(2.5) \quad \begin{aligned} & 2 \int_{\Omega} a_{i,j}^{\alpha,\beta}(u(x), x) D_{\alpha} \Delta_m u^i(x) D_{\beta} \phi^j(x) dx \\ & = 2 \int_{\Omega} [a_{i,j}^{\alpha,\beta}(x, u(x)) - a_{i,j}^{\alpha,\beta}(u(x + he_m), x + he_m)] D_{\alpha} u^i(x + he_m) D_{\beta} \phi^j(x) dx \\ & - \int_{\Omega} a_{i,j,k}^{\alpha,\beta}(x + he_m, u(x + he_m)) D_{\alpha} u^i(x + he_m) D_{\beta} u^j(x + he_m) \phi^k(x) dx \\ & + \int_{\Omega} a_{i,j,k}^{\alpha,\beta}(x, u(x)) D_{\alpha} u^i(x) D_{\beta} u^j(x) \phi^k(x) dx \end{aligned}$$

after subtracting (2.4) from (2.3) .

We now substitute $\Delta_m u(x)\zeta^2(x)$ into $\phi(x)$ in (2.5) , where $\zeta(x) \in \dot{C}^\infty(\Omega_\delta; R)$ is defined by

$$(2.6) \quad \zeta(x) = \begin{cases} 1 & : \Omega_{4\delta} \\ 0 & : \Omega / \Omega_{3\delta} \end{cases} \quad \text{with} \quad |D\zeta(x)| \leq \frac{2}{\delta}.$$

Then we have

$$(2.7) \quad \begin{aligned} & 2 \int_{\Omega} a_{i,j}^{\alpha,\beta}(u(x), x) D_{\alpha}(\Delta_m u^i(x)) D_{\beta}(\Delta_m u^j(x)) \zeta^2(x) dx \\ & + 4 \int_{\Omega} a_{i,j}^{\alpha,\beta}(u(x), x) D_{\alpha}(\Delta_m u^i(x)) D_{\beta} \zeta(x) \Delta_m u^j(x) \zeta(x) dx \\ & = 2 \int_{\Omega} [a_{i,j}^{\alpha,\beta}(x, u(x)) - a_{i,j}^{\alpha,\beta}(u(x + he_m), x + he_m)] D_{\alpha} u^i(x + he_m) \\ & \quad [D_{\beta}(\Delta_m u^j(x)) \zeta^2(x) + 2 \Delta_m u^j(x) D_{\beta} \zeta(x) \zeta(x)] dx \\ & - \int_{\Omega} a_{i,j,k}^{\alpha,\beta}(x + he_m, u(x + he_m)) D_{\alpha} u^i(x + he_m) D_{\beta} u^j(x + he_m) \Delta_m u^k(x) \zeta^2(x) dx \\ & + \int_{\Omega} a_{i,j,k}^{\alpha,\beta}(x, u(x)) D_{\alpha} u^i(x) D_{\beta} u^j(x) \Delta_m u^k(x) \zeta^2(x) dx \end{aligned}$$

Here, we estimate the left-hand side of (2.7) , which we call (L) , from below . First , by using (1.2) , we have

$$(2.8) \quad \begin{aligned} (L) & \geq 2\lambda \int_{\Omega} |D(\Delta_m u(x))|^2 \zeta^2(x) dx \\ & - 4nN\Lambda \int_{\Omega} |D(\Delta_m u(x))| |\zeta(x)| |\Delta_m u(x)| |D\zeta(x)| dx . \end{aligned}$$

Second , applying the *Schwarz inequality* to the second term of (2.8) with $\varepsilon = \frac{\lambda}{2nN\Lambda}$, we have

$$(2.9) \quad \begin{aligned} (L) & \geq 2\lambda \int_{\Omega} |\Delta_m(Du(x))|^2 \zeta^2(x) dx \\ & - 2\varepsilon nN\Lambda \int_{\Omega} |\Delta_m(Du(x))|^2 \zeta^2(x) dx \\ & - \frac{2nN\Lambda}{\varepsilon} \int_{\Omega} |D\zeta(x)|^2 |\Delta_m u(x)|^2 dx \\ & \geq \lambda \int_{\Omega} |\Delta_m(Du(x))|^2 \zeta^2(x) dx \\ & - 2 \frac{(nN\Lambda)^2}{\lambda} \int_{\Omega} |D\zeta^2(x)| |\Delta_m u(x)|^2 dx . \end{aligned}$$

On the other hand , we perform the estimates of the right-hand side of (2.7) , which we call

(R) .

$$\begin{aligned}
(R) = & -2 \int_{\Omega} \int_0^1 \frac{da_{i,j}^{\alpha,\beta}}{dt}(x + t h e_m, u(x) + t \Delta_m u(x)) dt D_{\alpha} u^i(x + h e_m) \\
& [D_{\beta}(\Delta_m u^j(x)) \zeta^2(x) + 2 \Delta_m u^j(x) D_{\beta} \zeta(x) \zeta(x)] dx \\
& - \int_{\Omega} a_{i,j,k}^{\alpha,\beta}(x + h e_m, u(x + h e_m)) D_{\alpha} u^i(x + h e_m) D_{\beta} u^j(x + h e_m) \Delta_m u^k(x) \zeta^2(x) dx \\
(2.10) \quad & + \int_{\Omega} a_{i,j,k}^{\alpha,\beta}(x, u(x)) D_{\alpha} u^i(x) D_{\beta} u^j(x) \Delta_m u^k(x) \zeta^2(x) dx
\end{aligned}$$

By using (1.4) and the boundedness of u , and applying the *Schwarz inequality* to (2.10), we have

$$\begin{aligned}
(R) \leq & C_1 \int_{\Omega} (h + |\Delta_m u(x)|) \zeta(x) dx. \\
(2.11) \quad & + C_1 \int_{\Omega} (h + |\Delta_m u(x)| + |\Delta_m u(x)|^2) [|Du(x + h e_m)|^2 + |Du(x)|^2] \zeta(x)^2 dx.
\end{aligned}$$

Thus, by combining (2.9) with (2.11), we have the following :

$$\begin{aligned}
& \int_{\Omega_{4\delta}} |\Delta_m(Du(x))|^2 dx \leq C_2 \int_{\Omega_{3\delta}} |\Delta_m u(x)|^2 dx \\
& + C_2 \int_{\Omega_{3\delta}} (h + |\Delta_m u(x)|) dx \\
(2.12) \quad & + C_2 \int_{\Omega_{3\delta}} (h + |\Delta_m u(x)|) [|Du(x)|^2 dx + |Du(x + h e_m)|^2] dx
\end{aligned}$$

Here, it is a well-known fact that a minimizer u satisfies a *so-called Caccioppoli inequality* (see [Gm]) : There exists a positive constant C , depending only on $n, N, \lambda, \Lambda, M$ such that

$$(2.13) \quad \int_{B_R} |Du(x)|^2 dx \leq \frac{C}{R^2} \int_{B_{2R}} |u(x) - u_R|^2 dx$$

holds for any ball $B_{2R} \subset \subset \Omega$ with $0 < \forall R < \delta$. A direct application of the above inequality to *Gering inequality* due to *F.W.Gering* [Ge] (see also [Gm]) leads to the following: There exists a positive number p ($p > 2$), which can be supposed to satisfy $p < 4$ and C depending only on $n, N, \lambda, \Lambda, \Omega, M$ such that $Du(x)$ belongs to $L_{loc}^p(\Omega; R^N)$ and moreover

$$(2.14) \quad \left(\frac{1}{|\tilde{\Omega}|} \int_{\tilde{\Omega}} |Du(x)|^p dx \right)^{\frac{1}{p}} \leq \left(\frac{1}{|\Omega|} \int_{\Omega} |Du(x)|^2 dx \right)^{\frac{1}{2}}$$

holds for $\forall \tilde{\Omega} \subset \subset \Omega$.

Thus, we apply *Hölder inequality* to the second term of the right-hand side of (2.12) and we have

$$\begin{aligned}
& \int_{\Omega_{4\delta}} |\Delta_m(Du(x))|^2 dx \leq C_4 \int_{\Omega_{2\delta}} [h + |\Delta_m u(x)| + |\Delta_m u(x)|^2] dx \\
(2.15) \quad & + C_4 \left[\int_{\Omega_{2\delta}} |D_m u(x)|^p dx \right]^{\frac{2}{p}} \left[\int_{\Omega_{2\delta}} |\Delta_m u(x)|^{\frac{p}{p-2}} dx \right]^{\frac{p-2}{p}}.
\end{aligned}$$

In addition , by using (2.13),(2.14) and the boundedness of u , we have

$$(2.16) \quad \begin{aligned} \int_{\Omega_{4\delta}} |\Delta_m(Du(x))|^2 dx &\leq C_4 \int_{\Omega_{2\delta}} [h + |\Delta_m u(x)| + |\Delta_m u(x)|^2] dx \\ &+ C_5 \left[\int_{\Omega_{2\delta}} |\Delta_m u(x)|^{\frac{p}{p-2}} dx \right]^{\frac{p-2}{p}}. \end{aligned}$$

Since $2 < p < 4$ implies $\frac{p}{p-2} > 2$ it follows from the boundedness of $u(x)$ that

$$(2.17) \quad \begin{aligned} \int_{\Omega_{4\delta}} |\Delta_m(Du(x))|^2 dx &\leq C_4 \int_{\Omega_{2\delta}} [h + |\Delta_m u(x)| + |\Delta_m u(x)|^2] dx \\ &+ C_6 \left[\int_{\Omega_{2\delta}} |\Delta_m u(x)|^2 dx \right]^{\frac{p-2}{p}}. \end{aligned}$$

Also , from *Newton-Leibnitz formula* and a *Caccioppoli inequality* , we obtain

$$(2.18) \quad \int_{\Omega_{2\delta}} |\Delta_m u(x)|^2 dx \leq C_7 h^2 \int_{\Omega_\delta} |Du(x)|^2 dx \leq C_8 h^2.$$

Consequently , from (2.17) and (2.18) , we deduce

$$(2.19) \quad \int_{\Omega_{4\delta}} |\Delta_m(Du(x))|^2 dx \leq C_8 h^{\frac{2}{p}(p-2)}.$$

¹ Also , for any fixed unit vector e one can easily prove

$$(2.20) \quad \int_{\Omega_{4\delta}} |\Delta_e(Du(x))|^2 dx \leq C_8 h^{\frac{2}{p}(p-2)}$$

PROOF OF THEOREM 2

From (2.19) , we obtain an equi-continuity of a sequence of minimizers in $H_{loc}^{1,2}(\Omega; R^N)$. Also , it follows from (2.14) that a sequence of minimizers satisfies a uniform boundedness in $H_{loc}^{1,2}(\Omega; R^N)$. Thus we obtain the assertion of this theorem from *Rellich - Kondrachev theorem* , (see [Ad]) .

PROOF OF THEOREM 3

The proof of this Theorem is based on estimate (2.20) and the following lemma due to [Gi] (see also [Gm]) .

¹The estimate (2.19) and (2.20) play an important role in the proofs of the Theorem 2 and Theorem 3

LEMMA 3.1.

Let v be a function in $L^1_{loc}(\Omega)$ and β be any number satisfying $n - 2\alpha < \beta < n$. Set

$$(2.21) \quad E_\beta = \{x \in \Omega : \limsup_{\rho \rightarrow +0} \rho^{-\beta} \int_{B_\rho(x)} |v(y)| dy > 0\}.$$

Then, we have

$$(2.22) \quad H^{(\beta)}(E_\beta) = 0.$$

First, to apply Lemma 3.1 to the proof of Theorem 3 we construct a support function defined as follows: For $\rho_k = \delta(\frac{1}{2})^{k+1}$ ($k = 1, 2, \dots$) with $\delta = \text{dist}(\tilde{\Omega}, \partial\Omega)$ and a sequence $\{e_k\}_{k \geq 1}$ of unit vectors in R^n we define

$$(2.23) \quad \varphi_k(y) = \rho_k^{-(n-\beta)-\varepsilon} |Du(y + \rho_k e_k) - Du(y)|^2 \quad \text{with} \quad \varepsilon = \frac{1}{2}(2\alpha - (n - \beta)).$$

When we set

$$(2.24) \quad \phi_k(y) = \sum_{j=1}^k \varphi_j(y),$$

one easily finds that the function $\phi_k(y)$ is a non-decreasing function for k and the following

$$(2.25) \quad \begin{aligned} \int_{\tilde{\Omega}} \phi_k(y) dy &= \sum_{j=1}^k \int_{\tilde{\Omega}} \varphi_j(y) dy \\ &= \sum_{j=1}^k \rho_j^{-(n-\beta)-\varepsilon} \int_{\tilde{\Omega}} |Du(y + \rho_j e_j) - Du(y)|^2 dy \leq C_8 \sum_{j=1}^k \rho_j^{2\alpha-(n-\beta)-\varepsilon} \\ &\leq C_8 \sum_{j=1}^k \delta^{\frac{1}{2}(2\alpha-(n-\beta))} \left(\frac{1}{2}\right)^{\frac{1}{2}\{2\alpha-(n-\beta)\}} = C_8 \delta^{\frac{1}{2}(2\alpha-(n-\beta))} \sum_{j=1}^k 2^{-\frac{1}{2}\{2\alpha-(n-\beta)\}} \leq C_9 < \infty. \end{aligned}$$

follows from (2.20) and assumption of $2\alpha - (n - \beta) = \beta - (n - 2\alpha) > 0$.

Thus $\{\phi_k\}_{k \geq 1}$ is a sequence of measurable functions and moreover, putting $\phi_\infty(y) = \lim_{k \rightarrow \infty} \phi_k(y)$, we obtain from *Beppo-Levi Theorem*

$$(2.26) \quad \int_{\tilde{\Omega}} \phi_\infty(y) dy = \lim_{k \rightarrow \infty} \int_{\tilde{\Omega}} \phi_k(y) dy \leq C_{10}.$$

Consequently, $\phi_\infty(y)$ is an integrable function on $\tilde{\Omega}$ and

$$(2.27) \quad \varphi_k(y) \leq \phi_\infty(y) \quad \text{for any } k \text{ and almost all } y \in \tilde{\Omega}.$$

To complete the proof of theorem, it is sufficient to show

$$(2.28) \quad S \subset E_\beta, \quad \text{namely,} \quad \text{if } x_0 \notin E_\beta, \quad \text{then } x_0 \notin S.$$

Now we fix $x_0 \notin E_\beta$, Then we show that the function

$$(2.29) \quad r \mapsto (Du)_{x_0, r} = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} (Du)(y) dy$$

is a continuous and bounded function in the open interval $(0, \delta)$ with $\delta = \text{dist}(x_0, \partial\Omega)$.

At first, we shall estimate $|(Du)_{x_0, R_i} - (Du)_{x_0, R_{i+1}}|$ ($i = 1, 2, \dots$). Also, by integrating the following (2.30) over $B_{R_i}(x_0)$ $R_i = \frac{\delta}{2}(\frac{1}{2})^i$ ($i = 1, 2, \dots$),

$$(2.30) \quad |(Du)_{x_0, R_i} - (Du)_{x_0, R_{i+1}}| \leq |(Du)_{x_0, R_i} - (Du)(x)| + |(Du)_{x_0, R_{i+1}} - (Du)(x)|.$$

we obtain

$$(2.31) \quad |B_{R_i}| \cdot |Du_{x_0, R_i} - Du_{x_0, R_{i+1}}| \leq \int_{B_{R_i}} |Du_{x_0, R_i} - Du(x)| dx + \int_{B_{R_i}} |Du_{x_0, R_{i+1}} - Du(x)| dx.$$

Next, dividing (2.31) by $|B_{R_i}|$ and by using Hölder inequality, we have

$$(2.32) \quad \begin{aligned} & |Du_{x_0, R_i} - Du_{x_0, R_{i+1}}| \\ & \leq \frac{1}{|B_{R_i}|} \int_{B_{R_i}} |Du_{x_0, R_i} - Du(x)| dx + \frac{1}{|B_{R_i}|} \int_{B_{R_i}} |Du_{x_0, R_{i+1}} - Du(x)| dx \\ & \leq \frac{1}{|B_{R_i}|} \int_{B_{R_i}} \left| \frac{1}{|B_{R_i}|} \int_{B_{R_i}} Du(y) dy - Du(x) \right| dx + \frac{1}{|B_{R_i}|} \int_{B_{R_i}} \left| \frac{1}{|B_{R_{i+1}}|} \int_{B_{R_{i+1}}} Du(y) dy - Du(x) \right| dx \\ & \leq \frac{1}{|B_{R_i}|^2} \int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)| dy + \frac{1}{|B_{R_i}| |B_{R_{i+1}}|} \int_{B_{R_i}} dx \int_{B_{R_{i+1}}} |Du(y) - Du(x)| dy \\ & \leq \frac{1 + 2^n}{|B_{R_i}|^2} \int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)| dy. \\ & \leq \frac{1 + 2^n}{|B_{R_i}|^2} \left[\int_{B_{R_i}} dx \int_{B_{R_i}} dy \right]^{\frac{1}{2}} \left[\int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)|^2 dy \right]^{\frac{1}{2}} \\ & \leq \frac{1 + 2^n}{|B_{R_i}|} \left[\int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)|^2 dy \right]^{\frac{1}{2}}. \end{aligned}$$

Here, we extend $Du(x)$ to be zero outside B_{R_i} and successively rewrite it to be $Du(x)$ for convenience. Then we continue the estimates of (2.32) as follows: From the change of variables,

$$\begin{cases} \bar{x} = x, \\ \bar{y} = y - x \end{cases}$$

we obtain

$$\begin{aligned}
& \frac{(1+2^n)}{|B_{R_i}|} \left[\int_{B_{R_i}} dx \int_{B_{R_i}} |(Du)(y) - (Du)(x)|^2 dy \right]^{1/2} \tag{2.33} \\
&= \frac{(1+2^n)}{|B_{R_i}|} \left[\int_{B_{R_i}} dx \int_{B_{2R_i}(x)} |(Du)(y) - (Du)(x)|^2 dy \right]^{1/2} \\
&= \frac{(1+2^n)}{|B_{R_i}|} \left[\int_{B_{R_i}} d\bar{x} \int_{B_{2R_i}(0)} |(Du)(\bar{x} + \bar{y}) - (Du)(\bar{x})|^2 d\bar{y} \right]^{1/2}.
\end{aligned}$$

By using *Fubini Theorem* and successively the mean value theorem, there exists a vector $\bar{y}_i^* \in R^n$ with $0 < |\bar{y}_i^*| < 2R_i$ such that

$$(2.34) \quad (2.33) = \left[\frac{c_{11}}{|B_{R_i}|} \int_{B_{R_i}} |(Du)(\bar{x} + \bar{y}_i^*) - (Du)(\bar{x})|^2 d\bar{x} \right].$$

From (2.32) and (2.34), we obtain

$$(2.35) \quad |(Du)_{x_0, R_i} - (Du)_{x_0, R_{i+1}}| \leq C_{12} \left[\frac{R_i^{n-\beta+\varepsilon}}{|B_{R_i}|} \int_{B_{R_i}} \frac{|(Du)(\bar{x} + \bar{y}_i^*) - (Du)(\bar{x})|^2}{|\bar{y}_i^*|^{n-\beta+\varepsilon}} d\bar{x} \right]^{\frac{1}{2}}.$$

Next we shall show that $\{Du_{x_0, r}\}$ ($r > 0$) is a *Cauchy filter*. Let r and R ($r < R$) be positive numbers sufficiently small and then we can take positive integer j and i ($i \leq j$) such that $R_{j+1} < r \leq R_j$ and $R_{i+1} < R \leq R_i$. We estimate $|Du_{x_0, r} - Du_{x_0, R}|$ by dividing it into the following three terms:

$$(2.36) \quad \begin{aligned} & |Du_{x_0, r} - Du_{x_0, R}| \\ & \leq |Du_{x_0, r} - Du_{x_0, R_j}| + |Du_{x_0, R_j} - Du_{x_0, R_i}| + |Du_{x_0, R_i} - Du_{x_0, R}| \end{aligned}$$

Thus, by the same way as above, for $0 < r < R < \delta$, the following holds:

$$(2.37) \quad |(Du)_{x_0, r} - (Du)_{x_0, R}| \leq C_{12} \sum_{k=i}^j R_k^{\varepsilon/2} \left[R_k^{-\beta} \int_{B_{R_k}} \frac{|(Du)(\bar{x} + \bar{y}_k^*) - (Du)(\bar{x})|^2}{|\bar{y}_k^*|^{n-\beta+\varepsilon}} d\bar{x} \right]^{\frac{1}{2}}.$$

We obtain from $\beta - (n - 2\alpha) \geq 0$,

$$\sum_{k=i}^j R_k^{\frac{\varepsilon}{2}} \leq R_i^{\frac{\varepsilon}{2}} \frac{1 - (\frac{\delta}{2})^{(j-i)\frac{\varepsilon}{2}}}{1 - (\frac{\delta}{2})^{\frac{\varepsilon}{2}}}$$

By noting

$$\frac{|(Du)(\bar{x} + \bar{y}_k^*) - (Du)(\bar{x})|^2}{|\bar{y}_k^*|^{n-\beta+\varepsilon}} \leq \phi_{\infty}(x) \quad a.e \quad \bar{x} \in \tilde{\Omega} \text{ and } k = 1, 2, \dots.$$

we can continue to estimate (2.37) as follows:

$$|(Du)_{x_0, r} - (Du)_{x_0, R}|$$

$$(2.38) \quad \leq C_{14} R^{\frac{\epsilon}{2}} \left[\operatorname{ess. sup}_{k>0} R_k^{-\beta} \int_{B_{R_k}} \phi_{\infty}(y) dy \right]^{\frac{1}{2}}.$$

Also, from (2.28), there exists a constant K such that

$$(2.39) \quad |(Du)_{x_0,r} - (Du)_{x_0,R}| \leq C_{14} K^{\frac{1}{2}} R^{\frac{\beta-(n-2\alpha)}{2}}$$

This shows that $\{(Du)_{x_0,r}\}_{r>0}$ is a *Cauchy filter*. Thus, $\lim_{R \rightarrow +0} (Du)_{x_0,R}$ surely exists. Also, from (2.39), we obtain

$$(2.40) \quad |(Du)_{x_0,r} - (Du)_{x_0,R}| \leq C_{14} K^{\frac{1}{2}} R^{\frac{\beta-(n-2\alpha)}{2}}.$$

Then

$$(2.41) \quad \lim_{R \rightarrow +0} |(Du)_{x_0,R}| \leq |(Du)_{x_0,\delta/4}| + C_{14} K^{\frac{1}{2}} (\delta/4)^{\frac{\beta-(n-2\alpha)}{2}}.$$

Consequently, $\lim_{R \rightarrow +0} (Du)_{x_0,R}$ exists and is finite. This shows $x_0 \notin S$.

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