

Local delta invariants of weak del Pezzo surfaces with the anti-canonical degree ≥ 5

Hiroto Akaike (Kyoto University, KTGU Researcher)

Let X be a log terminal Fano variety over a complex number field $\mathbb{C}$.

1 K-stability

The following problems were of importance.

When does X admit a weak Kähler Einstein (KE) metric?

Recently, the following was proved.

K-polystability $\iff \exists$ weak KE metric

In order to interpret the criterion for K-(poly)stability of X , the delta invariant is introduced in [3],[5]. It is known that

$$\begin{aligned}\delta(X) > 1 &\iff X:\text{K-stable} \\ &\iff X:\text{K-polystable and } \# \text{Aut}(X) < \infty.\end{aligned}$$

2 Local delta invariant

- Y : smooth variety
- $\sigma: Y \rightarrow X$: projective birational morphism
- $E \subset Y$: prime divisor

Let

$$\begin{aligned}A_X(E) &:= 1 + \text{ord}_E(K_Y - \sigma^* K_X), \\ S(E) &:= \frac{1}{(-K_X)^n} \int_0^\tau \text{vol}(\sigma^*(-K_X) - uE) du,\end{aligned}$$

where τ is the pseudo effective threshold of E with respect to $-K_X$, that is,

$$\tau := \sup\{u \in \mathbb{Q}_{\geq 0} \mid \sigma^*(-K_X) - uE \text{ is big}\}.$$

The **local delta invariant** $\delta_p(X)$ of X at $p \in X$ is as follows ([2]):

$$\delta_p(X) = \inf \left\{ \frac{A_X(E)}{S(E)} \mid E: \text{divisor}/X ; p \in C_X(E) \right\}.$$

Moreover, the **delta invariant** $\delta(X)$ of X is as follows ([3])

$$\delta(X) = \inf_{p \in X} \delta_p(X).$$

3 Main Results

It is known that there exist 7 (if $K^2 = 5$), 6 (if $K^2 = 6$), and 2 (if $K^2 = 7$) types of weak del Pezzo surface in terms of the configuration of negative curves, respectively.

A weak del Pezzo surface of anti-canonical degree 8 is one of the Hirzebruch surfaces of either degree 0, 1 or 2.

Main Result [1, Akaike]:

We determine the whole local delta invariants for all weak del Pezzo surfaces with the anti-canonical degree ≥ 5 .

4 Application

- (1) We get the delta invariants for any du Val del Pezzo surface with the anti-canonical degree ≥ 5 .
- (2) The estimation of the local delta invariant of weak del Pezzo surfaces is useful for the K-stability of higher dimensional Fano varieties (cf. [4, Lemma 24, 25]).

References

- [1] H. Akaike, Local delta invariants of weak del Pezzo surfaces with the anti-canonical degree ≥ 5 , arXiv:2304.09437[math.AG].
- [2] H. Abban, Z. Zhuang, K-stability of Fano varieties via admissible flags, Forum. of Mathematics. Pi. **10**: Paper No. e15 (2022).
- [3] H. Blum, M. Jonsson, Thresholds, valuations, and K-stability, Adv. Math. **365** (2020), 107062.
- [4] I. Cheltsov, K. Fujita, T. Kishimoto, T. Okada, K-stable divisors in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(1, 1, 2)$, arXiv:2206.08539v2 [math.AG]
- [5] K. Fujita, Y. Odaka, On the K-stability of Fano varieties and anticanonical divisors, Tohoku Math. J. **70** (2018), 511–521.