

Intertwining operator and C_2 -cofiniteness of modules

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Abstract

Let V be a vertex operator algebra and T a V -module. We show that if there are C_2 -cofinite V -modules U and W and a surjective (logarithmic) intertwining operator $\mathcal{Y}$ of type $\left(\begin{smallmatrix} T \\ U \quad W \end{smallmatrix}\right)$, then T is also C_2 -cofinite. So, when V is simple and $V' \cong V$, then if one of V -modules is C_2 -cofinite, then so is V .

1 Introduction

A vertex algebra was introduced by axiomatizing the concept of a Chiral algebra in conformal field theory by Borchers [1]. It is a triple $(V, Y, 1)$ satisfying the several axioms, where V is a graded vector space $V = \bigoplus_{i \in \mathbb{Z}} V_i$ over the complex number field $\mathbb{C}$, $Y(v, z) = \sum_{m \in \mathbb{Z}} v_m z^{-m-1} \in \text{End}(V)[[z, z^{-1}]]$ denotes a vertex operator of $v \in V$ on V , $1 \in V_0$ is a specified element called the vacuum. When V has another specified element $\omega \in V_2$ and V has a lower bound of weights and all homogeneous subspaces are of finite dimensional, then we call V a vertex operator algebra. We set $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-1}$.

For a VOA V -module W , we define $C_2(W) = \{v_{-2}u \mid v, u \in V, \text{wt}(v) \geq 1\}$. When $C_2(W)$ has a finite co-dimension in W , W is called to be C_2 -cofinite. A concept of C_2 -cofiniteness is originally introduced by Zhu [8] as a technical assumption to prove a modular invariance property of the space of the trace functions on modules. However, we are now recognizing the real meaning and the importance of C_2 -cofiniteness. For example, V is C_2 -cofinite if and only if all V -modules are $\mathbb{N}$ -gradable. (See [2] and [7] for the proof.) We will use this fact frequently in this paper.

Our main result in this paper is the following:

Theorem 1 *Let U be a vertex operator algebra of CFT-type. Let A, B, C be simple $\mathbb{N}$ -graded U -modules and $\mathcal{I}$ a surjective (formal power series) intertwining operator of type $\left(\begin{smallmatrix} C \\ A \quad B \end{smallmatrix}\right)$. If both of A and B are C_h -cofinite as U -modules for $h = 1, 2$, then so is C .*

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2 Preliminary

From the axiom of VOAs, for $v \in V_r$ and $u \in V_n$, we have $v_m u \in V_{r-m-1+n}$. Hence there is an integer N such that $v_s u = 0$ for any $s > N$. This property is called a truncation property. In this paper, we will say that “ v is truncated at u ” to simplify the terminology,

Set $V^* = \text{Hom}(V, \mathbb{C})$ and define a pairing $\langle \cdot, \cdot \rangle$ on $V^* \times V$ by $\langle \xi, v \rangle = \xi(v)$ for $\xi \in V^*$ and $v \in V$. For $T \subseteq V$, $\text{Annh}(T)$ denotes an annihilator of T , that is, $\text{Annh}(T) = \{\xi \in V^* \mid \langle \xi, t \rangle = 0 \text{ for all } t \in T\}$. For $v \in V$ and $m \in \mathbb{Z}$, an action v_m^* on V^* is defined by

$$\langle (\sum_{m \in \mathbb{Z}} v_m^* z^{-m-1}) \xi, w \rangle = \langle \xi, Y(e^{L(1)z} (-z^{-2})^{L(0)} v, z^{-1}) w \rangle$$

for $w \in V$ and $\xi \in \text{Hom}(V, \mathbb{C})$, where $Y^*(v, z) = \sum_{m \in \mathbb{Z}} v_m^* z^{-m-1}$ is called an adjoint operator of v . An important fact is that $(\oplus_{m \in \mathbb{Z}} \text{Hom}(V_m, \mathbb{C}), Y^*)$ becomes a V -module as they proved in [3]. This module is called a restricted dual of V and denoted by V' . In particular, $Y^*(\cdot, z)$ satisfy the Borcherds identity:

$$\sum_{i=0}^{\infty} \binom{m}{i} (u_{r+i}^* v^*)_{m+n-i} \xi = \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} \{u_{r+m-i}^* v_{n+i}^* \xi - (-1)^r v_{r+n-i}^* u_{m+i}^* \xi\} \quad (2.1)$$

for any $m, n, r \in \mathbb{Z}$, $v, u \in V$, $\xi \in V'$. We note $V' = \oplus_{n \in \mathbb{Z}} V_n$ and $V^* = \prod_{n \in \mathbb{Z}} V_n$. Therefore we can express $\xi \in V^*$ by $\prod_n \xi_n$ with $\xi_n \in \text{Hom}(V_n, \mathbb{C})$. We call that $\xi \in V^*$ is “ $L(0)$ -free” if $\dim \mathbb{C}[L(0)]\xi = \infty$, that is, $\xi_m \neq 0$ for infinitely many m . We note that any $\mathbb{N}$ -gradable module does not contain any $L(0)$ -free elements.

Let go back to (2.1). If $\xi \in \text{Hom}(V_t, \mathbb{C})$, then all terms in (2.1) have the same weight $\text{wt}(a) + \text{wt}(b) - r - m - n - 2 + t$ and so the Borcherds’ identity is also well-defined on V^* , as Li has pointed out in [5]. However, V^* is not a V -module because of failure of truncation properties. In order to find a V -module in V^* , we will start our arguments from one point ξ in V^* .

Lemma 2 *If u and v are truncated at ξ , then $v_m u$ is also truncated at ξ for any m . In particular, if all elements in Ω of V are truncated at ξ and $\langle \Omega \rangle_{VA} = V$, then all elements in V are truncated at ξ , where $\langle \Omega \rangle_{VA}$ denotes a vertex subalgebra generated by Ω .*

[Proof] By the assumption, there is an integer N such that $u_n \xi = v_n \xi = u_n v = 0$ for $n \geq N$. We assert that for $s \in \mathbb{N}$ and $n \geq 2N + s$, we have $(u_{N-s} v)_n \xi = 0$. Suppose false and let s be a minimal counterexample. Substituting $r = N - s$, $n = N + s + p$, $m = N + q$ in (2.1) with $p, q \geq 0$, we have

$$\begin{aligned} [\text{LeftSide}] &= \sum_{i=0}^{\infty} \binom{N+q}{i} (u_{N-s+i} v)_{2N+q+s+p-i} \xi = \sum_{i=0}^s \binom{N+q}{i} (u_{N-(s-i)} v)_{2N+s-i+p+q} \xi \\ &= (u_{N-s} v)_{2N+s+p+q} \xi \end{aligned}$$

by the minimality of s . On the other hand, we have:

$$[\text{RightSide}] = \sum_{i=0}^{\infty} (-1)^i \binom{N-s}{i} (u_{2N-s+q-i} v_{N+s+p+i} \xi - (-1)^{N-s} v_{2N-s+p-i} u_{N+q+i} \xi) = 0,$$

which contradicts the choice of s . ■

Since $v_n u_m \xi = u_m v_n \xi + \sum_{i=0}^{\infty} \binom{n}{i} (v_i u)_{n+m-i} \xi$, the above lemma also implies:

Lemma 3 *If v and u are truncated at ξ , then v is truncated at $u_m \xi$ for any m . In particular, if all elements of V are truncated at ξ , then $\langle u_{m_1}^1 \cdots u_{m_k}^k \xi \mid u^i \in V, m_i \in \mathbb{Z} \rangle_{\mathbb{C}}$ is a V -module.*

As Buhl has shown in [2], if V is C_2 -cofinite, then all V -modules are $\mathbb{N}$ -gradable and so there are no $L(0)$ -free elements at which all elements in V are truncated. Namely, we have proved the following, which we will frequently use.

Lemma 4 *Let V be a C_2 -cofinite vertex operator algebra and $\xi \in V^*$. If $\Omega \subseteq V$ generates V as a vertex subalgebra and all elements of Ω are truncated on ξ , then ξ is not $L(0)$ -free.*

For $A, B \subseteq V$, we will often use the notation $A_{(m)}B$ to denote a subspace spanned by $\{a_m b \mid a \in A, b \in B\}$. We note that if A is a $\mathbb{C}[L(-1)]$ -module, then $A_{(-2-m)}B \subseteq A_{(-2)}B$ for $m \in \mathbb{N}$ since $(L(-1)a)_{-m}b = ma_{-m-1}b$ for $a \in A$ and $b \in B$. Not only V , we use this notation for a pair (U, W) of a VOA U and its module W . For example, we set $C_2(W) = U_{(-2)}^+ W$, where $U^+ = \bigoplus_{k=1}^{\infty} U_k$. We also set $C_1(W) = U_{(-1)}^+ W$. We say that W is C_h -cofinite as a U -module if $\dim W/C_h(W) < \infty$ for $h = 1, 2$. We note any VOA U is C_1 -cofinite as a U -module and so this definition is not equal to the ordinary C_1 -cofiniteness.

We start the proof of Theorem 1. Namely, we will prove:

Theorem 1 *Let U be a vertex operator algebra of CFT-type. Let A, B, C be simple $\mathbb{N}$ -graded U -modules and $\mathcal{I}$ a surjective (formal power series) intertwining operator of type $\begin{pmatrix} C \\ A \quad B \end{pmatrix}$. If both of A and B are C_h -cofinite as U -modules for $h = 1, 2$, then so is C .*

We note that if U is of CFT-type and an $\mathbb{N}$ -graded U -module $A = \bigoplus_{k=0}^{\infty} A_{r+k}$ is C_1 -cofinite, then $\dim A_{r+k} < \infty$ for any k since $A_{r+k} \cap C_1(A) = \sum_{s=1}^{k-1} (U_s)_{-1} A_{r+k-s}$ has a finite codimension in A_{r+k} .

In the remainder part of this section, we assume the hypotheses of Theorem 1. Since A and B are C_h -cofinite, there are finite dimensional subspaces $F^1 \subseteq A$ and $F^2 \subseteq B$ such that $A = U_{(-h)}^+ A + F^1$ and $B = U_{(-h)}^+ B + F^2$. Let c_A and c_B be conformal weights of A and B , respectively. We may assume that there is an integer N such that $F^1 = \bigoplus_{k=0}^N A_{c_A+k}$ and $F^2 = \bigoplus_{k=0}^N B_{c_B+k}$. Fix bases $\{p^i \mid i \in I\}$ of F^1 and $\{q^j \mid j \in J\}$ of F^2 . In order to prove Theorem 1, we prove the following lemma by applying an idea in [4] to $(C/U_{(-h)}^+ C)^*$.

Lemma 5 *For $p \in A, q \in B$ and $\theta \in \text{Ann}_h(U_{(-h)}^+ C) \cap C'$,*

$$F(\theta, p, q; z) := \langle \theta, \mathcal{I}(p, z)q \rangle$$

is a linear combination of $\{F(\theta, p^i, q^j; z) \mid i \in I, j \in J\}$ with coefficients in $\mathbb{C}[z, z^{-1}]$ and we may choose these coefficients independently of the choice of θ .

[Proof] We will prove the assertion by the induction on the total weight $\text{wt}(p) + \text{wt}(q)$. If $\text{wt}(p) > N + c_B$, then $p = \sum_k u^k_{-h} a^k$ for some $u^k \in U$ and $a^k \in A$. We note this expression does not depend on the choice of θ . So we may assume $p = u_{-h}a$ with $u \in U$ and $a \in A$. Then for $\theta \in \text{Ann}(U_{(-h)}^+C)$, we have:

$$\begin{aligned} \langle \theta, \mathcal{I}(p, z)q \rangle &= \langle \theta, \mathcal{I}(u_{-h}a, z)q \rangle \\ &= \langle \theta, Y^-(L(-1)^{h-1}u, z)\mathcal{I}(a, z)q + \mathcal{I}(a, z)Y^+(L(-1)^{h-1}u, z)q \rangle \\ &= \langle \theta, \mathcal{I}(a, z)Y^+(L(-1)^{h-1}u, z)q \rangle, \end{aligned}$$

where $Y^-(v, z) = \sum_{m < 0} v_m z^{-m-1}$ and $Y^+(v, z) = \sum_{m \geq 0} v_m z^{-m-1}$. This is a reduction on the sum of weights because $Y^+(L(-1)^{h-1}u, z)q$ is a sum of finite terms and all weights of the coefficients are less than $\text{wt}(u) + \text{wt}(q)$.

Similarly, if $\text{wt}(q) > N + c_B$, then we may assume $q = u_{-h}b$ with $u \in U$ and $b \in B$ and

$$\begin{aligned} \langle \theta, \mathcal{I}(p, z)q \rangle &= \langle \theta, \mathcal{I}(p, z)u_{-h}b \rangle \\ &= \langle \theta, u_{-h}\mathcal{I}(p, z)b \rangle + \sum_{i=0}^{\infty} \binom{-h}{i} z^{-h-i} \langle \theta, \mathcal{I}(u_i p, z)b \rangle \\ &= \sum_{i=0}^{\infty} \binom{-h}{i} z^{-h-i} \langle \theta, \mathcal{I}(u_i p, z)b \rangle. \end{aligned}$$

Again, these process do not depend on the choice of θ and this is also a reduction on the weights because $\text{wt}(u_i p) + \text{wt}(b) < \text{wt}(u_{-h}b) + \text{wt}(p)$ for $i \geq 0$. Therefore, $\langle \theta, \mathcal{I}(p, z)q \rangle$ is a linear combination of $\{\langle \theta, \mathcal{I}(p^i, z)q^j \rangle \mid i \in I, j \in J\}$ with coefficients in $\mathbb{C}[z, z^{-1}]$. We note the coefficients do not depend on the choice of θ . ■

Now we are able to prove Theorem 1. By the proof of the above lemma,

$$\frac{d}{dz} F(\theta, p^s, q^t; z) = F(\theta, L(-1)p^s, q^t; z)$$

is a linear combination of $\{F(\theta, p^i, q^j; z) \mid i \in I, j \in J\}$ with coefficients in $\mathbb{C}[z, z^{-1}]$ for any $s \in I, t \in J$ and all coefficients do not depend on the choice of θ . Therefore, there is a differential linear equation such that $F(\theta, p^s, q^t)$ are all its solutions for any $s \in I, t \in J$ and θ . Furthermore, since $\{\mathcal{I}(p, z)q \mid p \in A, q \in B, z \in \mathbb{Z}\}$ spans C modulo $U_{(-h)}^+C$ and $\langle \theta, \mathcal{Y}(p, z)q \rangle$ are a linear sum of $\langle \theta, \mathcal{I}(p^i, z)q^j \rangle$, $\theta \in C' \cap \text{Ann}(U_{(-2)}^+C) \rightarrow \prod_{i \in I, j \in J} \langle \theta, \mathcal{I}(p^i, z)q^j \rangle$ is injective. Therefore, we have $\dim C/U_{(-h)}C < \infty$. This completes the proof of Theorem 1.

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